PROPERTIES OF TWO COLLINEAR SPACES WHERE COMMON TETRAHEDRON IS IRREGULAR-ORTHOCENTRIC

Aleksandar ČUČAKOVIĆ and Magdalena DIMITRIJEVIĆ
Faculty of Civil Engineering, University of Belgrade, Serbia

We were studying the case of two collinear corresponding projectively extended euclidean spaces $\Sigma \rightarrow \overline{\Sigma} = \Sigma$, where four „double“ (autocollinear) points are vertexes of irregular-orthocentric tetrahedron. A general collineation between two projectively extended affine spaces Σ and $\overline{\Sigma}$ is defined e.g. by the tetrahedron $D_1 D_2 D_3 D_4$ together with the „vanishing plane“ R in the space Σ, which is mapped to the infinite plane $\overline{R}\infty$ in space $\overline{\Sigma}$. Let the tetrahedron be positioned in a way that one of its sides $D_1 D_2 D_3$ lays in horizontal plane. The vanishing plane $\overline{Q}$ of the space $\overline{\Sigma}$ is determined by the known rule – edges of tetrahedron penetrate vanishing planes at equal distances from vertexes of tetrahedron. This paper confirms the existence of previously found significant invariants of two collinear spaces. This case of two collinear spaces does not have a focus. Therefore we tried to figure out some invariants which would function as focus surrogates within a framework of two collinear spaces described above. It is found out that in two collinear corresponding (euclidean) spaces there is a pair of corresponding straight lines $l \rightarrow \bar{l}$. Line l of the space Σ passes through the orthocenter of the tetrahedron and it is perpendicular to the vanishing plane $\overline{Q}$ of the space $\overline{\Sigma}$. Line $\bar{l}$ of the space $\overline{\Sigma}$ also passes through the orthocenter of the fixed tetrahedron and it is perpendicular to the vanishing plane R of the space Σ. Furthermore, through each of corresponding points, on corresponding lines $l \rightarrow \bar{l}$, passes three mutually orthogonal straight lines, forming rectangular corresponding trihedrons. One can state that the mentioned coresponding straight lines l and $\bar{l}$ in case of a general autocollineation of a euclidean space, can be used to establish corresponding coordinate systems with orthogonal axes.

Key words: Collinear spaces, orthocentric tetrahedron, projectively corresponding rectangular trihedron

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1. INTRODUCTION

We started from known significant invariants of two collinear corresponding projectively extended euclidean spaces in most general terms. The study objective is the case of two collinear corresponding spaces when common collinear tetrahedron is irregular and orthocentric. It is well known that in all cases of two general projectively extended collinear spaces focuses do not exist. It prompted an alternative approach: to try out invariants–focus surrogates.

1.1 Common orthocentric tetrahedron

A general colineation between two projectively extended affine spaces Σ and $\overline{\Sigma}$ *i*s defined by common irregular orthocentric tetrahedron $D_1D_2D_3D_4$ and vanishing plane R (Fig. 1). Vanishing plane $\overline{Q}$ is determined by condition that two vanishing planes of two collinear corresponding spaces penetrate edges of common tetrahedron on equal distances from two vertexes on the same edge.

orthocenter. Regarding the fact that the edges of orthocentric autocollinear tetrahedron are orthogonal and bypassing, therefore orthocenter can be defined as intersecting point of mutual perpendicular straight lines of three pairs of orthogonal bypassing edges of autocollinear tetrahedron.

1.2 Main perpendicular straight lines to vanishing planes

It is well known that in two collinear corresponding spaces exist a bundle of parallel straight lines of the vertex $\{N\ \infty\}$, of the space Σ, perpendicular to the vanishing plane R, which are transforming into a bundle of straight lines $\{\overline{N}\}$, with the vertex $\overline{N}$ in vanishing plane $\overline{Q}$, in a way that only one straight line from the bundle is perpendicular to vanishing plane $\overline{Q}$. It is correct to state that exists only one straight line $n \in \Sigma$ perpendicular to vanishing plane R, which corresponds with straight line $\overline{n} \in \overline{\Sigma}$ perpendicular to vanishing plane $\overline{Q}$.

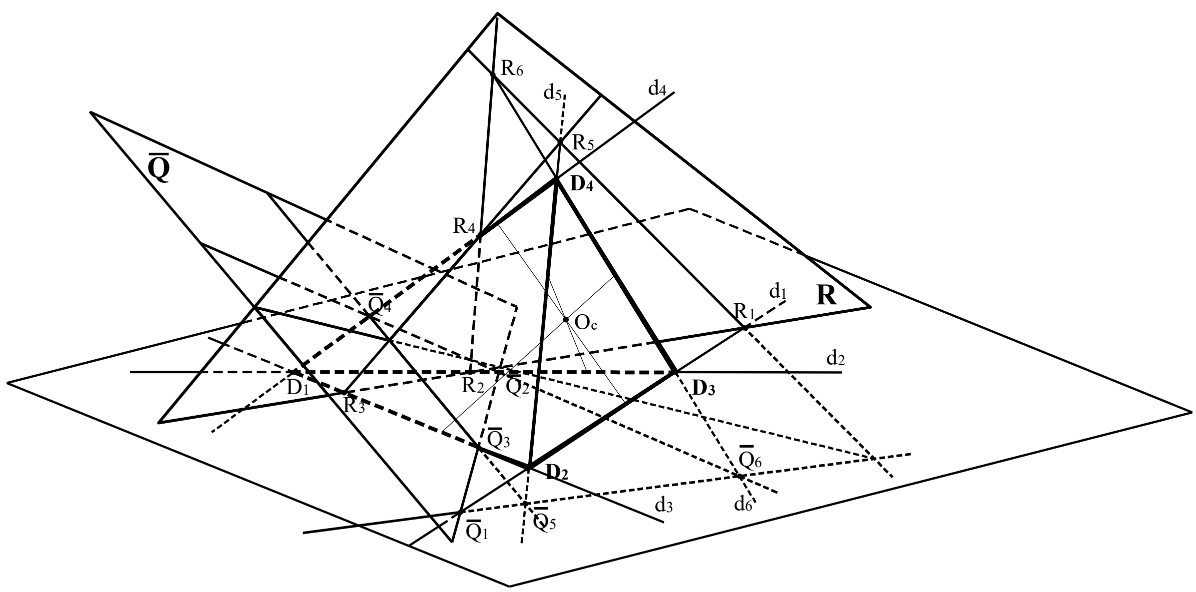

Fig. 1

To determine orthocenter of common tetrahedron we connected each vertex with orthocenter of the opposite side (triangle) of tetrahedron. Crossing point of this four lines (perpendiculars on plains of tetrahedron) is

These corresponding perpendicular straight lines of two collinear corresponding spaces are labeled as main perpendiculars of vanishing planes R and $\overline{Q}$. [5]

2. SPECIAL LINES OF TWO COLLINEAR CORRESPONDING SPACES

2.1 Determination of the main perpendiculars to the vanishing plains

Straight line n_4 is set through the vertex D_4 of common tetrahedron, perpendicular to vanishing plane R, (Fig.2.) in a direction of infinite point $N\infty$ **.** Straight line n_4 intersects side $D_1D_2D_3$ of common tetrahedron in point N_4. Following well known approach [4] for determination of corresponding points of two collocal collinear plains, one defines the corresponding point $\overline{N}_4$ for the point N_4 , in twofold plane $D_1D_2D_3$. Connection line $\overline{N}_4D_4$, the straight line $\overline{n}_4$, intersects vanishing plane $\overline{Q}$ in point $\overline{N}$, the foot of the main perpendicular straight line $\overline{n} \perp \overline{Q}$ **.** Then one sets through the vertex D_4 a straight line $\overline{m}_4$, perpendicular to the vanishing plane $\overline{Q}$, in a direction of point $\overline{M}\infty$. Straight line $\overline{m}_4$ intersects twofold side $D_1D_2D_3$ of common tetrahedron in point $\overline{M}_4$. Then one defines corresponding point $\overline{M}_4$ for the point M_4 , in twofold plane $D_1 D_2 D_3$. The connection line M_4D_4, the straight line m_4, intersects vanishing plane R in point M**,** the foot of the main perpendicular straight line $n \perp R$ of the space Σ, which is in turn perpendicular to the vanishing plane R.

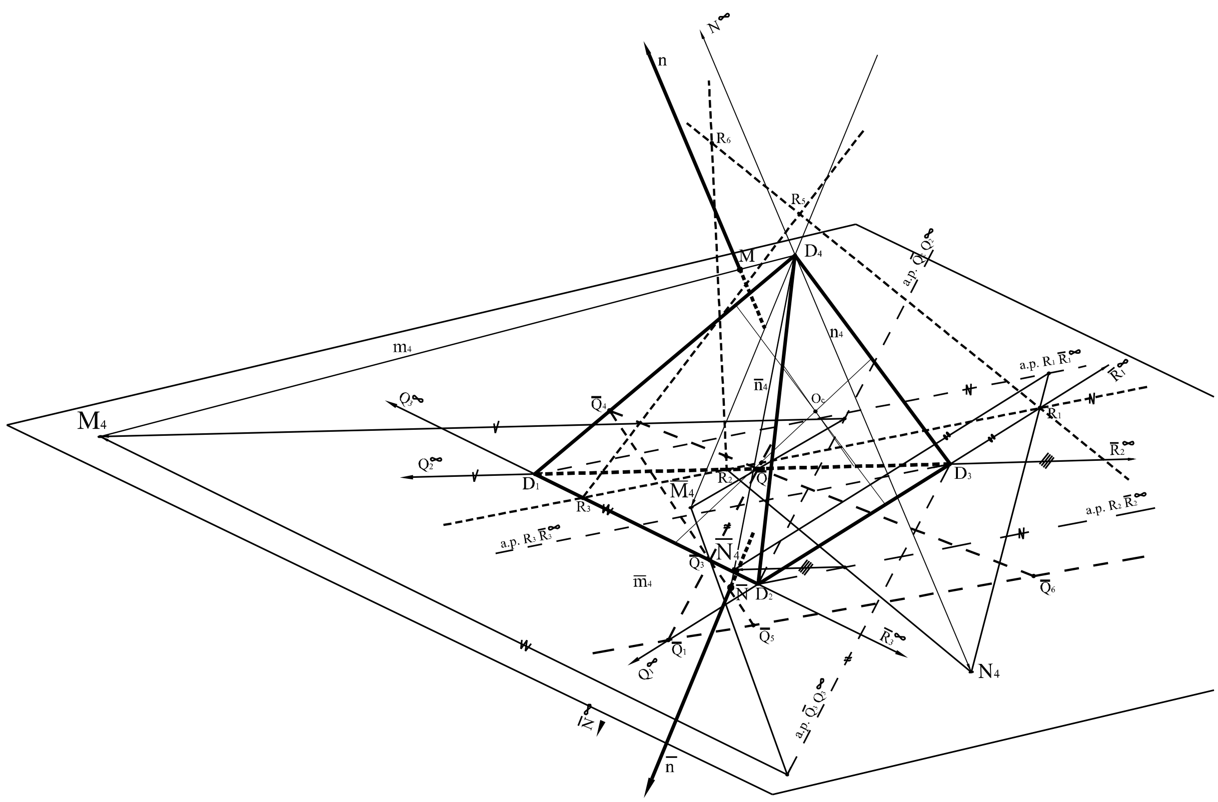

Fig.2

2.2 Two corresponding special straight lines l and $\bar{\text{l}}$

For determination of straight lines l and $\bar{l}$ in two collinear spaces Σ i $\bar{\Sigma}$ are used infinite points $\bar{Y}\infty \equiv N\infty$ (in space $\bar{\Sigma}$) and $W\infty \equiv \bar{M}\infty$ (in space Σ) of the main perpendicular straight lines (Fig. 3). Point $\bar{Y}\infty$, in the space $\bar{\Sigma}$, coincides with point $N\infty \in n$ of the space Σ. Connection line $\bar{Y}\infty\ D_4$ intersects side of common tetrahedron $D_1D_2D_3$ in the point $\bar{Y}_1$. In the plane $D_1D_2D_3$ is determined corresponding point Y_1. Connection line D_4Y_1 intersects vanishing plane R in point Y. Connection of point Y and point $W\infty \equiv \bar{M}\infty$ is the special line $l \perp \bar{Q}$ of the space Σ, and passes through orthocenter Oc of common tetrahedron.

Then, one adopts point $W\infty$ of the space Σ, which coincides with point $\bar{M}\infty \in \bar{n}$, of the space $\bar{\Sigma}$. Connection line $W\infty\ D_4$ intersects tetrahedron's side $D_1D_2D_3$ in the point W_1. Corresponding point $\bar{W}_4$ is determined in the plane $D_1D_2D_3$. Connection line $D_4\bar{W}_4$ intersects vanishing plane $\bar{Q}$ in the point $\bar{W}$. Connection of point $\bar{W}$ and point $\bar{Y}\infty \equiv N\infty$ is special straight line $\bar{l} \perp R$ of the space $\bar{\Sigma}$, and passes through orthocenter Oc of common tetrahedron. Two corresponding collinear special straight lines l and $\bar{l}$, in two generaly collinear spaces, can be quickly defined by setting perpendiculars: $l \perp \bar{Q}$ and $\bar{l} \perp R$ starting from orthocenter of common tetrahedron.

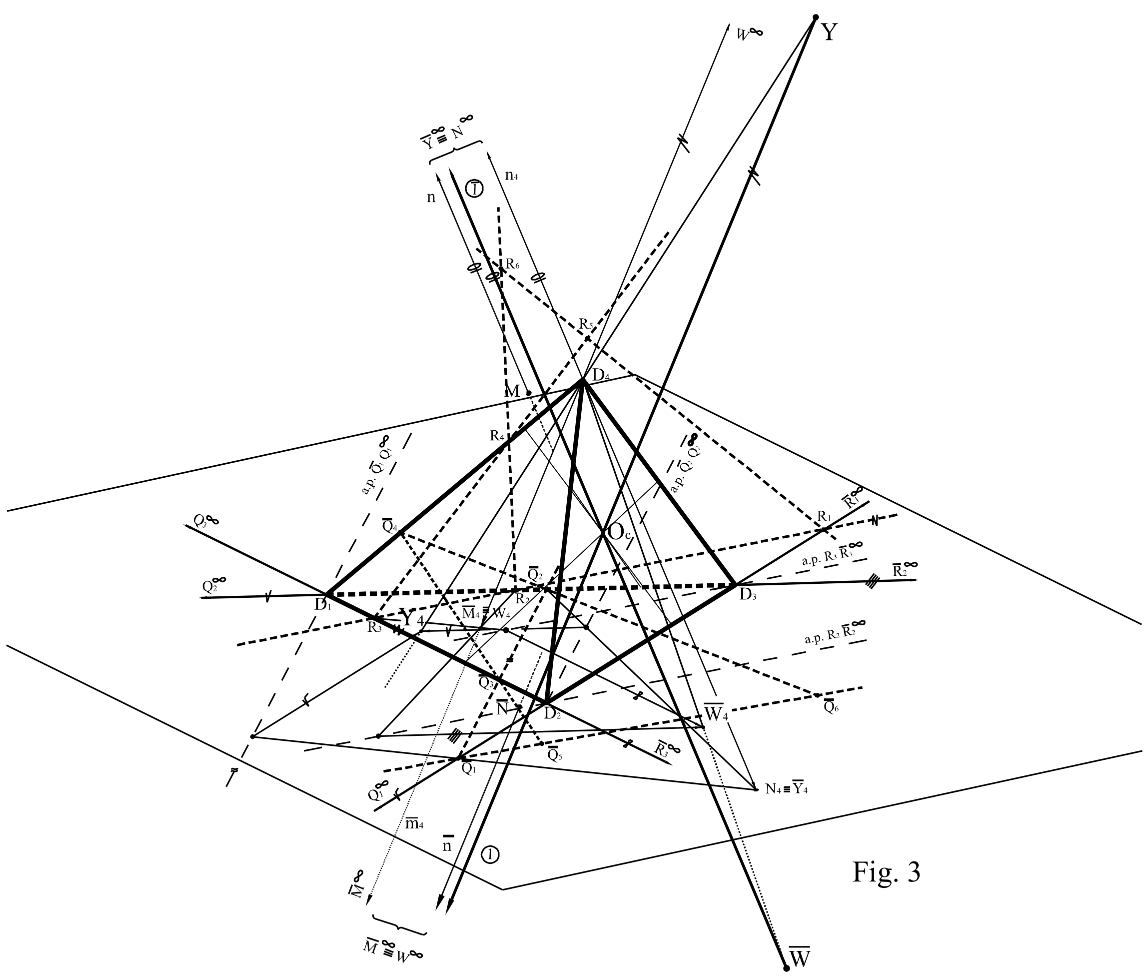

Fig. 3

2.3 Coincident infinite lines of opposite spaces in direction of their vanishing planes

We know that only infinite straight line $r\infty \in R$ of the space Σ, transforms into infinite straight line $\overline{r}\,\infty \in \overline{Q}$ of the space $\overline{\Sigma}$.

If infinite straight line $\overline{f}\,\infty \in \overline{\Sigma}$ coincides with line $r\infty \in \Sigma$, (Fig. 4.) the corresponding straight line $f \in \Sigma$ would be finite one in the vanishing plane R. Straight line $\overline{a}$, parallel to vanishing line $R_3R_4R_5$ of the plane $D_1D_2D_4$ is set through the vertex D_4.

Connection line AB is a finite straight line f (vanishing line) of the space Σ.

If infinite straight line $g\infty \in \Sigma$ coincides with $\overline{r}\,\infty \in \overline{\Sigma}$, the corresponding straight line $\overline{g} \in \overline{\Sigma}$, would be finite one (vanishing line) in vanishing plane $\overline{Q}$. Through the vertex D_4 is set straight line c parallel to vanishing line $\overline{Q}_3\,\overline{Q}_4\,\overline{Q}_5$ of plane $D_1D_2D_4$. Straight line c intersects plane $D_1D_2D_3$ in the point 3. Point 3 corresponds with point $\overline{3}$. Connection line $\overline{3}D_4$ is line $\overline{c}$, which intersects vanishing plane $\overline{Q}$ in point C.

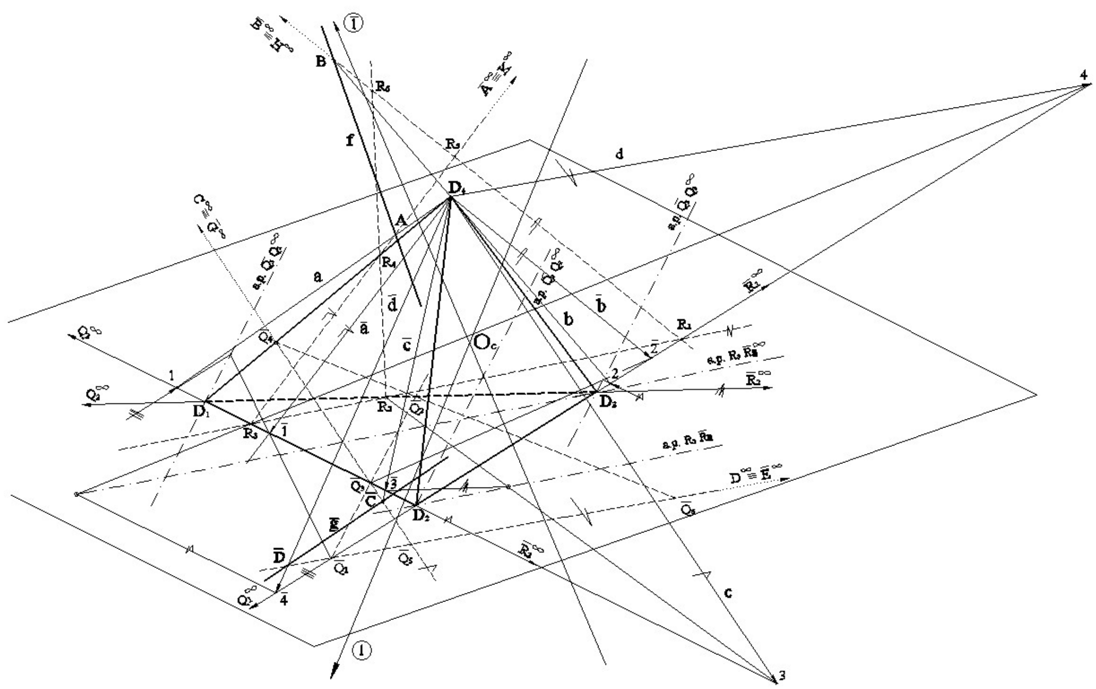

Fig. 4

The straight line $\overline{a}$, parallel to line $R_3R_4R_5$ intersects plane $D_1D_2D_3$ in point $\overline{1}$. Point $\overline{1}$ corresponds with point 1. Connection line $1D_4$ is straight line a which intersects vanishing plane R in the point A. The straight line $\overline{b}$, parallel to the vanishing line $R_1R_5R_6$ of the plane $D_2D_3D_4$, passes through vertex D_4. Line $\overline{b}$ intersects plane $D_1D_2D_3$ in point $\overline{2}$. Line b is connection of it's corresponding point 2 and vertex D_4. It intersects vanishing plane R in point B.

Then through the vertex D_4 is set straight line d parallel to the vanishing line $\overline{Q}_1\overline{Q}_5\overline{Q}_6$ of plane $D_2D_3D_4$. Line d intersects plain $D_1D_2D_3$ in point 4. Line $\overline{d}$ is a connection of it's corresponding point $\overline{4}$ and vertex D_4. It intersects vanishing plane $\overline{Q}$ in the point D. Connection line CD is a finite straight line $\overline{g}$ (vanishing line) of the space $\overline{\Sigma}$.

2.4 Corresponding perpendiculars on common tetrahedron's twofold planes

Through vertex D_4 is set (Fig. 5) a straight line perpendicular to plane $D_1D_2D_3$, in direction of point $P\infty$. Point P_4 is it's intersection through plane $D_1D_2D_3$. Using perspective corresponding pencils of straight lines (for the point P_4), in twofold plane $D_1D_2D_3$, one determines corresponding point $\overline{P}_4$. Connection line $D_4\overline{P}_4$ intersects the vanishing plane $\overline{Q}$ in point $\overline{P}$. $\overline{P}$ is vertex for the bundle of straight lines of the space $\overline{\Sigma}$, corresponding to a bundle of parallel straight lines with vertex $P\infty$ in space Σ. Perpendicular straight line on twofold plane $D_1D_2D_3$, set through point $\overline{P}$, passes through point L - intersection point of the special straight line $\bar{l}$ and twofold plane $D_1D_2D_3$.

Automatically, by adopting $P\infty$ of the space Σ one obtains coinciding point $\overline{S}\infty$ of the space $\overline{\Sigma}$. Connection line $\overline{S}\infty D_4$ intersects plane $D_1D_2D_3$ of common tetrahedron in point $\overline{S}_4 \equiv P_4$. According to a known principles, one determines point S_4, corresponding to $\overline{S}_4$ in twofold plane $D_1D_2D_3$. The connection line S_4D_4 intersects vanishing plane R in the point S. The perpendicular straight line on twofold plane $D_1D_2D_3$ set through point S, passes through the point $\overline{L}$ point - intersection of the special straight line l and twofold plane $D_1D_2D_3$.

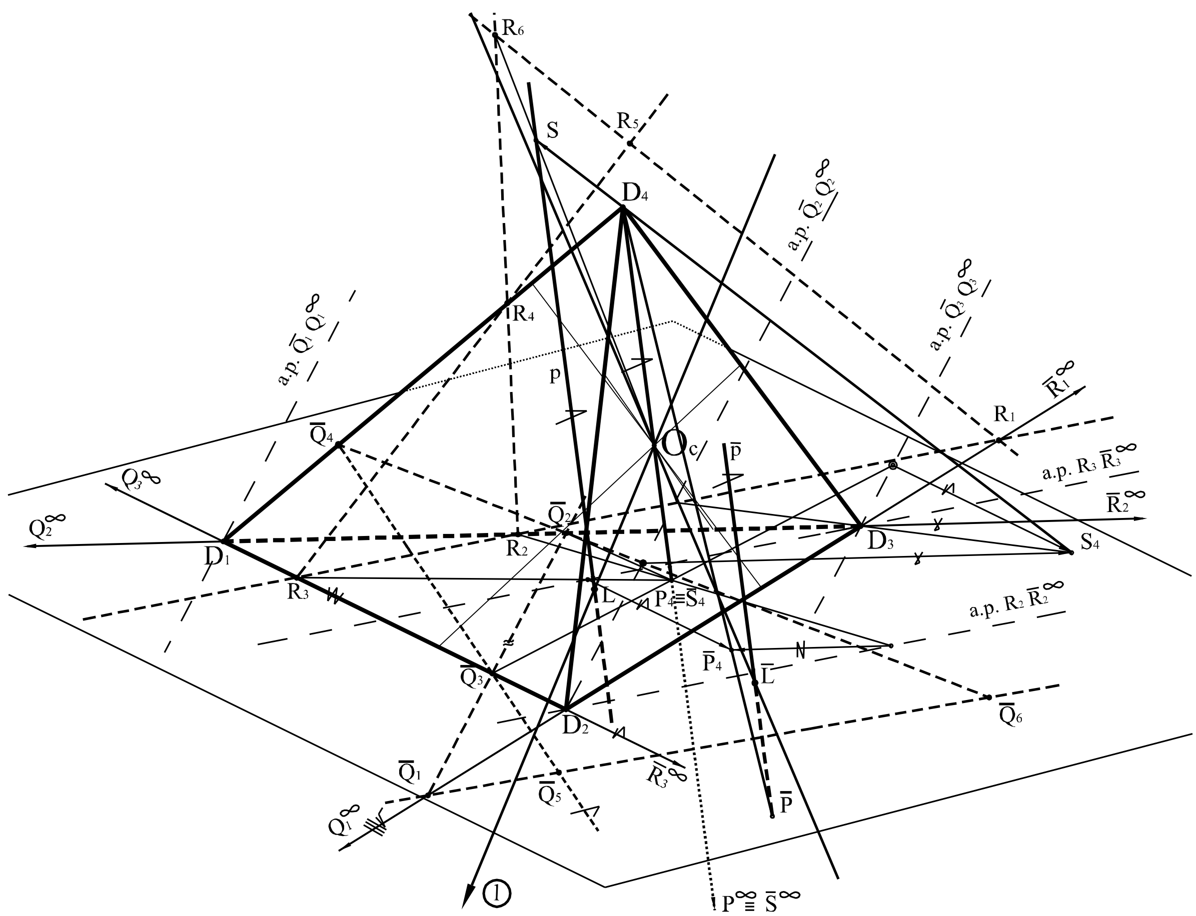

Fig.5

2.5 Corresponding straight lines perpendicular to pair of corresponding parallel planes

Pair of corresponding parallel straight lines $S_1 S_3$ and $\bar{S}_1 \bar{S}_3$ is determined in twofold plane $D_1D_2D_3$. Two parallel straight lines are carriers for two perspectively corresponding pencils of planes which plane of perspectivity passes through vertex D_4 of common tetrahedron, opposite to side $D_1D_2D_3$ [2, p.52, concl. 22]. Plane of perspectivity (Fig. 6) for parallel straight lines $S_1 S_3$ and $\bar{S}_1 \bar{S}_3$ is defined by two axes of perspectivity: one for pair of points S_1 and $\bar{S}_1$ (vertexes of corresponding pencils of straight lines) in twofold plane $D_2D_3D_4$, and another, for pair of points S_3 and $\bar{S}_3$ in twofold plane $D_1D_2D_4$ (since both axes of perspectivity passes through vertex D_4 they define a plane). Two planes T and $\bar{T}$, parallel to the plane of perspectivity, are set through parallel lines $S_1 S_3$ and $\bar{S}_1 \bar{S}_3$.

Through vertex D_4 is set straight line perpendicular to the plane T. It's intersection point through twofold plane $D_1D_2D_3$ is point Sn. The point $\bar{S}n$ has it's corresponding point Sn in the plane $D_1D_2D_3$. Connection line $D_4\bar{S}n$ intersects vanishing plane $\bar{Q}$ in point $\bar{S}$. Point $\bar{S} \in \bar{\Sigma}$ is vertex of bundle of straight lines corresponding to vertex $S\infty \in \Sigma$ of bundle of parallel straight lines perpendicular to the plane T**.**

Straight line $n\bar{t} \perp \bar{T}$, set through point $\bar{S}$, passes through point $\bar{V}$ (in plane $\bar{T}$), also the intersection point of special line $\bar{l}$ and plain $\bar{T}$. Straight line $n\bar{t}$ intersects plain $D_1D_2D_3$ in point $\bar{K}$. It's corresponding point K is defined in the plane $D_1D_2D_3$. Straight line $nt \perp T$, set through point K, passes through point V (in plane T), intersection point of special line l and plain T.

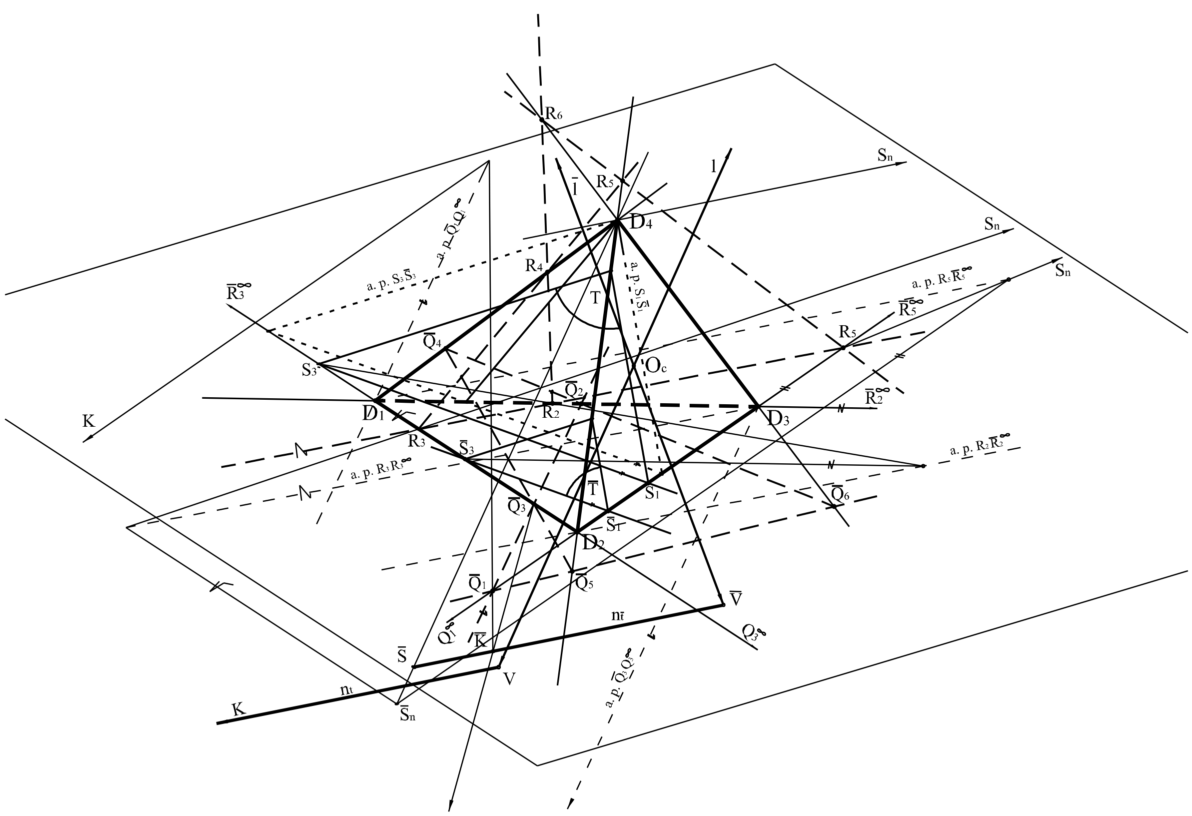

Fig. 6

Through corresponding pair of points V and $\overline{V}$, the intersection points of straight lines l and $\bar{l}$ through corresponding parallel planes T and $\overline{T}$, one can set pair of corresponding perpendiculars $n\,t$ and $n\,\bar{t}$. It indicates that one pair of corresponding parallel planes passes through each pair of corresponding points on special straight lines l and $\bar{l}$. Also, through each pair of this corresponding points, one can set pair of corresponding perpendiculars (of parallel plains).

3. CONCLUSION

3.1 The position of special lines *l* and $\bar{l}$

When analysing the position of straight lines l and $\bar{l}$ in relation to the bypassing main perpendicular lines $n \in \Sigma$ and $\bar{n} \in \overline{\Sigma}$ (Fig. 3) in compliance with the fact that two collinear corresponding spaces have no focuses, one can state : focuses would exist only in a case if straight line l would intersect perpendicular line n and corresponding line $\bar{l}$ would intersect perpendicular line $\bar{n}$.

3.2 Relations between special lines (l , $\bar{l}$) and vanishing lines (f , $\bar{g}$)

Special lines l and $\bar{l}$ in relation to the obtained vanishing lines f and $\bar{g}$ (Fig. 4) of corresponding infinite lines $\bar{f}\infty \equiv r\infty$ and $g\infty \equiv \bar{r}\infty$ of two collinear corresponding spaces, are orthogonal and by passing : the line l is orthogonal and by passing to the vanishing line $f \in \Sigma$, and line $\bar{l}$ is orthogonal and bypassing to the vanishing line $\bar{g} \in \overline{\Sigma}$.

3.3 Pairs of corresponding intersecting points of special lines (l , $\bar{l}$) on twofold sides of common tetrahedron

Intersection points of the special lines l and $\bar{l}$ through twofold planes (the sides of common tetrahedron) are, in fact, pairs of corresponding points of two collinear spaces, also the foots of corresponding perpendiculars (Fig.5) of twofold planes – i.e the sides of common tetrahedron, and there are four such pairs. In each of coresponding points in twofold plane exists a pair of orthogonal lines corresponding to another orthogonal pair.

Those corresponding pairs of orthogonal straight lines in twofold plains common tetrahedron form together, with mentioned perpendiculars, corresponding rectangle trihedrons.

3.4 Pairs of corresponding intersecting points of special lines (l , $\bar{l}$) and approximate pair of parallel plains

Intersection points of the special lines l and $\bar{l}$ through approximate pair of parallel plains (Fig. 6) are corresponding points of two collinear spaces, also the foots of corresponding parallel perpendiculars of those plains. In each of coresponding points (in parallel planes) exists a pair of orthogonal lines corresponding to another orthogonal pair. Using the same analogy from conclusion 3.3, in this pair of points exists pair of corresponding rectangle trihedrons.

After all studies, regarding the pair of a straight lines l and $\bar{l}$, could be concluded that special straight lines l and $\bar{l}$ function as carriers of two corresponding sequence of points where through the pairs of corresponding points pass by three mutually orthogonal straight lines i.e. axes of corresponding rectangle coordinate systems (trihedrons).

Previous research indicated that each corresponding pair of points, in collinear corresponding spaces, can be a pair of corresponding vertexes of bundles of straight lines, containing corresponding pairs of orthogonal trihedrons (rectangle coordinate systems). This invariants of collinear corresponding spaces could be of great significance in transforming general cases of surfaces.

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ABOUT THE AUTHORS

1. Aleksandar Čučaković, DIA, PhD is associeted prof. at the Faculty of Civil Engineering, University of Belgrade in Serbia. His research interests are in: transformations of collinear spaces, constructive elaboration of ruled surfaces, geometry in computer graphics. He can be reached by e-mail: cucak@grf.bg.ac.yu or through postal address: Faculty of Civil Engineering, Institute of Math., Phys. and Descriptive Geometry, Bul. Kralja Aleksandra 73/I, 11000 Belgrade, Serbia.

2. Magdalena Dimitrijević, DIA, MSc is teaching assistant at the Faculty of Civil Engineering, Univercity of Belgrade in Serbia. Her research interests are in transformations of collinear spaces, constructive elaboration of surfaces and applicability of computer graphics in geometry education for engineers. She can be reached by e-mail**:** dim@grf.bg.ac.yu or through postal address: Faculty of Civil Engineering, Institute of Math., Phys. and Descriptive Geometry, Bul. Kralja Aleksandra 73/I, 11000 Belgrade, Serbia.