

COEFFICIENT INEQUALITIES FOR UNIVALENT STARLIKE FUNCTIONS

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ABSTRACT. Let \mathcal{A} be the class of analytic functions in the unit disk \mathbb{D} with the normalization $f(0) = f'(0) - 1 = 0$. In this paper the authors discuss necessary and sufficient coefficient conditions for $f \in \mathcal{A}$ of the form

$$\left(\frac{z}{f(z)}\right)^\mu = 1 + b_1z + b_2z^2 + \dots$$

to be starlike in \mathbb{D} and more generally, starlike of some order β , $0 \leq \beta < 1$. Here μ is a suitable complex number so that the right hand side expression is analytic in \mathbb{D} and the power is chosen to be the principal power. A similar problem for the class of convex functions of order β is open.

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1. Introduction and main results

Let \mathcal{A} denote the family of all normalized analytic functions f ($f(0) = 0 = f'(0) - 1$) defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane \mathbb{C} , and

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is one-to-one in } \mathbb{D}\}.$$

A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps \mathbb{D} onto a convex domain, denoted by $f \in \mathcal{K}$, is called a convex function. The

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analytic condition for the class $\mathcal{S}^*(\beta)$ of starlike functions f of order β can be written in the form

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D},$$

where $0 \leq \beta < 1$. It is well-known that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. We denote by \mathcal{S}_p the class of spirallike functions. Analytically, the class \mathcal{S}_p is characterized as follows:

$$\mathcal{S}_p := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\theta} \frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}, \text{ for some } \theta \in (-\pi/2, \pi/2) \right\}.$$

Clearly $\mathcal{S}^* \subset \mathcal{S}_p$. A spirallike function is not necessarily close-to-convex and hence need not belong to \mathcal{S}^* . Moreover, close-to-convex function is not necessarily spirallike, see [3, 4]. The above analytic characterizations are deduced by using the corresponding geometric criterion for $f(z)$ in the disk $|z| < r$. Various aspects of these and many other special subclasses of \mathcal{S} are presented in [3, 4]. Finally, we now introduce the classes

$$\mathcal{U}(\lambda, \mu) := \left\{ f \in \mathcal{A} : \frac{f(z)}{z} \neq 0 \text{ and } \left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \mathbb{D} \right\}$$

and $\mathcal{U}(\lambda) := \mathcal{U}(\lambda, 1)$. It is known [2, 7] (see also [10]) that functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \leq 1$ but not necessarily univalent if $\lambda > 1$. It is also known [5] that $\mathcal{U}(1, -1) \not\subset \mathcal{S}^*$ and V. Singh [12] gave an estimate for the radius of starlikeness (which is surprisingly close to unity) of $\mathcal{U}(1, -1)$. More recently, Obradović [8] and, Ponnusamy and Singh [9] proved that

$$\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^* \quad \text{if } \mu < 1 \quad \text{and} \quad 0 \leq \lambda \leq \frac{1 - \mu}{\sqrt{(1 - \mu)^2 + \mu^2}}.$$

In a recent paper, R. Fournier and S. Ponnusamy [6] extended it as follows:

LEMMA A. *Let $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) < 1$. Then*

- (a) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^*$ iff $0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2 + |\mu|^2}}$
- (b) $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}_p$ iff $0 \leq \lambda \leq \min\left(1, \frac{|1-\mu|}{|\mu|}\right)$.

It is worth to recalling the following implications from the last two results:

- (c) $\mathcal{U}(1, \mu) \subset \mathcal{S}^*$ iff $\mu = 0$
- (d) $\mathcal{U}(1, \mu) \subset \mathcal{S}_p$ iff $\operatorname{Re}(\mu) \leq \frac{1}{2}$.

Moreover, each $f \in \mathcal{S}$ can be written in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \dots, \quad z \in \mathbb{D}. \tag{1.1}$$

Many interesting results concerning functions of this form are known in the literature, see for instance [1, 11]. We begin the discussion by recalling the well-known Area Theorem [3: Theorem 11, p. 193, Vol. 2] and the case $\mu = 1$ of this result has a special role in geometric function theory.

LEMMA B. *If $f \in \mathcal{S}$, $\mu > 0$, and if*

$$\left(\frac{z}{f(z)}\right)^\mu = 1 + b_1z + b_2z^2 + \dots, \tag{1.2}$$

then

$$\sum_{k=1}^\infty (k - \mu)|b_k|^2 \leq \mu.$$

In [11], Reade et. al. have obtained the following sufficient condition for a function f of the form (1.1) to be in $\mathcal{S}^*(\beta)$.

LEMMA C. *Suppose that f is of the form (1.1). If*

$$\sum_{k=1}^\infty (k - 1 + \beta)|b_k| \leq \begin{cases} (1 - \beta) - (1 - 2\beta)|b_1| & \text{for } 0 \leq \beta \leq \frac{1}{2} \\ 1 - \beta & \text{for } \frac{1}{2} < \beta < 1, \end{cases}$$

then $f \in \mathcal{S}^*(\beta)$.

In this article, we generalize this result in the context of Lemma B (see Corollary 3 with $\mu = 1$). We now state our first result which provides a necessary condition for f of the form (1.2) to belong to $\mathcal{S}^*(\beta)$. Unlike the original proof of Lemma B, proof in this case is a bit straightforward.

THEOREM 1. *Every $f \in \mathcal{S}^*(\beta)$ which has the form (1.2) with $\mu > 0$ necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^\infty (k - (1 - \beta)\mu)|b_k|^2 \leq (1 - \beta)\mu. \tag{1.3}$$

Proof. Let $f \in \mathcal{S}^*(\beta)$. Next, we observe that

$$\frac{zf'(z)}{f(z)} = \frac{\left(\frac{z}{f(z)}\right)^\mu - \frac{1}{\mu}z\left(\left(\frac{z}{f(z)}\right)^\mu\right)'}{\left(\frac{z}{f(z)}\right)^\mu}$$

and the inequality $|\zeta - 1| < |\zeta + 1 - 2\beta|$ is equivalent to $\text{Re } \zeta > \beta$. In view of this observation, it can be easily seen that $f \in \mathcal{S}^*(\beta)$ is equivalent to the inequality

$$\left|\frac{1}{\mu}z\left(\left(\frac{z}{f(z)}\right)^\mu\right)'\right| < \left|2(1 - \beta)\left(\frac{z}{f(z)}\right)^\mu - \frac{1}{\mu}z\left(\left(\frac{z}{f(z)}\right)^\mu\right)'\right|.$$

Using the representation (1.2), the last inequality takes the form

$$\left| \sum_{k=1}^{\infty} k b_k z^k \right| < \left| 2(1 - \beta)\mu - \sum_{k=1}^{\infty} (k - 2(1 - \beta)\mu) b_k z^k \right|. \quad (1.4)$$

Therefore, with $z = r e^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the inequality (1.4) gives

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} k b_k z^k \right|^2 d\theta < \frac{1}{2\pi} \int_0^{2\pi} \left| 2(1 - \beta)\mu - \sum_{k=1}^{\infty} (k - 2(1 - \beta)\mu) b_k z^k \right|^2 d\theta$$

or equivalently,

$$\sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k} < 4(1 - \beta)^2 \mu^2 + \sum_{k=1}^{\infty} (k - 2(1 - \beta)\mu)^2 |b_k|^2 r^{2k}$$

which, upon simplification, implies that

$$\sum_{k=1}^{\infty} (k - (1 - \beta)\mu) |b_k|^2 r^{2k} < (1 - \beta)\mu.$$

Allowing $r \rightarrow 1^-$, we obtain the desired inequality. \square

Using the method of proof of Theorem 1, we can easily establish the following general result, and so we omit the details.

COROLLARY 1. *Every $f \in \mathcal{S}^*(\beta)$ which has the form (1.2) with $\operatorname{Re} \mu > 0$ necessarily satisfies the coefficient inequality*

$$\sum_{k=1}^{\infty} (k \operatorname{Re} \mu - (1 - \beta)|\mu|^2) |b_k|^2 \leq (1 - \beta)|\mu|^2. \quad (1.5)$$

There is some difficulty in obtaining an analog of Theorem 1 for functions $f \in \mathcal{K}(\beta)$ of the form (1.2) and hence, this remains an open problem. Here $\mathcal{K}(\beta)$, $0 \leq \beta < 1$, denotes the class of convex functions f of order β defined by the analytic condition

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > \beta, \quad z \in \mathbb{D},$$

so that $\mathcal{K} := \mathcal{K}(0)$. However, if $\beta = 0$ in Theorem 1, then the coefficient inequality (1.3) is same as in Lemma B for the case of univalent functions. At this place it is worth remembering that for this problem, the Koebe function is extremal for both \mathcal{S} and \mathcal{S}^* . However, for all other values of $\beta > 0$, Theorem 1 provides an improved inequality. For instance, we have

COROLLARY 2. Every $f \in \mathcal{S}^*(1/2)$ which has the form (1.2) with $\mu > 0$ necessarily satisfies the coefficient inequality

$$\sum_{k=1}^{\infty} (2k - \mu) |b_k|^2 \leq \mu.$$

In connection with Lemma B, it is natural to ask for a sufficient condition in terms of the coefficients b_k implying the univalence of the corresponding f in the unit disk.

THEOREM 2. Let $\mu \in \mathbb{C}$ with $\operatorname{Re}(\mu) < 1$. Assume that $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$, and has the form (1.2). If

$$\sum_{k=1}^{\infty} |k - \mu| |b_k| \leq \lambda |\mu|.$$

Then we have the following

- (a) $f \in \mathcal{S}^*$ if $0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2 + |\mu|^2}}$
- (b) $f \in \mathcal{S}_p$ if $0 \leq \lambda \leq \min\left(1, \frac{|1-\mu|}{|\mu|}\right)$.

Proof. We observe that

$$\left(\frac{z}{f(z)}\right)^\mu - \left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) = \frac{1}{\mu} z \left(\left(\frac{z}{f(z)}\right)^\mu\right)' = \frac{1}{\mu} \sum_{k=1}^{\infty} k b_k z^k$$

so that

$$\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1 = - \sum_{k=1}^{\infty} \frac{k - \mu}{\mu} b_k z^k$$

and therefore,

$$\begin{aligned} \left| \left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1 \right| &= \left| \sum_{k=1}^{\infty} \frac{k - \mu}{\mu} b_k z^k \right| \\ &\leq \sum_{k=1}^{\infty} |(k - \mu)/\mu| |b_k|, \quad z \in \mathbb{D}. \end{aligned}$$

The desired conclusion follows from Lemma A and the hypotheses. □

THEOREM 3. Assume that $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$, and has the form (1.2) with $\mu > 0$. If

$$\sum_{k=1}^{\infty} (|k - 2(1 - \beta)\mu| + k) |b_k| \leq 2(1 - \beta)\mu, \tag{1.6}$$

then $f \in \mathcal{S}^*(\beta)$.

Proof. Without loss of generality assume that $b_k \neq 0$ at least for a $k \geq 1$. In order to prove (1.6) implies $f \in \mathcal{S}^*(\beta)$, it suffices to show that (1.4) holds. By (1.6), it follows that

$$\begin{aligned} \left| \sum_{k=1}^{\infty} k b_k z^k \right| &< \sum_{k=1}^{\infty} k |b_k| \\ &\leq 2(1-\beta)\mu - \sum_{k=1}^{\infty} |k - 2(1-\beta)\mu| |b_k| \\ &\leq \left| 2(1-\beta)\mu - \sum_{k=1}^{\infty} (k - 2(1-\beta)\mu) b_k z^k \right| \end{aligned}$$

and therefore the inequality (1.4) holds. Thus, $f \in \mathcal{S}^*(\beta)$. □

If $0 < \mu \leq 1$, then Theorem 3 can be simplified. A simplification gives

COROLLARY 3. *Suppose that f is of the form (1.2) with $0 < \mu \leq 1$. If*

$$\begin{aligned} &\sum_{k=1}^{\infty} (k - (1-\beta)\mu) |b_k| \\ &\leq \begin{cases} (1-\beta)\mu - (1-2\beta)\mu |b_1| & \text{for } 1 - (1/\mu) \leq \beta \leq 1 - 1/(2\mu) \\ (1-\beta)\mu & \text{for } 1 - 1/(2\mu) \leq \beta < 1, \end{cases} \end{aligned}$$

then $f \in \mathcal{S}^*(\beta)$.

Clearly the choice $\lambda = 1$ of Theorem 2 yields that $\mu = 0$ and hence, we have only a trivial function, namely $f(z) = z$. On the other hand, in the case of $0 < \mu \leq 1/2$, one can refine Theorem 2(b) through coefficient condition as follows. Note that spiral-like function is not necessarily starlike.

COROLLARY 4. *Suppose that f is of the form (1.2) with $0 < \mu \leq 1/2$. If*

$$\sum_{k=1}^{\infty} (k - \mu) |b_k| \leq \mu,$$

then $f \in \mathcal{S}^*$.

Proof. Set $\beta = 0$ in Corollary 3. □

Because of its natural form, the coefficient inequalities of Corollaries 3 and 4 are easy to apply in many special situations.

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In order to state our next result we need to introduce the definition of Hadamard product or convolution $f \star g$ of two convergent power series $f(z) := \sum_{n=0}^{\infty} a_n(f)z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g)z^n$ in the unit disk $|z| < 1$. This is defined by the power series

$$f \star g(z) := \sum_{n=0}^{\infty} a_n(f)a_n(g)z^n.$$

It is clear that $f \star g$ is also a member of the class of analytic functions in \mathbb{D} . It is possible to formulate our next result.

THEOREM 4. *Let $0 < \mu \leq 1/2$, $f, g \in \mathcal{S}$ such that $\left(\frac{z}{f(z)}\right)^\mu \star \left(\frac{z}{g(z)}\right)^\mu \neq 0$ for $z \in \mathbb{D}$. Then the function F defined by*

$$\frac{z}{F(z)} = \left(\frac{z}{f(z)}\right)^\mu \star \left(\frac{z}{g(z)}\right)^\mu$$

belongs to the class $\mathcal{S}^(1 - \mu)$.*

Epecially, for $\mu = 1/2$ we have

$$f, g \in \mathcal{S} \implies F(z) = \frac{z}{\left(\frac{z}{f(z)}\right)^{\frac{1}{2}} \star \left(\frac{z}{g(z)}\right)^{\frac{1}{2}}} \in \mathcal{S}^*(1/2)$$

whenever $\left(\frac{z}{f(z)}\right)^{\frac{1}{2}} \star \left(\frac{z}{g(z)}\right)^{\frac{1}{2}} \neq 0$ for $z \in \mathbb{D}$.

Proof. By hypothesis, we may let

$$\begin{aligned} \left(\frac{z}{f(z)}\right)^\mu &= 1 + b_1z + b_2z^2 + \dots \\ \left(\frac{z}{g(z)}\right)^\mu &= 1 + c_1z + c_2z^2 + \dots \end{aligned} \tag{1.7}$$

Then by Lemma B we have

$$\sum_{k=1}^{\infty} (k - \mu)|b_k|^2 \leq \mu \quad \text{and} \quad \sum_{k=1}^{\infty} (k - \mu)|c_k|^2 \leq \mu. \tag{1.8}$$

As

$$\begin{aligned} \frac{z}{F(z)} &= \left(\frac{z}{f(z)}\right)^\mu \star \left(\frac{z}{g(z)}\right)^\mu \\ &= 1 + (b_1c_1)z + (b_2c_2)z^2 + \dots, \end{aligned}$$

from (1.8) we get, by means of the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{k=1}^{\infty} (k - \mu) |b_k c_k| &\leq \left(\sum_{k=1}^{\infty} (k - \mu) |b_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (k - \mu) |c_k|^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{\mu} \sqrt{\mu} = \mu. \end{aligned}$$

Since F is of the form (1.1) (with the new coefficients $b_k c_k$ in place of b_k 's), according to Lemma C with $\beta = 1 - \mu$, the last inequality implies that $F \in \mathcal{S}^*(1 - \mu)$. \square

Example 1. Let $f, g \in \mathcal{S}^*(1/2)$ and have the form (1.1). Then, according to Corollary 2 with $\mu = 1$, we have

$$\sum_{k=1}^{\infty} (2k - 1) |b_k|^2 \leq 1$$

and

$$\sum_{k=1}^{\infty} (2k - 1) |c_k|^2 \leq 1.$$

Because $f, g \in \mathcal{S}^*(1/2)$, by the Marx-Strohhäcker theorem

$$\operatorname{Re} \left(\frac{f(z)}{z} \right) > \frac{1}{2} \quad \text{and} \quad \operatorname{Re} \left(\frac{g(z)}{z} \right) > \frac{1}{2}, \quad z \in \mathbb{D},$$

and thus, $f(z) + g(z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$. Now, we consider the function

$$F(z) = \frac{2f(z)g(z)}{f(z) + g(z)}.$$

Then $F \in \mathcal{A}$. For $0 < r \leq 1$, we introduce $G(z) = r^{-1}F(rz)$. We are interested in determining range values of r for which $G \in \mathcal{S}^*(1/2)$. In order to solve this, we form

$$\begin{aligned} \frac{z}{G(z)} &= \frac{rz}{F(rz)} = \frac{1}{2} \left(\frac{rz}{f(rz)} + \frac{rz}{g(rz)} \right) \\ &= 1 + \sum_{k=1}^{\infty} \frac{b_k + c_k}{2} r^k z^k. \end{aligned}$$

Then, according to Corollary 3 (with $\mu = 1$ and $\beta = 1/2$), it suffices to show that

$$\sum_{k=1}^{\infty} (2k - 1) |b_k + c_k| r^k \leq 2.$$

Again, in view of the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} & \sum_{k=1}^{\infty} (2k-1) |b_k + c_k| r^k \\ & \leq \sum_{k=1}^{\infty} (2k-1) |b_k| r^k + \sum_{k=1}^{\infty} (2k-1) |c_k| r^k \\ & \leq \left(\left(\sum_{k=1}^{\infty} (2k-1) |b_k|^2 \right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} (2k-1) |c_k|^2 \right)^{\frac{1}{2}} \right) \left(\sum_{k=1}^{\infty} \frac{r^{2k}}{2k-1} \right)^{\frac{1}{2}} \\ & \leq 2 \left(\sum_{k=1}^{\infty} \frac{r^{2k}}{2k-1} \right)^{\frac{1}{2}} = 2 \left(\frac{r}{2} \log \frac{1+r}{1-r} \right)^{\frac{1}{2}} \leq 2 \end{aligned}$$

for $0 < r \leq r_0$, where $r_0 \approx 0.83356$ is the root of the equation

$$\frac{r}{2} \log \frac{1+r}{1-r} = 1$$

in the interval $(0, 1)$. Thus, $G \in \mathcal{S}^*(1/2)$ for $0 < r \leq r_0$ and hence, we conclude that $F \in \mathcal{S}^*(1/2)$ in the disk $|z| < r_0$.

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