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HANKEL DETERMINANTS OF SECOND AND THIRD ORDER FOR THE CLASS $\mathcal{S}$ OF UNIVALENT FUNCTIONS

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ABSTRACT. In this paper we give the upper bounds of the Hankel determinants of the second and third order for the class $\mathcal{S}$ of univalent functions in the unit disc.

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and let $\mathcal{S}$ be the class of univalent functions in the unit disc $\mathbb{D}$. Let $\mathcal{S}^\star$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ which are starlike and convex in $\mathbb{D}$, respectively, and let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1 \qquad (z \in \mathbb{D}).$$

(see [5–7]).

The *qth* Hankel determinant for a function f from $\mathcal{A}$ is defined for $q \geq 1$, and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

Thus, the second Hankel determinant is

$$H_2(2) = a_2a_4 - a_3^2 \tag{1}$$

and the third is

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

The concept of Hankel determinant finds its application in the theory of singularities (see [1]) and in the study of power series with integral coefficients.

For some subclasses of the class $\mathcal{S}$ of univalent functions the sharp estimation of $|H_2(2)|$ are known. For example, for the classes $\mathcal{S}^\star$ and $\mathcal{U}$ we have that $|H_2(2)| \leq 1$ (see [3, 8]), while $|H_2(2)| \leq \frac{1}{8}$ for the class $\mathcal{K}$ ([3]). Finding sharp estimates of the third order Hankel determinant turns out to be more complicated, so very few are known. An overview of results on the upper

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bound of $|H_3(1)|$ can be found in [10], while new non-sharp upper bounds for different classes and conjectures about the sharp ones are given in [9].

In this paper we give an upper bound of $|H_2(2)|$ and $|H_3(1)|$ for the class $\mathcal{S}$. Namely, we have:

Theorem 1. *For the class $\mathcal{S}$ we have*

$$|H_2(2)| \le A, \quad \text{where} \quad 1 \le A \le \frac{11}{3} = 3,66\dots$$

and

$$|H_3(1)| \le B, \quad \text{where} \quad \frac{4}{9} \le B \le \frac{32+\sqrt{285}}{15} = 3.258796\dots$$

P r o o f. In the proof of this theorem we will use mainly the notations and results given in the book of N. A. Lebedev ([4]).

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t)-f(z)}{t-z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are called Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([2, 4]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{2}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{3}$$

belongs to $\mathcal{S}$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 + c_5 z^5 + \dots$$

belongs to the class $\mathcal{S}$. Then for the function f_2 we have the appropriate Grunsky's coefficients of the form $\omega^{(2)}_{2p-1,2q-1}$ and the inequality (2) has the form

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega^{(2)}_{2p-1,2q-1} x_{2p-1} \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{4}$$

As it has been shown in [4: p. 57], if f is given by (3) then the coefficients a_2, a_3, a_4 and a_5 are expressed by Grunsky's coefficients $\omega^{(2)}_{2p-1,2q-1}$ of the function f_2 given by (3) in the following way (in the next text we omit upper index 2 in $\omega^{(2)}_{2p-1,2q-1}$):

$$\begin{aligned}
a_2 &= 2\omega_{11}, \\
a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\
a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3 \\
a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{15}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4 \\
0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}.
\end{aligned} \tag{5}$$

Now, from (1) and (5), we have

$$H_2(2) = 4\omega_{11}\omega_{33} + 4\omega_{11}^2\omega_{13} - 4\omega_{13}^2 - \frac{7}{3}\omega_{11}^4 = 4\omega_{11}\omega_{33} - \frac{4}{3}\omega_{11}^4 - \left(2\omega_{13} - \omega_{11}^2\right)^2,$$

and from here

$$|H_2(2)| \le 4|\omega_{11}||\omega_{33}| + \frac{4}{3}|\omega_{11}|^4 + \left|2\omega_{13} - \omega_{11}^2\right|^2. \tag{6}$$

Since for the class $\mathcal{S}$ we have $|a_3 - a_2^2| \le 1$ (see [2]) and since from (5)

$$|2\omega_{13} - \omega_{11}^2| = |a_3 - a_2^2|,$$

then

$$|2\omega_{13} - \omega_{11}^2| \le 1. \tag{7}$$

On the other hand, from (4) for $x_{2p-1} = 0$, $p = 3, 4, \ldots$, we have

$$|\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 \le |x_1|^2 + \frac{|x_3|^2}{3}. \tag{8}$$

From (8) for $x_1 = 1$, $x_3 = 0$ and since $\omega_{31} = \omega_{13}$, we have

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 \le 1,$$

which implies

$$|\omega_{13}|^2 \le \frac{1}{3}(1 - |\omega_{11}|^2). \tag{9}$$

Also, for $x_1 = 0$, $x_3 = 1$ we obtain

$$|\omega_{31}|^2 + 3|\omega_{33}|^2 \le \frac{1}{3}$$

and so

$$|\omega_{33}| \le \frac{1}{3}\sqrt{1 - 3|\omega_{31}|^2} \le \frac{1}{3}. \tag{10}$$

Finally, from (6), (7), (9) and (10), we have

$$|H_2(2)| \le \frac{4}{3}|\omega_{11}| + \frac{4}{3}|\omega_{11}|^4 + 1 \le \frac{11}{3},$$

because from (5) we have that

$$|a_2| = |2\omega_{11}| \le 2 \implies |\omega_{11}| \le 1.$$

Since $\mathcal{S}^*$ and $\mathcal{U}$ are both subsets of $\mathcal{S}$ with 1 as a sharp upper bound of $|H_2(2)|$, we have that on the class $\mathcal{S}$, $|H_2(2)| \ge 1$.

As for Hankel determinant of the third order, by using (5), we can write

$$\begin{aligned} H_3(1) &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \\ &= -8\omega_{13}^3 + 2\omega_{11}^4\omega_{13} + \frac{8}{3}\omega_{11}^3\omega_{33} - 4\omega_{33}^2 - \frac{4}{9}\omega_{11}^6 \\ &\quad + 4\omega_{13}\omega_{35} + 10\omega_{13}\omega_{15}^2 - 5\omega_{11}^2\omega_{15}^2 - 2\omega_{11}^2\omega_{35} \\ &= -2\omega_{13}\left(4\omega_{13}^2 - \omega_{11}^4\right) - \left(2\omega_{33} - \frac{2}{3}\omega_{11}^3\right)^2 + (2\omega_{35} + 5\omega_{15}^2)(2\omega_{13} - \omega_{11}^2), \end{aligned}$$

and from here

$$|H_3(1)| \le \underbrace{2|\omega_{13}|\left|4\omega_{13}^2-\omega_{11}^4\right|}_{B_1}+\underbrace{\left|2\omega_{33}-\frac{2}{3}\omega_{11}^3\right|^2}_{B_2}+\underbrace{|2\omega_{35}+5\omega_{15}^2||(2\omega_{13}-\omega_{11}^2|}_{B_3}$$
$$= B_1+B_2+B_3.$$

By using the relations (7) and (9), we obtain

$$\begin{aligned} B_1 &= 2|\omega_{13}|\left|2\omega_{13}-\omega_{11}^2\right|\left|2\omega_{13}+\omega_{11}^2\right| \\ &\le 2|\omega_{13}|\left|2\omega_{13}+\omega_{11}^2\right| \\ &\le 2|\omega_{13}|\left(2|\omega_{13}|+|\omega_{11}|^2\right) \\ &= 4|\omega_{13}|^2+2|\omega_{13}||\omega_{11}|^2 \\ &\le \frac{2}{3}\left[2\left(1-|\omega_{11}|^2\right)+\sqrt{3}|\omega_{11}|^2\sqrt{1-|\omega_{11}|^2}\right] \\ &=: \frac{2}{3}\varphi(|\omega_{11}|^2), \end{aligned}$$

where

$$\varphi(t)=2(1-t)+\sqrt{3}t\sqrt{1-t}, \qquad 0\le t\le 1.$$

It is easily to show that the function φ decreases on $(0,1)$ and has maximum $\varphi(0)=2$, which implies

$$B_1 \le \frac{2}{3}\varphi(0)=\frac{4}{3}. \tag{11}$$

From the last equation in the relation (5), we have

$$2\omega_{33}-\frac{2}{3}\omega_{11}^3 = 2\omega_{15}-2\omega_{11}\omega_{13},$$

and from here

$$\left|2\omega_{33}-\frac{2}{3}\omega_{11}^3\right| \le 2|\omega_{15}|+2|\omega_{11}||\omega_{13}|. \tag{12}$$

Similarly as in (8), we have

$$|\omega_{11}x_1+\omega_{31}x_3|^2+3|\omega_{13}x_1+\omega_{33}x_3|^2+5|\omega_{15}x_1+\omega_{35}x_3|^2 \le |x_1|^2+\frac{|x_3|^2}{3}. \tag{13}$$

If we put $x_1=1$ and $x_3=0$, then we get

$$|\omega_{11}|^2+3|\omega_{13}|^2+5|\omega_{15}|^2 \le 1,$$

and so

$$|\omega_{15}| \le \frac{1}{\sqrt{5}}\sqrt{1-|\omega_{11}|^2-3|\omega_{13}|^2}. \tag{14}$$

From (12) and (14), we have

$$\begin{aligned} \left|2\omega_{33}-\frac{2}{3}\omega_{11}^3\right| &\le \frac{2}{\sqrt{5}}\left(\sqrt{1-|\omega_{11}|^2-3|\omega_{13}|^2}+\sqrt{5}|\omega_{11}||\omega_{13}|\right) \\ &=: \frac{2}{\sqrt{5}}\psi(|\omega_{11}|,|\omega_{13}|), \end{aligned}$$

where

$$\psi(t,s)=\sqrt{1-t^2-3s^2}+\sqrt{5}ts, \qquad 0\le t\le 1, \quad 0\le s\le \frac{1}{\sqrt{3}}\sqrt{1-t^2}.$$

It is an elementary fact to find that in cited domain $\max\psi = 1$ attained for $t = s = 0$, which implies

$$B_2 = \left|2\omega_{33} - \frac{2}{3}\omega_{11}^3\right|^2 \le \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}. \tag{15}$$

From relation (13) we also have

$$5|\omega_{15}x_1 + \omega_{35}x_3|^2 \le |x_1|^2 + \frac{|x_3|^2}{3}.$$

If we put in the previous relation $x_1 = 5\omega_{15}$, $x_3 = 2$, and then use (14) we receive

$$|2\omega_{35} + 5\omega_{15}^2|^2 \le 5|\omega_{15}|^2 + \frac{4}{15} \le 1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 + \frac{4}{15} \le \frac{19}{15},$$

which finally gives

$$B_3 = |2\omega_{35} + 5\omega_{15}^2| \cdot |2\omega_{13} - \omega_1^2| \le \sqrt{\frac{19}{15}} \tag{16}$$

(in the last step we have used the relation (7)). By using relations (11), (15) and (16), we obtained

$$|H_3(1)| \le B_1 + B_2 + B_3 \le \frac{4}{3} + \frac{4}{5} + \sqrt{\frac{19}{15}} = \frac{32+\sqrt{285}}{15}.$$

The function defined by $\frac{zf'(z)}{f(z)} = \frac{1+z^3}{1-z^3}$ where $a_2 = a_3 = a_5 = 0$, $a_4 = \frac{2}{3}$ is starlike (thus univalent) and $H_3(1) = -\frac{4}{9}$. Therefore on the class $\mathcal{S}$,

$$|H_3(1)| \ge \frac{4}{9}.$$

□

REFERENCES

[1] DIENES, P.: *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable*, New York-Dover: Mineola, NY, USA, 1957.

[2] DUREN, P. L.: *Univalent Function*, Springer-Verlag, New York, 1983.

[3] JANTENG, A.—HALIM, S. A.—DARUS, M.: *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. **1**(13) (2007), 619–225.

[4] LEBEDEV, N. A.: *Area Principle in the Theory of Univalent Functions*, Nauka, Moscow, 1975 (in Russian).

[5] OBRADOVIĆ, M.—PONNUSAMY, S.: *New criteria and distortion theorems for univalent functions*, Complex Variables Theory Appl. **44** (2001), 173–191.

[6] OBRADOVIĆ, M.—PONNUSAMY, S.: *On the class $\mathcal{U}$*, Proc. 21st Annual Conference of the Jammu Math. Soc. and a National Seminar on Analysis and its Application, 2011, pp. 11–26.

[7] OBRADOVIĆ, M.—PONNUSAMY, S.—WIRTHS, K. J.: *Geometric studies on the class $\mathcal{U}(\lambda)$*, Bull. Malays. Math. Sci. Soc. **39**(3) (2016), 1259–1284.

[8] OBRADOVIĆ, M.—TUNESKI, N.: *Some properties of the class $\mathcal{U}$*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **73**(1) (2019), 49–56.

[9] OBRADOVIĆ, M.—TUNESKI, N.: *New upper bounds of the third Hankel determinant for some classes of univalent functions*, submitted, `https://arxiv.org/abs/1911.10770`.

[10] SHI, L.—SRIVASTAVA, H. M.—ARIF, M.—HUSSAIN, S.—KHAN, H.: *An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function*, Symmetry **11** (2019), 598.

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