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GEOMETRICALLY NONLINEAR ANALYSIS OF LAMINATED COMPOSITE PLATES

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ABSTRACT. The low mass density and the high tensile strength, usually expressed through the specific modulus of elasticity and the specific strength, have made composite materials lighter and stronger compared with most traditional materials (such as steel, concrete, wood, etc.) and have increased their application not only for secondary, but during the last two decades also for primarily structural members in aerospace and automotive industry, ship building industry and bridge design. Although weight saving has eliminated constrain of slenderness and thickness and has made possible use of very thin plate elements, they have become susceptible to large deflections. In such cases, the geometry of structures is continually changing during the deformation and geometrically nonlinear analysis should be adopted. In this paper the geometrically nonlinear laminated plate finite element model is obtained using the principle of virtual displacement. With the layerwise displacement field of Reddy [1], nonlinear Green-Lagrange small strain large displacements relations (in the von Karman sense) and linear elastic orthotropic material properties for each lamina, the 3D elasticity equations are reduced to 2D problem and the nonlinear equilibrium integral form is obtained. The obtained displacement dependent secant stiffness matrix is utilized in Direct interation procedure for the numerical solution of nonlinear finite element equilibrium equations. The originally coded MATLAB computer program for the finite element solution is used to verify the accuracy of the numerical model, by calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different load direction (unloading/loading). The obtained results are compared with available results from the literature and the linear solutions from the previous paper [2].

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Abstract. The low mass density and the high tensile strength, usually expressed through the specific modulus of elasticity and the specific strength, have made composite materials lighter and stronger compared with most traditional materials (such as steel, concrete, wood, etc.) and have increased their application not only for secondary, but during the last two decades also for primarily structural members in aerospace and automotive industry, ship building industry and bridge design. Although weight saving has eliminated constrain of slenderness and thickness and has made possible use of very thin plate elements, they have become susceptible to large deflections. In such cases, the geometry of structures is continually changing during the deformation and geometrically nonlinear analysis should be adopted. In this paper the geometrically nonlinear laminated plate finite element model is obtained using the principle of virtual displacement. With the layerwise displacement field of Reddy [1], nonlinear Green-Lagrange small strain large displacements relations (in the von Karman sense) and linear elastic orthotropic material properties for each lamina, the 3D elasticity equations are reduced to 2D problem and the nonlinear equilibrium integral form is obtained. The obtained displacement dependent secant stiffness matrix is utilized in Direct interation procedure for the numerical solution of nonlinear finite element equilibrium equations. The originally coded MATLAB computer program for the finite element solution is used to verify the accuracy of the numerical model, by calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different load direction (unloading/loading). The obtained results are compared with available results from the literature and the linear solutions from the previous paper [2].

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1. Introduction

The low mass density (ρ) and the high tensile strength (σ_u), usually expressed through the specific modulus of elasticity (E/ρ) and the specific strength (σ_u/ρ) have made composite materials lighter and stronger compared with most traditional materials (such as steel, concrete, wood, etc.) and have increased their application not only for secondary, but during the last two decades also for primarily structural members in aerospace and automotive industry, ship building industry and bridge design. The advanced mechanical properties of composite materials, which are resulted in large weight savings, have given designers more flexibility in finding efficient solution for specific problem, but have also required formulation of mathematical model able to present their complex anisotropic nature. Although weight saving has eliminated constrain of slenderness and thickness and has made possible use of very thin plate elements, they have become susceptible to large deflections [4,5]. In such cases, the geometry of structures is continually changing during the deformation and geometrically nonlinear analysis should be adopted. The geometrically nonlinear analysis seems also to be necessary for obtaining the structural response of unsymmetrical laminated composite materials [6]. Namely, the nonlinear response of these laminates is present even for small displacements, due to complex coupling between in-plane and out-of plane deformation.

A considerable amount of research work has been carried out so far on the nonlinear analysis of laminated plates. Among the published works, the von Karman plate theory of plates undergoing large deflections has attracted outstanding attention and a number of papers have been published. The first authors investigating the nonlinear response using the von Karman nonlinear theory [7, 8] were: Leissa, Bennett, Bert, Chandra and Raju, Zaghloul and Kennedy, Chia and Prabhakara, Noor and Hartley, and in the last decades Han, Tabiei and Park, Singh, Lal and Kumar, Reddy and Chao, Zhang Kim and others.

Mechanical response of laminated composite material is generally 3D problem of nonlinear mechanics. However, due to its mathematical complexity, analytical solutions using 3D theory of elasticity are usually difficult and some times even impossible to achieve, while numerical solutions are computationally inefficient and constrained to very specific domains. Thus, whenever possible, refined simplified mathematical models, with acceptable accuracy in a field of applications, should be used. It is shown that the Equivalent Single Layer theories (ESL) may give acceptable results when analyzing global response, such as gross deflections and gross stresses, critical buckling loads and fundamental frequencies of thin to moderate thick laminated composite plates [9]. However, a continuous displacement function in ESL is not able to accurately present the discontinuous zigzag variation of displacements in highly anisotropic plates and give adequate stress distribution at local or ply level [2]. A compromise between 3D theory of elasticity and ESL theories is then achieved with the use of Layer Wise theories (LW). In LW theories the in-plane displacement field, assumed for each layer, is interpolated through the thickness by appropriate layerwise Lagrange interpolation function or Heaviside step function [3], thus replacing 3D laminated element with N+1 2D plate elements (N is number of layers), which fulfills the continuity of displacement functions at the interfaces between adjacent layers.

From the continuum mechanics it is known that two different level of geometrical nonlinearity may be modeled, which are: geometrically nonlinear models with small strain and large displacements (von Karman theory) and geometrically nonlinear models with large strains. In the first case, the geometry of the structure before deformation remains unchanged after the deformation. However, the structure is subjected to large displacements and the equilibrium is achieved on the configuration displaced from the undeformed one. In the second case the geometry of the structure is changing during the deformation and the equilibrium is achieved on the deformed configuration. In both cases equilibrium equations are nonlinear.

In order to formulate nonlinear finite element model of laminated structures, which will be able to represent two above mentioned levels of geometrical nonlinearity, two distinct approaches have been reported in the literature [3]. The first approach is based on laminate theory, in which 3D elasticity equations are reduced to 2D equations through certain kinematical assumptions and homogenization through the thickness. In this approach only first type of nonlinearity or small strain, large displacement assumption may be included. The finite elements based on such an assumptions are named the laminated elements. The second approach is based on 3D continuum formulation (total and updated Lagrange formulation) and both types on nonlinearity may be included. Finite elements based on this approach are called the continuum elements.

The aim of the author's research on composite materials so far was to implement Layerwise theory of Reddy or Generalized Layerwise Plate Theory-GLPT [1] on different levels of analysis of laminated composite plates. The previous work has been concerned with the linear analysis [2], and the linear laminated plate element of GLPT has been formulated, while in the present paper the GLPT nonlinear laminated plate element with von Karman geometrical nonlinearity is presented.

In this paper the mathematical and numerical model for geometrically nonlinear, small strain, large displacements problem of laminated composite plates is presented. The 3D elasticity equations are reduced to 2D problem using kinematical assumptions based on layerwise displacement field of Reddy (GLPT). With the assumed displacement field, nonlinear Green-Lagrange small strain large displacements relations and linear orthotropic material properties for each lamina, the principle of virtual displacement (PVD) is used to derive the weak form of the problem. The weak form or nonlinear integral equilibrium equations are discretized using isoparametric finite element approximation. The obtained nonlinear incremental algebric equilibrium equations are solved using direct iteration procedure. The originally coded MATLAB computer program for the finite element solution is used to investigate the effects of geometrical nonlinearity on displacement and stress field of thin and thick, isotropic, orthotropic and anisotropic laminated composite plates with various boundary conditions and loading direction (loading/unloading). The accuracy of the numerical model is verified by being compared with available results from the literature and the linear solutions from the previous paper [2]. The appropriate conclusions are derived.

2. Theoretical formulation

2.1 Displacement field

In the LW theory of Reddy [1] or Generalized Layerwise Plate Theory (GLPT), in-plane displacements components (u,v) are interpolated through the thickness using 1D linear Lagrangian interpolation function $\Phi^I(z)$, while transverse displacement component w is assumed to be constant through the plate thickness.

$$\begin{aligned} u_1(x,y,z) &= u(x,y) + \sum_{I=1}^{N+1} U^I(x,y)\cdot\Phi^I(z) \\ u_2(x,y,z) &= v(x,y) + \sum_{I=1}^{N+1} V^I(x,y)\cdot\Phi^I(z), \\ u_3(x,y,z) &= w(x,y) \end{aligned} \tag{1}$$

2.2 Strain-displacement relations

The Green Lagrange strain tensor associated with the displacement field Eq.(1) can be computed using von Karman strain-displacement relation to include geometric nonlinearities as follows:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \sum_{I=1}^{N+1} \frac{\partial U^I}{\partial x}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
\varepsilon_{yy} &= \frac{\partial u_2}{\partial y} + \frac{1}{2}\left(\frac{\partial u_3}{\partial y}\right)^2 = \frac{\partial v}{\partial y} + \sum_{I=1}^{N+1} \frac{\partial V^I}{\partial y}\Phi^I + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\
\gamma_{xy} &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x}\frac{\partial u_3}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N+1}\left(\frac{\partial U^I}{\partial y} + \frac{\partial V^I}{\partial x}\right)\Phi^I + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}, \qquad (2) \\
\gamma_{xz} &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \sum_{I=1}^{N+1} U^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial x}, \\
\gamma_{yz} &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \sum_{I=1}^{N+1} V^I \frac{d\Phi^I}{dz} + \frac{\partial w}{\partial y}.
\end{aligned}$$

2.3 Constitutive equations

For Hook's elastic material, the stress-strain relations for k-th orthotropic lamina have the following form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\ Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}^{(k)} \times \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)}. \qquad (3)$$

Where:
$\boldsymbol{\sigma}^{(k)} = \{\sigma_{xx} \quad \sigma_{yy} \quad \tau_{xy} \quad \tau_{xz} \quad \tau_{yz}\}^{(k)T}$ and $\boldsymbol{\varepsilon}^{(k)} = \{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \gamma_{xy} \quad \gamma_{xz} \quad \gamma_{yz}\}^{(k)T}$ are stress and strain components respectively, and $\mathbf{Q}_{ij}^{(k)}$ are transformed elastic coefficients, of k-th lamina in global coordinates.

2.4 Equilibrium equations

Equilibrium equations may be obtained from the Principle of Virtual Displacements (PVD), in which sum of external virtual work done on the body and internal virtual work stored in the body should be equal zero:

$$0=\iint_{\Omega}\left[\left(\{\delta\varepsilon^0\}^T+\{\delta\varepsilon^m\}^T\right)\{N^0\}+\{\delta\varepsilon^I\}^T\{N^I\}+\delta u\, q_x^0+\delta v\, q_y^0+\delta w\, q_z^0\right]dxdy$$
$$-\oint_{\Gamma}\delta u_n\, N_{nn}\, ds-\oint_{\Gamma}\delta u_s\, N_{ns}\, ds-\oint_{\Gamma}\delta w\left(Q_n+P_n\right)ds-\oint_{\Gamma}\delta U_n^I\, N_{nn}^I\, ds-\oint_{\Gamma}\delta U_s^I\, N_{ns}^I ds \tag{4}$$

where $\{q_x^0, q_y^0, q_z^0\}$ is distributed load in x, y, z directons, while internal forces are:

$$\begin{Bmatrix}\{\mathbf{N}^0\}\\ \{\mathbf{N}^I\}\end{Bmatrix}=\begin{bmatrix}[\mathbf{A}] & [\mathbf{B}^I]\\ [\mathbf{B}^I] & \sum_{J=1}^{N}[\mathbf{D}^{JI}]\end{bmatrix}\begin{Bmatrix}\{\boldsymbol{\varepsilon}^0\}+\{\boldsymbol{\varepsilon}^m\}\\ \{\boldsymbol{\varepsilon}^I\}\end{Bmatrix} \tag{5}$$

Where **A**, **B**, $\mathbf{B^I}$, $\mathbf{D^{JI}}$ matrices are given in [12], while internal force vectors are:

$$\{\mathbf{N}^0\}=\{N_{xx}\quad N_{yy}\quad N_{xy}\quad Q_x\quad Q_y\}^T,\qquad \{\mathbf{N}^I\}=\{N_{xx}^I\quad N_{yy}^I\quad N_{xy}^I\quad Q_x^I\quad Q_y^I\}^T$$

$$N_{nn}=N_{xx}n_x+N_{xy}n_y,\quad N_{ns}=N_{xy}n_x+N_{yy}n_y,\quad Q_n=Q_xn_x+Q_yn_y,$$

$$P_n=\left(N_{xx}\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)n_x+\left(N_{xy}\frac{\partial w}{\partial x}+N_{yy}\frac{\partial w}{\partial y}\right)n_y,$$

$$N_{nn}^I=N_{xx}^In_x+N_{xy}^In_y,\quad N_{ns}^I=N_{xy}^In_x+N_{yy}^In_y.$$

and strain vectors are:

$$\{\boldsymbol{\varepsilon}^0\}=\left\{\frac{\partial u}{\partial x}\quad \frac{\partial v}{\partial y}\quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\quad \frac{\partial w}{\partial x}\quad \frac{\partial w}{\partial y}\right\}^T,$$

$$\{\boldsymbol{\varepsilon}^m\}=\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\quad \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial x}\quad 0\quad 0\right\}^T,$$

$$\{\boldsymbol{\varepsilon}^I\}=\left\{\frac{\partial U^I}{\partial x}\quad \frac{\partial V^I}{\partial y}\quad \frac{\partial U^I}{\partial y}+\frac{\partial V^I}{\partial x}\quad U^I\quad V^I\right\}^T.$$

3. Finite Element Model

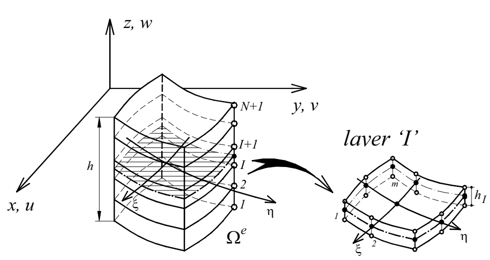

Figure 1. Plate finite element with n layers and m nodes

The GLPT finite element consists of middle surface plane and I=1, N+1 planes through the plate thickness Figure 1. The element requires only the C^0 continuity of major unknowns, thus in each node only displacement components are adopted, that are (u, v, w) in the middle surface element nodes and (U^I, V^I) in the I-th plane element nodes. The generalized displacements over element Ω^e can be expressed as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} u_j \Psi_j \\ \sum_{j=1}^{m} v_j \Psi_j \\ \sum_{j=1}^{m} w_j \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\boldsymbol{\Psi}_j]^e \{\mathbf{d}_j\}^e \quad \begin{Bmatrix} U^I \\ V^I \end{Bmatrix}^e = \begin{Bmatrix} \sum_{j=1}^{m} U_j^I \Psi_j \\ \sum_{j=1}^{m} V_j^I \Psi_j \end{Bmatrix}^e = \sum_{j=1}^{m} [\overline{\boldsymbol{\Psi}}_j]^e \{\mathbf{d}_j^I\}^e \tag{6}$$

where $\{\mathbf{d}_j\}^e = \{u_j^e \quad v_j^e \quad w_j^e\}^T, \{\mathbf{d}_j^I\}^e = \{U_j^I \quad V_j^I\}^T$ are displacement vectors, in the middle plane and I-th plane, respectively, Ψ_j^e are interpolation functions, while $[\boldsymbol{\Psi}_j]^e, [\overline{\boldsymbol{\Psi}}_j]^e$ are interpolation function matrix for the j-th node of the element Ω^e, given in [2].
Substituting element displacement field Eq.(6) in to weak form Eq.(4), the nonlinear laminated finite element is obtained:

$$[\mathbf{K}_{NL}]^e \cdot \{\mathbf{d}\}^e = \{\mathbf{f}\}^e \tag{7}$$

where secant stiffness matrix is:

$$[\mathbf{K}_{NL}]^e = \begin{bmatrix} [\mathbf{K}^{11}]^e & [\mathbf{K}^{12}]^e \\ [\mathbf{K}^{12}]^e & [\mathbf{K}^{22}]^e \end{bmatrix}$$

$$[\mathbf{K}^{11}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{\Omega^e} [\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e] + \underline{[\mathbf{H}_i^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{j\,NL}^e]} + \underline{2\,[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_j^e]} + \underline{2\,[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{A}] \cdot [\mathbf{H}_{j\,NL}^e]} \, d\Omega^e$$

$$[\mathbf{K}^{12}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{\Omega^e} \left([\mathbf{H}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e] + \underline{2\,[\mathbf{H}_{i\,NL}^e]^T \cdot [\mathbf{B}^I] \cdot [\overline{\mathbf{H}}_j^e]} \right) d\Omega^e$$

$$[\mathbf{K}^{21}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{\Omega^e} \left([\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_j^e] + \underline{[\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{B}^I] \cdot [\mathbf{H}_{j\,NL}^e]} \right) d\Omega^e$$

$$[\mathbf{K}^{22}]^e = \sum_{i=1}^{m}\sum_{j=1}^{n} \int_{\Omega^e} [\overline{\mathbf{H}}_i^e]^T \cdot [\mathbf{D}^{IJ}] \cdot [\overline{\mathbf{H}}_j^e] \, d\Omega^e \tag{8}$$

and external force vectors $\{\mathbf{f}\}^e = \begin{Bmatrix} \{\mathbf{f}^0\}^e \\ \{\mathbf{f}^I\}^e \end{Bmatrix}$ are:

$$\{\mathbf{f}^0\}^e = \sum_{i=1}^{m} \left[\int_{\Omega^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} q_x^0 \\ q_y^0 \\ q_z^0 \end{Bmatrix} d\Omega^e + \oint_{\Gamma^e} [\mathbf{\Psi}_i^e]^T \begin{Bmatrix} N_{nn} \\ N_{ns} \\ Q_n + P_n \end{Bmatrix} d\Gamma^e \right]$$

$$\{\mathbf{f}^I\}^e = \sum_{i=1}^{m} \left[\int_{\Omega^e} [\overline{\mathbf{\Psi}}_i^e]^T \cdot \begin{Bmatrix} q_x^I \\ q_y^I \end{Bmatrix} d\Omega^e + \oint_{\Gamma^e} [\overline{\mathbf{\Psi}}_i^e]^T \cdot \begin{Bmatrix} N_{nn}^I \\ N_{ns}^I \end{Bmatrix} d\Gamma^e \right] \tag{9}$$

while:

$$[\mathbf{H}_j^e]_j^e = \begin{bmatrix} \frac{\partial \Psi_j^e}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \Psi_j^e}{\partial y} & 0 \\ \frac{\partial \Psi_j^e}{\partial y} & \frac{\partial \Psi_j^e}{\partial x} & 0 \\ 0 & 0 & \frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & \frac{\partial \Psi_j^e}{\partial y} \end{bmatrix}, \quad [\mathbf{H}_{j\,NL}^e] = \frac{1}{2} \begin{bmatrix} 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & \frac{\partial w}{\partial y}\frac{\partial \Psi_j^e}{\partial y} \\ 0 & 0 & \frac{\partial w}{\partial x}\frac{\partial \Psi_j^e}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \Psi_j^e}{\partial x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\overline{\mathbf{H}}_j^e]_j^e = \begin{bmatrix} \frac{\partial \Psi_j^e}{\partial x} & 0 \\ 0 & \frac{\partial \Psi_j^e}{\partial y} \\ \frac{\partial \Psi_j^e}{\partial y} & \frac{\partial \Psi_j^e}{\partial x} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{10}$$

With the known displacement field, the stress field over the element may be obtained as a part of a postprocessor, using strain displacement and constitutive relations, Eqs. (2), (3) as:

$$\{\mathbf{\sigma}_b\}_U^{(\kappa)\,e} = [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} \left([\mathbf{H}_{bj}] + [\mathbf{H}_{bj}^{NL}] \right) \{\mathbf{d}_j\}^e + [\mathbf{Q}_b]^{(k)} \sum_{j=1}^{m} [\overline{\mathbf{H}}_{b_j}] \{\mathbf{d}_j^I\}^e$$

$$\{\boldsymbol{\sigma}_b\}_O^{(\kappa)\,e} = [\mathbf{Q_b}]^{(k)} \sum_{j=1}^{m} \left([\mathbf{H_{bj}}] + [\mathbf{H_{bj}^{NL}}] \right) \{\mathbf{d}_j\}^e + [\mathbf{Q_b}]^{(k)} \sum_{j=1}^{m} [\overline{\mathbf{H}}_{\mathbf{b}_j}] \{\mathbf{d}_j^{I+1}\}^e$$

$$\{\boldsymbol{\sigma}_s\}_{const}^{(k)\quad e} = [\mathbf{Q_s}]^{(k)} \sum_{j=1}^{m} [\mathbf{H_{sj}}] \{\mathbf{d}_j\}^e + [\mathbf{Q_s}]^{(k)} \sum_{j=1}^{m} [\overline{\mathbf{H}}_{\mathbf{sj}}] \left(\{\mathbf{d}_j^{I+1}\}^e - \{\mathbf{d}_j^{I}\}^e \right) / h_k \qquad (11)_{1,2,,3}$$

where $\{\boldsymbol{\sigma}_b\}_U^{(\kappa)\,e}$ and $\{\boldsymbol{\sigma}_b\}_O^{(\kappa)\,e}$ are in-plane normal stresses $(\sigma_{xx}, \sigma_{yy}, \tau_{xy})$ at bottom and upper plane in k-th layer of plate element 'e', while $\{\boldsymbol{\sigma}_s\}_{const}^{(k)\quad e}$ are average transverse shear stresses (τ_{xz}, τ_{yz}) in k-the layer of plate element.

4. Numerical results and discussion

Based on the previously derived laminated finite element model for the geometrically nonlinear analysis of laminated composite plates, the original computer program is coded using MATLAB programming language. The nonlinear finite element secant stiffness matrix is evaluated using Gauss–Legendre quadrature rule, which are 3x3 Gauss integration schemes or 2D quadratic Lagrange rectangular element for in-plane interpolation and 1D linear Lagrange element for through the thickness interpolation. The Direct iteration numerical method is used to solve nonlinear incremental equilibrium equations. The effects of plate thickness, lamination scheme, boundary conditions and load direction on nonlinear response of isotropic, orthotropic and anisotropic plates are analyzed. The accuracy of the present formulation is demonstrated through a number of examples and by comparison with results available from the literature.

The following boundary conditions at the plate edges are analyzed [10].

Simply supported (SS):

$$\textbf{SS:} \quad \begin{cases} x = 0, a: & v_0 = w_0 = V^I = N_{xx} = N_{xx}^I = 0 \\ y = 0, b: & u_0 = w_0 = U^I = N_{yy} = N_{yy}^I = 0 \end{cases} \quad I = 1, \ldots N+1 \qquad (12)$$

Simply supported-hinged (HH):

$$\textbf{HH:} \quad \begin{cases} x = 0, a: & u_0 = v_0 = w_0 = V^I = N_{xx}^I = 0 \\ y = 0, b: & u_0 = v_0 = w_0 = U^I = N_{yy}^I = 0 \end{cases} \quad I = 1, \ldots N+1 \qquad (13)$$

Clamped (CC):

$$\textbf{CC:} \quad \begin{cases} x = 0, a: & u_0 = v_0 = w_0 = U^I = V^I = 0 \\ y = 0, b: & u_0 = v_0 = w_0 = U^I = V^I = 0 \end{cases} \quad I = 1, \ldots N+1 \qquad (14)$$

When analyzing a quarter of a plate, boundary conditions in the plane of symmetry become:

For cross ply laminates:

$$\textbf{SS1:} \quad \begin{cases} x = a/2: & u_0 = U^I = N_{yy} = N_{yy}^I = 0 \\ y = b/2: & v_0 = V^I = N_{xx} = N_{xx}^I = 0 \end{cases} \quad I = 1, \ldots N+1 \qquad (15)$$

For angle ply laminates:

$$\textbf{SS2:}\quad \begin{cases} x = a/2: & v_0 = U^I = N_{xx} = N^I_{yy} = 0 \\ y = b/2: & u_0 = V^I = N_{yy} = N^I_{xx} = 0 \end{cases} \quad I = 1,\ldots N+1 \tag{16}$$

Example 4.1. A nonlinear bending of square, simply supported (SS1), isotropic plate, with $a = b = 10\,in$ and $h = 1\,in$ made of material:

$$E = 7.8\cdot 10^6 \text{ psi}, \quad \nu = 0.3 \tag{17}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4 / \left(E_2 h^4\right)$, the incremental load vector is chosen to be:

$$\{\Delta P\} = \{6.25, 6.25, 12.5, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0\}\cdot \overline{P} \tag{18}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}8$. The displacements and stresses are given in following nondimensional form:

$$\overline{w} = w_0 \cdot E_2 h^3 / \left(q_0 \cdot a^4\right), \quad \overline{\sigma}_{xx} = \sigma_{xx}\cdot (a/h)^2 \cdot 1/E \tag{19}$$

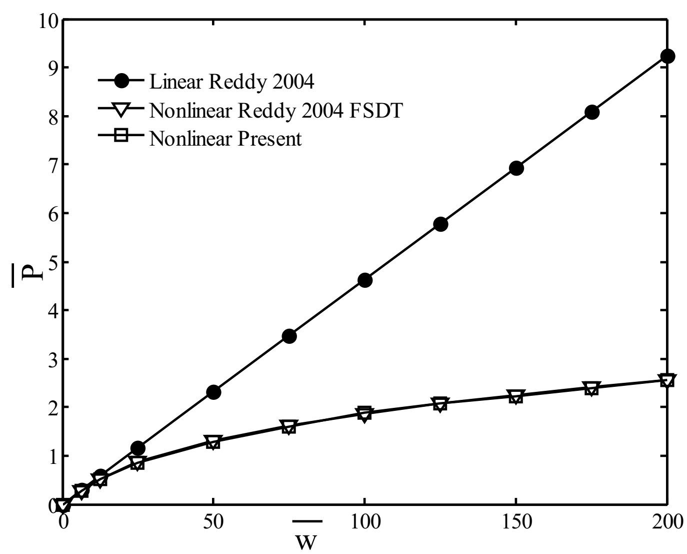

Figure 2. Nonlinear bending of square simply supported (SS1) isotropic plate with $a/h = 10$; central displacement versus load parameter

A 3x3 quarter plate laminated GLPT model is compared with 4x4 quadratic FSDT model [3]. The results for linear and nonlinear deflections are presented on Figure 2. It is shown that proposed GLPT model closely agree with FSDT model. The Figure 2 also demonstrates the physical nature of geometrically nonlinear response. The study has proved that depending of applied load level, the plate goes from the state of pure bending, at small displacement ($w \leq 0.30h$) to the phase of bending-stretching coupling, at large displacements. Namely, when the lateral displacement reaches approximately one half of plate thickness ($w \approx 0.5.h$), they take part in stretching, together with bending of the plate middle surface (nonlinear terms in Eq.(2)). This activates the tensile forces, thus enlarging the stiffness of the plates, and reducing displacements and stresses from the values predicted by linear theory. This may be the reason why this phenomena is also known as "plate stiffening" or "stress relaxation". Moreover, the activation of tensile forces in laminated composite plates is of utmost importance, due to their high available specific tensile strength.

Example 4.2. A nonlinear bending of square simply supported (SS1), orthotropic plate made of high modulus glass-epoxy fiber reinforced material:

$$E_1 / E_2 = 25, G_{12} / E_2 = 0.5, G_{13} / E_2 = 0.5, G_{23} / E_2 = 0.2, \quad \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{20}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4 / (E_2 h^4)$, the incremental load vector is chosen to be:

$$\{\Delta P\} = \{10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, 130, 140\} \cdot \overline{P} \tag{21}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}3$. The displacements and stresses are given in following nondimensional form:

$$\overline{w} = w_0 \cdot E_2 h^3 / (q_0 \cdot a^4)$$

$$(\overline{\sigma}_{xx}, \overline{\sigma}_{yy}, \overline{\tau}_{xy}) = (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \cdot \left(\frac{h}{a}\right)^2 \cdot \frac{1}{E_2}, \quad \overline{\tau}_{xz} = \tau_{xz} \cdot \frac{h}{a} \cdot \frac{1}{E_2} \tag*{(22)$_{1,2}$}$$

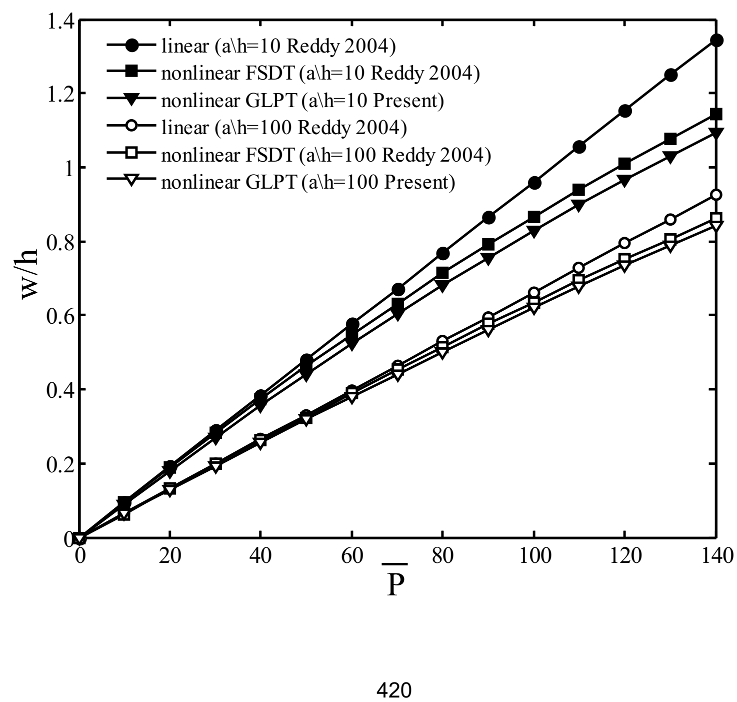

Figure 3. Nonlinear bending of square simply supported (SS1) orthotropic plate; central displacement versus load parameter

A 2x2 quarter plate laminated GLPT model is compared with 8x8 CPT nonconforming and 4x4 quadratic FSDT models [4]. The results for thick and thin plates (a/h=10 and a/h=100) of linear and nonlinear deflections are presented on Figure 3. It is shown that proposed GLPT model closely agree with CLPT and FSDT models. The more significant difference between linear and nonlinear solutions is observed for thick plates, while in thick plates larger lateral deflections have greater influence on nonlinear response, as it can be seen from the underlined nonlinear terms in Eq. (2).

Example 4.3. A nonlinear bending of square cross ply 0/90 and angle ply 45/-45 plates, with $a = b = 1$ and $h = 0.1$, with three different boundary conditions (SS, HH and CC, Eqs. 12, 13, 14) , made of material:

$$E_1 / E_2 = 40, G_{12} / E_2 = 0.6, G_{13} / E_2 = 0.6, G_{23} / E_2 = 0.5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{23}$$

subjected to uniform transverse pressure $\overline{q} = q(x, y) \cdot \left(\frac{a}{h}\right)^4 \cdot \frac{1}{E_2}$ are analyzed. The incremental load vector is:

$$\{\Delta\overline{q}\} = \{-100, -20, -20, -20, -20, 40, 20, 20, 20, 20\} \tag{24}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0,5$. The displacements and stresses are given in following nondimensional form:

$$\overline{w}_{LIN} = w \times \frac{h^3}{a^4}\frac{E_2}{q} \cdot 100, \quad (\overline{\sigma}_{xx}, \overline{\sigma}_{yy}) = (\sigma_{xx}, \sigma_{yy}) \times \left(\frac{a}{h}\right)^2 \cdot \frac{1}{E_2} \tag{25}$$

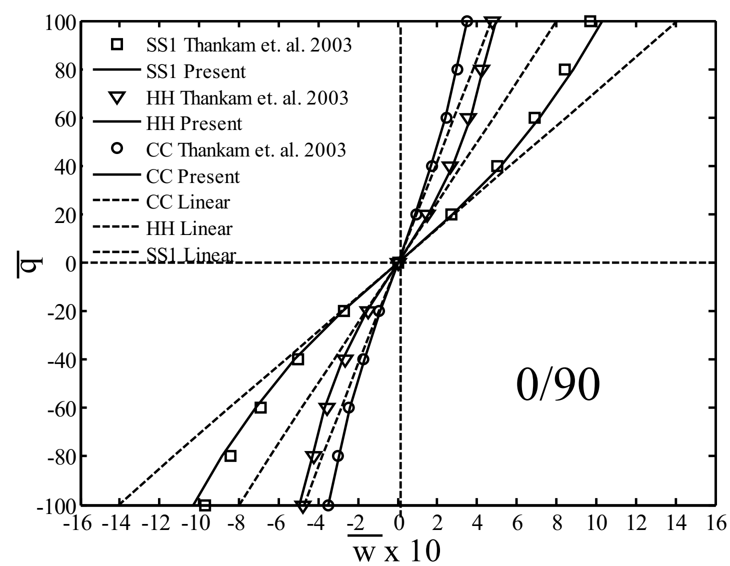

Figure 4. Nonlinear bending of square cross ply 0/90 plate with different boundary conditions and $a/h = 10$; central displacement versus load parameter

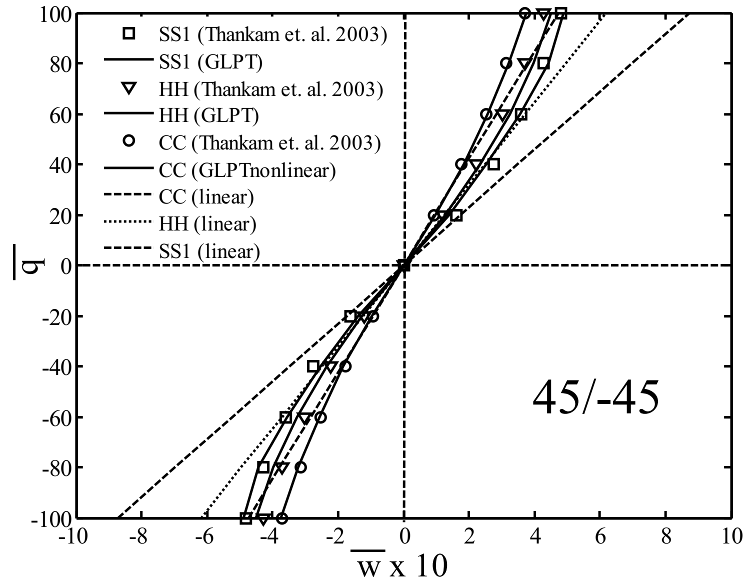

Figure 5. Nonlinear bending of square angle ply 45/-45 plate with different boundary conditions and $a/h = 10$; central displacement versus load parameter

A 2x2 quarter plate and 4x4 full plate laminated GLPT models are analyzed and compared with full 8x8 plate FSDT models (Thankam and Singh and Rao and Rath, A.K. 2003 [10]). The results for linear and nonlinear deflections are presented in Figures 4,5. It is shown that proposed GLPT model closely agree with FSDT model form literature, with the faster convergence. Also, the discrepancy between linear and nonlinear solutions are larger for flexible plates, which are the plates with simply supported boundary conditions, compared to hinged (HH) and clamped (CC) boundary conditions. The study has verified that the change in load direction gives symmetrical displacement field.

Example 4.4. A nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0/45/\text{-}45/90)_s$, laminated plate with $a = b = 1$ and $h = 0.1$, made of material:

$$E_1/E_2 = 40,\ G_{12}/E_2 = 0.6,\ G_{13}/E_2 = 0.6,\ G_{23}/E_2 = 0.5,\ \nu_{12} = \nu_{13} = \nu_{23} = 0.25 \tag{26}$$

subjected to uniform transverse pressure is analyzed. Using the load parameter $\overline{P} = q_0 \cdot a^4/(E_2 h^4)$, the incremental load vector is chosen to be:

$$\{\Delta q\} = \{50, 50, 50, 50, 50\} \cdot \overline{P} \tag{27}$$

with convergence tolerance $\varepsilon = 0.01$ and acceleration parameter $\gamma = 0{,}8$.

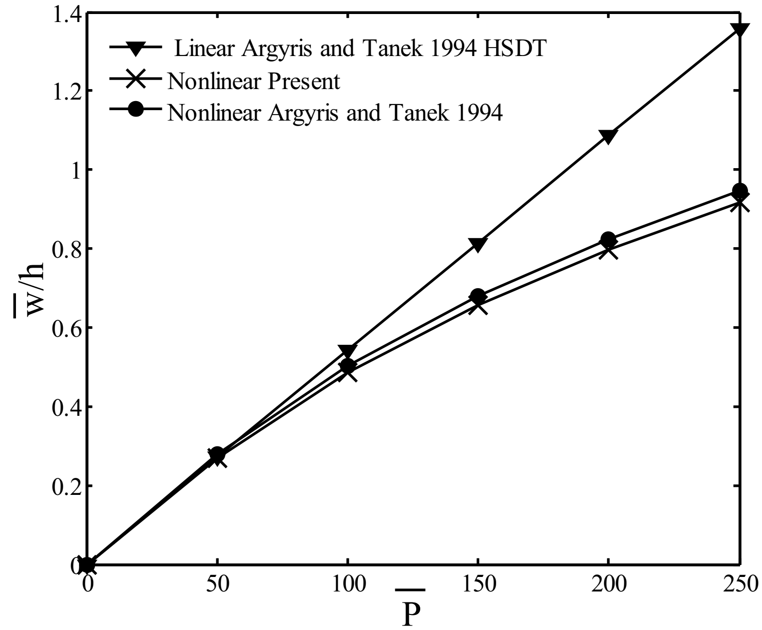

Figure 6. Nonlinear bending of square simply supported (SS1) general quasi-isotropic $(0/45/-45/90)_s$ laminated plate with $a/h=10$; central displacement versus load parameter

A 2x2 quarter plate continuum GLPT model is compared with 8x8 full plate HSDT model [11]. The results for linear and nonlinear deflections are presented in Figure 6. It is shown that proposed GLPT model closely agree with HSDT model form literature, with the faster convergence.

5. Conclusion

In this paper a laminated layerwise finite element model for geometrically nonlinear small strain, large deflection analysis of laminated composite plates is derived using the PVD. The accuracy of the model is verified calculating nonlinear response of plates with different mechanical properties, which are isotropic, orthotropic and anisotropic (cross ply and angle ply), different plate thickness, different boundary conditions and different load direction (unloading/loading). In despite of its mathematical complexity, proposed model has shown better convergence characteristics than ESL models of CLPT, FDST and HSDT, still with less computational cost than 3D elasticity model. Moreover, present model has no shear locking problems, compared to ESL models, or aspect ratio problems, as the 3D finite element may have when analyzing thin plate behavior. The analysis has also shown that the discrepancy of nonlinear from linear response is greater for flexible plates, such as thick compared to thin plates, or plates with SS compared to hinged (HH) and clamped (CC) boundary conditions. It is verified that the change of load direction (unloading/loading) gives symmetrical displacement field.

6. References

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