On the initial coefficients for certain class of functions analytic in the unit disc

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ON THE INITIAL COEFFICIENTS FOR CERTAIN CLASS OF FUNCTIONS ANALYTIC IN THE UNIT DISC

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Let function f be analytic in the unit disk $\mathbb{D}$ and be normalized so that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ In this paper we give sharp bounds of the modulus of its second, third and fourth coefficient, if f satisfies

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma\tfrac{1}{2}\pi \quad (z \in \mathbb{D})$$

for $0 < \alpha < 1$ and $0 < \gamma \le 1$.

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$.

A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* β, $0 < \beta \le 1$ if and only if

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \beta\tfrac{1}{2}\pi \quad (z \in \mathbb{D}).$$

We denote this class by $\mathcal{S}^\star_\beta$. If $\beta = 1$, then $\mathcal{S}^\star_1 \equiv \mathcal{S}^\star$ is the well-known class of *starlike functions*.

In [1] the author introduced the class $\mathcal{U}(\alpha, \lambda)$ $(0 < \alpha$ and $\lambda < 1)$ consisting of functions $f \in \mathcal{A}$ for which we have

$$\left|\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z) - 1\right| < \lambda \quad (z \in \mathbb{D}).$$

In the same paper it is shown that $\mathcal{U}(\alpha, \lambda) \subset \mathcal{S}^\star$ if

$$0 < \lambda \le \frac{1-\alpha}{\sqrt{(1-\alpha)^2 + \alpha^2}}.$$

The most valuable up to date results about this class can be found in [4, Chapter 12].

In the paper [2] the author considered univalence of the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma\tfrac{1}{2}\pi \quad (z \in \mathbb{D}) \tag{1}$$

for $0 < \alpha < 1$ and $0 < \gamma \le 1$, and proved the following theorem.

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Theorem A. *Let* $f \in \mathcal{A}$, $0 < \alpha < \frac{2}{\pi}$ *and let*

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma_\star(\alpha)\tfrac{1}{2}\pi \quad (z \in \mathbb{D}),$$

where

$$\gamma_\star(\alpha) = \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{\pi\alpha} - 1}\right) - \alpha\sqrt{\frac{2}{\pi\alpha} - 1}.$$

Then $f \in \mathcal{S}^\star_\beta$, *where*

$$\beta = \frac{2}{\pi}\arctan\sqrt{\frac{2}{\pi\alpha} - 1}.$$

2. Main result

In this paper we will give the sharp estimates for initial coefficients of functions $f \in \mathcal{A}$ which satisfied the condition (1). Namely, we have

Theorem 1. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *belong to the class* $\mathcal{A}$ *and satisfy the condition* (1) *for* $0 < \alpha < 1$ *and* $0 < \gamma \le 1$. *Then we have the following sharp estimations*:

(a) $|a_2| \le \dfrac{2\gamma}{1-\alpha}$;

(b) $|a_3| \le \begin{cases} \dfrac{2\gamma}{2-\alpha}, & 0 < \gamma \le \dfrac{(1-\alpha)^2}{3-\alpha}, \\ \dfrac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)}, & \dfrac{(1-\alpha)^2}{3-\alpha} \le \gamma \le 1; \end{cases}$

(c) $|a_4| \le \begin{cases} \dfrac{2\gamma}{3-\alpha}, & 0 < \gamma \le \gamma_\nu, \\ \dfrac{2\gamma}{3-\alpha}\left[\dfrac{1}{3} + \dfrac{2}{3}\dfrac{(\alpha^2-6\alpha+17)\gamma^2}{(1-\alpha)^3(2-\alpha)}\right], & \gamma_\nu \le \gamma \le 1; \end{cases}$

where

$$\gamma_\nu = \sqrt{\frac{(1-\alpha)^3(2-\alpha)}{\alpha^2 - 6\alpha + 17}}.$$

Proof. We can write the condition (1) in the form

$$\left(\frac{f(z)}{z}\right)^{-(1+\alpha)} f'(z) = \left(\frac{1+\omega(z)}{1-\omega(z)}\right)^{\gamma} \quad (= (1 + 2\omega(z) + 2\omega^2(z) + \cdots)^{\gamma}), \tag{2}$$

where ω is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$. If we denote by L and R the left- and right-hand side of equality (2), then we have

$$L = \left[1 - (1+\alpha)(a_2z + \cdots) + \binom{-(1+\alpha)}{2}(a_2z + \cdots)^2 + \binom{-(1+\alpha)}{3}(a_2z + \cdots)^3 + \cdots\right] \cdot (1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots)$$

and if we put $\omega(z) = c_1 z + c_2 z^2 + \cdots$:

$$\begin{aligned}R = 1 + \gamma[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots] \\ + \binom{\gamma}{2}[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots]^2 \\ + \binom{\gamma}{3}[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots]^3 + \cdots\end{aligned}$$

If we compare the coefficients on z, z^2, z^3 in L and R, then, after some calculations, we obtain

$$a_2 = \frac{2\gamma}{1-\alpha} c_1, \quad a_3 = \frac{2\gamma}{2-\alpha} c_2 + \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)} c_1^2, \quad a_4 = \frac{2\gamma}{3-\alpha}(c_3 + \mu c_1 c_2 + \nu c_1^3), \tag{3}$$

where

$$\mu = \mu(\alpha, \gamma) = \frac{2(5-\alpha)\gamma}{(1-\alpha)(2-\alpha)} \quad \text{and} \quad \nu = \nu(\alpha, \gamma) = \frac{1}{3} + \frac{2}{3}\frac{(\alpha^2 - 6\alpha + 17)\gamma^2}{(1-\alpha)^3(2-\alpha)}. \tag{4}$$

Since $|c_1| \le 1$, then by using (3) we easily obtain the result (a) from this theorem. Also, by using $|c_1| \le 1$ and $|c_2| \le 1 - |c_1|^2$, from (3) we have

$$\begin{aligned}|a_3| &\le \frac{2\gamma}{2-\alpha}|c_2| + \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)}|c_1|^2 \\ &\le \frac{2\gamma}{2-\alpha}(1 - |c_1|^2) + \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)}|c_1|^2 \\ &= \frac{2\gamma}{2-\alpha} + \frac{2\gamma}{2-\alpha}\left[\frac{(3-\alpha)\gamma}{(1-\alpha)^2} - 1\right]|c_1|^2\end{aligned}$$

and the result depends of the sign of the factor in the last bracket.

The main tool of our proof for the coefficient a_4 will be the results of [3, Lemma 2]. Namely, in that paper the authors considered the sharp estimate of the functional

$$\Psi(\omega) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$$

within the class of all holomorphic functions of the form

$$\omega(z) = c_1 z + c_2 z^2 + \cdots$$

and satisfying the condition $|\omega(z)| < 1$, $z \in \mathbb{D}$. In the same paper in Lemma 2, for ω of the previous type and for any real numbers μ and ν they give the sharp estimates $\Psi(\omega) \le \Phi(\mu, \nu)$, where $\Phi(\mu, \nu)$ is given in general form in Lemma 2, and here we will use

$$\Phi(\mu, \nu) = \begin{cases} 1, & (\mu, \nu) \in D_1 \cup D_2 \cup \{(2, 1)\}, \\ |\nu|, & (\mu, \nu) \in \bigcup_{k=3}^{7} D_k, \end{cases}$$

where

$$\begin{aligned}
D_1 &= \left\{(\mu,\nu) : |\mu| \le \tfrac{1}{2},\ -1 \le \nu \le 1\right\},\\
D_2 &= \left\{(\mu,\nu) : \tfrac{1}{2} \le |\mu| \le 2,\ \tfrac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \le \nu \le 1\right\},\\
D_3 &= \left\{(\mu,\nu) : |\mu| \le \tfrac{1}{2},\ \nu \le -1\right\},\\
D_4 &= \left\{(\mu,\nu) : |\mu| \ge \tfrac{1}{2},\ \nu \le -\tfrac{2}{3}(|\mu|+1)\right\},\\
D_5 &= \{(\mu,\nu) : |\mu| \le 2,\ \nu \ge 1\},\\
D_6 &= \left\{(\mu,\nu) : 2 \le |\mu| \le 4,\ \nu \ge \tfrac{1}{12}(\mu^2+8)\right\},\\
D_7 &= \left\{(\mu,\nu) : |\mu| \ge 4,\ \nu \ge \tfrac{2}{3}(|\mu|-1)\right\}.
\end{aligned}$$

In that sense, we need the values α and γ such that $0 < \mu \le \frac{1}{2}$, $\mu \le 2$, $\mu \le 4$, $\nu \le 1$. So, by using (4), we easily get the equivalence

$$\begin{aligned}
0 < \mu \le \tfrac{1}{2} &\Leftrightarrow \gamma \le \frac{(1-\alpha)(2-\alpha)}{4(5-\alpha)} := \gamma_{1/2};\\
\mu \le 2 &\Leftrightarrow \gamma \le \frac{(1-\alpha)(2-\alpha)}{5-\alpha} := \gamma_2;\\
\mu \le 4 &\Leftrightarrow \gamma \le \frac{2(1-\alpha)(2-\alpha)}{5-\alpha} := \gamma_4;\\
\nu \le 1 &\Leftrightarrow \gamma \le \sqrt{\frac{(1-\alpha)^3(2-\alpha)}{\alpha^2-6\alpha+17}} := \gamma_\nu.
\end{aligned}$$

It is easily to obtain that all values $\gamma_{1/2}$, γ_2, γ_4, γ_ν are decreasing functions of α, $0 < \alpha < 1$ and that

$$0 < \gamma_{1/2} < \tfrac{1}{10}, \quad 0 < \gamma_2 < \tfrac{2}{5}, \quad 0 < \gamma_4 < \tfrac{4}{5}, \quad 0 < \gamma_\nu < \sqrt{\tfrac{2}{17}} = 0.342997\ldots$$

Also, it is clear that

$$0 < \gamma_{1/2} < \gamma_2 < \gamma_4$$

and it is easy to obtain that

$$\gamma_{1/2} \le \gamma_\nu \quad \text{for} \quad \alpha \in (0, \alpha_\nu],$$

where $\alpha_\nu = 0.951226\ldots$ is the root of the equation $5\alpha^3 - 56\alpha^2 + 177\alpha - 122 = 0$ (of course $\gamma_\nu \le \gamma_{1/2}$ for $\alpha \in [\alpha_\nu, 1)$).

Further, the next relation is valid:

$$0 < \gamma_\nu < \gamma_2 < \gamma_4.$$

Case 1 ($0 < \gamma \le \gamma_\nu$). First, it means that $\nu \le 1$. If $0 < \gamma \le \gamma_{1/2}$, then $0 < \mu \le \frac{1}{2}$ and $0 < \nu \le 1$, which by [3, Lemma 2] gives $\Phi(\mu,\nu) = 1$. If $\gamma_{1/2} \le \gamma \le \gamma_\nu$, $\alpha \in (0, \alpha_\nu)$, then $\frac{1}{2} \le \mu < 2$, $0 < \nu \le 1$ and if we prove that

$$\tfrac{4}{27}(\mu+1)^3 - (\mu+1) \le \nu,$$

then also by [3, Lemma 2] we have $\Phi(\mu,\nu) = 1$. In that sense, let us denote

$$L_1 =: \tfrac{4}{27}(\mu+1)^3 - (\mu+1) \quad \text{and} \quad R_1 = \nu.$$

Since L_1 is an increasing function of μ for $\mu \geq \frac{1}{2}$ and since $\gamma \leq \gamma_\nu$, then

$$\mu \leq \frac{2(3-\alpha)\gamma_\nu}{(1-\alpha)(5-\alpha)} = 2\sqrt{\frac{(1-\alpha)(5-\alpha)^2}{(2-\alpha)(\alpha^2-6\alpha+17)}} < 2\sqrt{\frac{25}{34}} = \frac{10}{\sqrt{34}}$$

(because the function under the square root is decreasing) and so

$$L_1 < \frac{4}{27}\Big(\frac{10}{\sqrt{34}}+1\Big)^3 - \Big(\frac{10}{\sqrt{34}}+1\Big) = 0.249838\ldots,$$

while

$$R_1 = \nu = \frac{1}{3} + \frac{2}{3}\frac{(\alpha^2-6\alpha+17)\gamma^2}{(1-\alpha)^3(2-\alpha)} > \frac{1}{3} = 0.33\ldots$$

This implies the desired inequality.

Case 2 ($\gamma_\nu \leq \gamma \leq 1$). In this case we have that $\nu \geq 1$. If $\gamma_\nu \leq \gamma \leq \gamma_2$, $\alpha \in (0, 1)$, then $0 < \mu \leq 2$, $\nu \geq 1$, which by [3, Lemma 2] implies $\Phi(\mu, \nu) = \nu$. If $\gamma_2 \leq \gamma \leq \gamma_4$, $\alpha \in (0, 1)$, then $2 \leq \mu \leq 4$. Also, after some calculations, the inequality $\nu \geq \frac{1}{12}(\mu^2+8)$ is equivalent to

$$\frac{43-23\alpha+5\alpha^2-\alpha^3}{(1-\alpha)^3(2-\alpha)^2}\gamma^2 \geq 1.$$

Since $\gamma_2 \leq \gamma$, then the previous inequality is satisfied if

$$\frac{43-23\alpha+5\alpha^2-\alpha^3}{(1-\alpha)^3(2-\alpha)^2}\gamma_2^2 \geq 1.$$

But, the last inequality is equivalent to the inequality $\alpha^2-2\alpha-3 \leq 0$, which is really true for $\alpha \in (0, 1)$. By [3, Lemma 2] we also have $\Phi(\mu, \nu) = \nu$. Finally, if $\gamma \geq \gamma_4$, then $\mu \geq 4$ and if $\nu \geq \frac{2}{3}(\mu-1)$ we have (by using the same lemma) $\Phi(\mu, \nu) = \nu$. Really, the inequality $\nu \geq \frac{2}{3}(\mu-1)$ is equivalent with

$$2(\alpha^2-6\alpha+17)\gamma^2 - 4(1-\alpha)^2(5-\alpha)\gamma + 3(1-\alpha)^3(2-\alpha) \geq 0.$$

Since the discriminant of previous trinomial is

$$D = 8(1-\alpha)^3(\alpha^3-2\alpha^2+17\alpha-52) < 0$$

for $\alpha \in (0, 1)$, then the previous inequality is valid. By using (3) we have that $|a_4| \leq \gamma/(3-\alpha)$ (Case 1), or $|a_4| \leq \gamma/(3-\alpha)\nu$ (Case 2), and from there the statement of the theorem.

All results of Theorem 1 are the best possible as demonstrated by the functions f_i, $i = 1, 2, 3$, defined with

$$\left(\frac{z}{f_i(z)}\right)^{1+\alpha} f_i'(z) = \left(\frac{1+z^i}{1-z^i}\right)^\gamma,$$

where $0 < \alpha < 1$, $0 < \gamma \leq 1$. We have that

$$c_i = 1 \quad \text{and} \quad c_j = 0 \text{ when } j \neq i. \qquad \square$$

Remark 2. By using Theorem A we can conclude that it is sufficient to be $\gamma \leq \gamma_\star(\alpha)$ and $0 < \alpha < \frac{2}{\pi}$ for starlikeness of functions $f \in \mathcal{A}$ which satisfied the condition (1).

Also, these conditions imply that the modulus of the coefficients a_2, a_3, a_4 is bounded with some constants. Namely, from the estimates given in Theorem 1 we have, for example,

$$|a_2| \le \frac{2\gamma}{1-\alpha} \le \frac{2\gamma_\star(\alpha)}{1-\alpha}, \quad |a_3| \le \frac{2(3-\alpha)\gamma_\star^2(\alpha)}{(1-\alpha)^2(2-\alpha)},$$

etc.

We note that $\gamma_\star(\alpha) < 1-\alpha$ for $0 < \alpha < \frac{2}{\pi}$. Namely, if we put

$$\phi(\alpha) =: \gamma_\star(\alpha) - (1-\alpha),$$

then $\phi'(\alpha) = 1 - \sqrt{2/(\pi\alpha) - 1}$. It is easily to see that ϕ attains its minimum $\phi(1/\pi) = -\frac{1}{2}$ and since $\phi(0+) = 0$, $\phi\big(\frac{2}{\pi}-\big) = \frac{2}{\pi} - 1 < 0$, we have the desired inequality.

When $\alpha \to 0$, then $\gamma_\star(0+) = 1$, and from Theorem 1, we have the next estimates for $0 < \gamma \le 1$:

$$|a_2| \le 2\gamma \le 2, \quad |a_3| \le \begin{cases} \gamma, & 0 < \gamma \le \frac{1}{3}, \\ 3\gamma^2, & \frac{1}{3} \le \gamma \le 1, \end{cases}$$

and

$$|a_4| \le \begin{cases} \dfrac{2\gamma}{3}, & 0 < \gamma \le \sqrt{2/17}, \\ \dfrac{2\gamma(1+17\gamma^2)}{9 \le 4}, & \sqrt{2/17} \le \gamma \le 1. \end{cases}$$

This is the case when we have strongly starlike functions of order γ.

For $\gamma = 1$ in Theorem 1, i.e., if

$$\mathrm{Re}\bigg[\Big(\frac{z}{f(z)}\Big)^{1+\alpha} f'(z)\bigg] > 0, \quad z \in \mathbb{D},$$

we have

$$|a_2| \le \frac{2}{1-\alpha}, \quad |a_3| \le \frac{2(3-\alpha)}{(1-\alpha)^2(2-\alpha)}$$

and

$$|a_4| \le \frac{2}{3-\alpha}\bigg[\frac{1}{3} + \frac{2}{3}\frac{(\alpha^2 - 6\alpha + 17)}{(1-\alpha)^3(2-\alpha)}\bigg].$$

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