THE CLEAN GRAPH OF A COMMUTATIVE RING *

Zoran Petrović
Faculty of Mathematics, University of Belgrade, Belgrade, Serbia
Zoran Pucanović
Faculty of Civil Engineering, University of Belgrade, Belgrade, Serbia

Abstract

To gain a better understanding of clean rings and their relatives, the clean graph of a commutative ring with identity is introduced and its various properties established. Further investigation of clean graphs leads to additional results concerning other classes of rings.

Keywords: Clean rings; clean graphs; genus of a graph.

Mathematics Subject Classification: 13A99; 05C10.

1 Introduction

In this paper, R always denotes a commutative ring with identity, $\mathrm{U}(R)$ its set of units, $\mathrm{Id}(R)$ its set of idempotents, $\mathrm{Z}(R)$ its set of zero divisors. For simplicity, we introduce the notation $\mathrm{UI}(R) = \mathrm{U}(R) \cup \mathrm{Id}(R)$. For any integer $n \geq 1$, $\mathrm{U}_n(R)$ denotes the set of elements of the ring R which can be represented as a sum of $k \leq n$ units, while $\mathrm{U}'(R)$ denotes the elements in R which can be represented as a finite sum of units, i.e., $\mathrm{U}'(R) = \cup_{n=1}^{\infty} \mathrm{U}_n(R)$. The rings which are generated by their invertible elements are the subject of a lot of research. A ring R is *good*, or, according to Raphael [17], an *S-ring*, if $R = \mathrm{U}'(R)$. A ring is, according to Vámos [18], *n-good*, or according to Henriksen [10], *(S, n)-ring*, if $R = \mathrm{U}_n(R)$. The element $a \in R$ is said to be *clean* if there exists an idempotent $e \in R$ such that $a - e$ is invertible. If every element of R is clean then, according to Nicholson [14], R is said to be a *clean ring*. Xiao and Tong in [23] generalize these types of rings by introducing the notion of *n-clean* and *Σ-clean* rings in which, respectively, every element is the sum of an idempotent and n units, or the sum of an idempotent and finitely many units. The class of n-clean rings contains clean and n-good rings, while the class of Σ-clean rings contains all

*Address correspondence to Zoran Pucanović, Faculty of Civil Engineering, University of Belgrade, Bulevar kralja Aleksandra 73, 11000 Beograd, Serbia; E-mail: pucanovic@grf.bg.ac.rs

the previously described classes. Among other things, Xiao and Tong show that the group ring $\mathbb{Z}_{(p)}G$, where G is a cyclic group of order 3, is 2-clean for every prime $p \neq 2$. Since Han and Nicholson have shown earlier in [9] that the group ring $\mathbb{Z}_{(7)}G$ is not clean, we have a ring which is 2-clean, but not clean. On the other hand, Camillo and Yu have shown in [5] that every clean ring in which 2 is a unit must be 2-good, therefore 2-clean. Let us also mention the result of Henriksen [10] that $M_n(R)$ is 3-good for an arbitrary ring R and every $n \geq 2$.

The main idea of this paper is to associate a graph to a ring with identity in a way which will enable us to better grasp the properties of the above mentioned classes of rings. This approach has lead to interesting results both in algebra and graph theory (one may check some recent papers, for example [3], [4], [16], [21], [22]). We restrict ourselves to commutative rings in this paper although the same definition may be given for the non-commutative case as well.

The *clean graph* $\mathcal{C}\Gamma(R)$ of a commutative ring R is defined as follows:

$$V(\mathcal{C}\Gamma(R)) = R, \quad E(\mathcal{C}\Gamma(R)) = \{\{r_1, r_2\} : r_1 + r_2 \in \mathrm{UI}(R)\},$$

where $V(\Gamma)$ $(E(\Gamma))$ denotes the set of vertices (edges) of the graph Γ.

The relation with clean rings should be evident. Suppose that R is a clean ring. Then the clean graph $\mathcal{C}\Gamma(R)$ is connected and $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 4$. Namely, if $x, y \in R$, then $x = u_1 + e_1$, $y = u_2 + e_2$ for some $u_1, u_2 \in \mathrm{U}(R)$, $e_1, e_2 \in \mathrm{Id}(R)$. Then

$$x = u_1 + e_1 — (-u_1) — 0 — (-u_2) — u_2 + e_2 = y$$

is a path from x to y in $\mathcal{C}\Gamma(R)$, from which the claim follows. Actually, a more precise result holds; we are going to prove that $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$ for every clean ring R (see Theorem 2.5).

Let us briefly mention some other notation and notions from graph theory which we will use. For us, a graph Γ is always a simple non-directed graph without loops. The degree of a vertex x, denoted by $\deg(x)$, is the number of vertices adjacent to the vertex x. Vertices x and y of a graph Γ are *connected* if there is a path in Γ which connects them. If every two vertices in Γ are connected, we say that the graph Γ is *connected*. For vertices $x, y \in \Gamma$, one defines the *distance* $d(x, y)$ as the length of a shortest path between x and y, if the vertices x and y are connected, and $d(x, y) = \infty$ if they are not. The *diameter* of a graph Γ is $\mathrm{diam}(\Gamma) = \sup\{d(x, y) | x, y \in \Gamma\}$. A connected graph of diameter 1 with n vertices is *complete*. If the set of vertices of Γ is the disjoint union of two sets A and B, such that $|A| = m$ and $|B| = n$ and two vertices are adjacent if and only if they are not in the same set, then Γ is a *complete bipartite* graph. For complete and complete bipartite graphs, we use the standard notation K^n and $K^{m,n}$, where we allow that m or n need not be finite. A graph is of *genus* 0 or *planar* if it can be embedded into the plane in such a way that different edges do not intersect each other. If one cannot accomplish this in the plane, but on the torus, we say that the graph is of genus 1 or *toroidal*.

Some of our main results are in section 2, where we discuss the structure and connectivity of the clean graph. We determine conditions under which $\mathcal{C}\Gamma(R)$

is complete and prove that $\mathcal{C}\Gamma(R)$ is connected iff R is additively generated by its idempotents and units. We find and prove a condition for finiteness of the diameter of $\mathcal{C}\Gamma(R)$, and in addition to that, determine when this diameter is ≤ 2. Among other things, we prove that all commutative Artinian and clean rings have connected clean graph of diameter ≤ 2. In section 3, we discuss the genus of $\mathcal{C}\Gamma(R)$ when R is a finite ring. We prove that $\mathcal{C}\Gamma(R)$ is planar iff R is isomorphic to one of the following rings: $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{Z}_4$, $\mathbb{F}_2\times\mathbb{F}_2$, or $\mathbb{F}_2[X]/(X^2)$, and that it is toroidal iff R is isomorphic to one of the following rings: $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{Z}_8$, $\mathbb{F}_2[X]/(X^3)$, $\mathbb{F}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X,X^2)$, $\mathbb{Z}_4[X]/(2X,X^2-2)$, $\mathbb{F}_2\times\mathbb{F}_3$, $\mathbb{F}_2\times\mathbb{Z}_4$, $\mathbb{F}_2\times\mathbb{F}_4$, or $\mathbb{F}_2\times\mathbb{F}_2[X]/(X^2)$.

2 On the structure, connectivity and the diameter of $\mathcal{C}\Gamma(R)$

The main difficulty in determining the structure of $\mathcal{C}\Gamma(R)$ stems from the irregular nature of the sum of units or idempotents. However, in some special cases, for example when the ring R is quasi-local, the structure of $\mathcal{C}\Gamma(R)$ may be described. Let us first determine when our graph is complete.

Theorem 2.1 *$\mathcal{C}\Gamma(R)$ is complete if and only if R is a field or a Boolean ring.*

Proof. Suppose that $\mathcal{C}\Gamma(R)$ is complete and that R is neither a field nor a Boolean ring. Therefore, $R=\mathrm{U}(R)\cup\mathrm{Id}(R)$ ($\mathcal{C}\Gamma(R)$ is complete, so every element is adjacent to 0), $\mathrm{Id}(R)\neq R$ (R is not Boolean) and $R\setminus\{0\}\neq\mathrm{U}(R)$ (R is not a field). We conclude that there exist $x\in\mathrm{U}(R)\setminus\mathrm{Id}(R)$ and $y\in\mathrm{Id}(R)\setminus(\mathrm{U}(R)\cup\{0\})$. If $xy\in\mathrm{U}(R)$, then $y\in\mathrm{U}(R)$, a contradiction. We conclude that $xy\in\mathrm{Id}(R)$. So, $(xy)^2=xy$, and, since $x\in\mathrm{U}(R)$ and $y\in\mathrm{Id}(R)$, $xy=y$. Therefore $y(1-x)=0$, and since $y\neq 0$, $1-x\notin\mathrm{U}(R)$; so $1-x\in\mathrm{Id}(R)$. Then $x=1-(1-x)\in\mathrm{Id}(R)$, and this contradicts our choice of x.

The reverse implication is trivial. □

The question that we are mostly concerned with in this section is the connectivity of $\mathcal{C}\Gamma(R)$ and the computation of its diameter. Albeit somewhat unexpected, it turns out that the graph $\mathcal{C}\Gamma(R)$ of a finite ring R is always connected of diameter at most 2. In the case of non-finite rings, there are some other possibilities. Let us look at the simplest example, the ring of rational integers. The ring $\mathbb{Z}$ is Σ-clean, and it is not n-clean for any n; it is a good ring which is not n-good. We have $\mathrm{UI}(\mathbb{Z})=\{-1,0,1\}$. Therefore, the only vertices which are adjacent to the vertex m are $-m$, $1-m$ and $-1-m$. The distance between two vertices $d(m,n)$ is consequently $||m|-|n||$ or $||m|-|n||+1$. Thus $\mathcal{C}\Gamma(\mathbb{Z})$ (Fig. 1) is connected and does not have finite diameter.

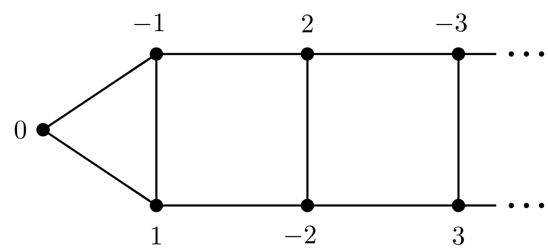


Figure 1. $\mathcal{C}\Gamma(\mathbb{Z})$

Let us now prove a theorem on the connectivity of $\mathcal{C}\Gamma(R)$ for an arbitrary commutative ring R. In order to do that, we introduce the notion of (k,m)-clean elements.

Definition. Let k and m be non-negative integers. An element $x \in R$ is (k,m)-clean if $x = e_1 + \cdots + e_k + u_1 + \cdots + u_m$, where $e_i \in \mathrm{Id}(R)$, and $u_j \in \mathrm{U}(R)$.

Thus, the clean elements are those which are $(1,1)$-clean, the n-clean elements are $(1,n)$-clean, while n-good elements are those which are $(0,n)$-clean. When it does not lead to any confusion, we write a (k,m)-clean element x in the form $x = \sum_{i=1}^{n} x_i$, where $x_i \in \mathrm{UI}(R)$ and $n = k+m$. Let

$$E_n(R) = \left\{ \sum_{i=1}^{k} x_i : x_i \in \mathrm{UI}(R), k \le n \right\}, \quad E(R) = \cup_{n=1}^{\infty} E_n(R).$$

A ring R is an E_n-ring for some n if $R = E_n(R)$, and R is an E-ring if $R = E(R)$. So, E-rings are exactly those rings which are additively generated by their idempotents and units. The ring of rational integers $\mathbb{Z}$ is an example of an E-ring which is not an E_n-ring for any n; therefore the class of E_n-rings is a proper subclass of the class of E-rings. These notions are closely related to the question of connectedness and the question of finite diameter of $\mathcal{C}\Gamma(R)$. Namely, it turns out that $\mathcal{C}\Gamma(R)$ is connected if and only if R is an E-ring, while its diameter is finite if and only if R is an E_n ring for some n.

Before we prove this, let us first establish a few useful lemmas. In these proofs, we use the fact that units and idempotents are clean. This is clear for units, while if $e \in \mathrm{Id}(R)$, we have $e = (2e-1) + (1-e)$, where $2e - 1 \in \mathrm{U}(R)$ ($(2e-1)^{-1} = 2e-1$). This representation of idempotents is unique, i.e., if $e = u + f$, for some $u \in \mathrm{U}(R)$, $f \in \mathrm{Id}(R)$, then $u = 2e-1$ and $f = 1-u$. The proof may be found in [1, Lemma 6].

Lemma 2.1 *Let $x = \sum_{i=1}^{m} e_i$, where $e_i \in \mathrm{Id}(R)$. Then there exists a path from x to 0 in $\mathcal{C}\Gamma(R)$.*

Proof. We get a path from x to 0 by using the well-known property of idempotents: $e \in R$ is an idempotent iff $1-e$ is an idempotent. Here is one path:

$$\sum_{i=1}^{m} e_i — \left(-\sum_{i=2}^{m} e_i\right) — \left(1+\sum_{i=3}^{m} e_i\right) — \left(-\sum_{i=3}^{m} e_i\right) — \left(1+\sum_{i=4}^{m} e_i\right) — \cdots$$

$$\cdots — (1+e_m) — (-e_m) — 1 — 0.$$

Note that $d\left(\sum_{i=1}^{m} e_i, 0\right) \leq 2m-1$. □

Lemma 2.2 *Let $x, y \in R$. If $d(x+y, 0) = r$, then $d(x,y) \leq r+1$.*

Proof. Suppose that

$$(x+y) — s_1 — s_2 — \cdots — s_{r-1} — 0$$

is a path of length r in $\mathcal{C}\Gamma(R)$. In this case,

$$x — (y+s_1) — (-y+s_2) — \cdots — ((-1)^{r-1}y + s_{r-1}) — (-1)^r y — (-1)^{r+1} y$$

is also a path in $\mathcal{C}\Gamma(R)$. We conclude that $d(x,y) \leq r$ if r is an even integer, and $d(x,y) \leq r+1$ if r is an odd integer. □

Lemma 2.3 *Let $x \in R$ be such that $d(x,0) = n$ in $\mathcal{C}\Gamma(R)$. Then there exists k and m such that x is (k,m)-clean. Moreover, we have*

$$x = \sum_{i=1}^{k+m} x_i, \text{ where } x_i \in \mathrm{UI}(R) \text{ and } k+m \leq \left\lfloor \frac{3n}{2} \right\rfloor.$$

Proof. Let $0 — s_1 — s_2 — \cdots — s_{n-1} — x$ be a path of length n in $\mathcal{C}\Gamma(R)$. So, the elements $s_1, s_1+s_2, \ldots, s_{n-1}+x$ belong to $\mathrm{UI}(R)$. The element x may be represented in the form

$$x = (x+s_{n-1}) - (s_{n-1}+s_{n-2}) + \cdots + (-1)^{n-1}s_1.$$

This, however, may not be the representation we need, since if $s_{i-1}+s_i$ is an idempotent, the element $-(s_{i-1}+s_i)$ need not be. The number of such elements is at most $\left\lfloor \frac{n}{2} \right\rfloor$, and every one of them may be replaced by two elements which are in $\mathrm{UI}(R)$: $-(s_{i-1}+s_i) = (1-(s_{i-1}+s_i)) + (-1)$. □

Theorem 2.2 *Let R be a commutative ring. Then $\mathcal{C}\Gamma(R)$ is connected if and only if R is an E-ring.*

Proof. Suppose first that R is an E-ring, i.e., that every element in R may be represented as a sum of elements from $\mathrm{UI}(R)$, and let $x, y \in R$. According to Lemma 2.2, it is enough to show that there is a path from $x+y$ to 0. Suppose that $x+y$ is (m,n)-clean, i.e., $x+y=\sum_{i=1}^{n} u_i+\sum_{j=1}^{m} e_j$, where $u_i \in U(R)$, $e_j \in \mathrm{Id}(R)$. We have the following path in $\mathcal{C}\Gamma(R)$:

$$x+y=\sum_{i=1}^{n} u_i+\sum_{i=1}^{m} e_i \;—\; \left(-\sum_{i=2}^{n} u_i-\sum_{i=1}^{m} e_i\right) \;—\; \left(\sum_{i=3}^{n} u_i+\sum_{i=1}^{m} e_i\right) \;—\; \cdots$$

$$\cdots \;—\; (-1)^{n-1}\left(u_n+\sum_{i=1}^{m} e_i\right) \;—\; (-1)^n \sum_{i=1}^{m} e_i.$$

If n is even, according to Lemma 2.1, we may extend this path to 0. If n is odd, then since $-\sum_{i=1}^{m} e_i$ is adjacent to $\sum_{i=1}^{m} e_i$, we can also extend this path to 0.

To prove the reverse implication, let us assume that $\mathcal{C}\Gamma(R)$ is connected, and let $x \neq 0$ be an element in R. Since our graph is connected, there exists a path from x to 0 in $\mathcal{C}\Gamma(R)$, and let the length of that path be n. By Lemma 2.3, there exists k and m such that x is (k,m)-clean. Therefore, an arbitrary element of the ring R is a sum of finitely many units and idempotents; so R is an E-ring. □

Theorem 2.3 *Let R be a commutative ring. Then* $\mathrm{diam}(\mathcal{C}\Gamma(R))$ *is finite if and only if R is an E_n-ring for some n.*

Proof. Let $\mathrm{diam}(\mathcal{C}\Gamma(R))=d$, and $0 \neq x \in R$. We have that $d(x,0) \leq d$; so according to Lemma 2.3, $x=\sum_{i=1}^{n} x_i$, where $x_i \in \mathrm{UI}(R)$ and $n \leq \left\lfloor \frac{3d}{2} \right\rfloor$.

Suppose now that R is an E_n-ring for some n. Naturally, R is then an E-ring; so the graph $\mathcal{C}\Gamma(R)$ is connected. Let $x,y \in R$. Since R is an E_n-ring, the element $x+y$ is (k,m)-clean for some k and m such that $k+m \leq n$. According to Lemma 2.1 and the proof of the previous theorem, the length of the path from $x+y=\sum_{i=1}^{k} e_i+\sum_{i=1}^{m} u_i$ to 0 is at most $2k+m$. According to Lemma 2.2, $d(x,y) \leq 2k+m+1 \leq 2n+1$; so $\mathrm{diam}(\mathcal{C}\Gamma(R))$ is finite. □

Theorem 2.4 *Let R be a quasi-local ring and $x,y \in R$. Then there exists $z \in R$ such that $x+z \in \{0,1\}$ and $z+y \in \mathrm{U}(R)$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal in the quasi-local ring R and $x,y \in R$. We check all cases:

(1) $x,y \in \mathfrak{m}$: $z=1-x$.
(2) $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$ (or $x \notin \mathfrak{m}$, $y \in \mathfrak{m}$): $z=-x$.
(3) $x \notin \mathfrak{m}$, $y \notin \mathfrak{m}$. We have two cases.
  (3.1) $y-x \notin \mathfrak{m}$: $z=-x$;
  (3.2) $y-x \in \mathfrak{m}$: $z=1-x$. □

Corollary 2.1 *Let $R=R_1 \times \cdots \times R_n$ be a product of quasi-local rings. Then $\mathcal{C}\Gamma(R)$ is connected with* $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$.

Proof. This follows directly from the previous theorem. □

Lemma 2.4 *Suppose that R satisfies:*

$$(\gamma) \qquad (\forall x \in R)(\exists y \in \mathrm{UI}(R))\; x+y \in \mathrm{UI}(R).$$

Then $\mathcal{C}\Gamma(R)$ is connected with $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$.

Proof. Let $x, z \in R$ and $x \neq z$. We prove that $d(x, z) \leq 2$. The condition (γ) holds for $x - z$ as well; so there exists $y \in \mathrm{UI}(R)$ such that $(x - z)$ — y — 0 is a path in $\mathcal{C}\Gamma(R)$. The path we are looking for is the path: x — $(y - z)$ — z. □

Remark 2.1 *It is clear that if* $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$*, then the condition* (γ) *must hold. Namely, if* $x \in R$*, since* $d(x, 0) \leq 2$ *there must exist* $y \in \mathrm{UI}(R)$ *such that* $x + y \in \mathrm{UI}(R)$.

Theorem 2.5 *Let R be a commutative ring. Then* $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$ *if R is clean or 2-good.*

Proof. We only need to check that the condition (γ) is satisfied.

If R is clean and $x \in R$, then $x = e + u$, where $e \in \mathrm{Id}(R)$ and $u \in \mathrm{U}(R)$. The element $y = -u$ shows that condition (γ) is satisfied.

Similarly, if R is 2-good and $x \in R$, we have that $x = u + v$, where $u, v \in \mathrm{U}(R)$. We take $y = -u$. □

It is known that the class of commutative clean rings contains rings of dimension 0, quasi-local rings, as well as commutative von Neumann regular rings [1, Corollary 11, Proposition 2, Theorem 10.]. We also know that homomorphic images and direct product of clean rings are also clean, and that the ring of formal power series $R[[X]]$ is clean if and only if R is clean [1, Proposition 12]). We see that the large class of rings has connected clean graph with diameter $\mathrm{diam}(\mathcal{C}\Gamma(R)) \leq 2$. There remains a question: *Does there exist a commutative E_n-ring R whose clean graph is connected and such that* $\mathrm{diam}(\mathcal{C}\Gamma(R)) > 2$*?*

3 The genus of the clean graph

The notation we use in this section is standard: $\gamma(G)$ is the genus of the graph G, i.e., it is the smallest n such that G may be embedded in S_n, where S_n is an orientable surface of genus n. If H is a subgraph of G, then $\gamma(H) \leq \gamma(G)$. By $\deg(v)$ we denote the number of edges incident to v. Graphs of genus 0 are planar, and graphs of genus 1 are toroidal graphs.

First we show that for a finite commutative ring R, the graph $\mathcal{C}\Gamma(R)$ is planar if and only if $|R| \leq 4$. By a well-known theorem of Kuratowski [12], a graph G is planar if and only if it does not contain a subdivision of K^5 or $K^{3,3}$.

In what follows, we often use the fact that any finite commutative ring with identity is isomorphic to a finite direct product of local rings. In order to simplify notation, we will frequently write that $R = R_1 \times \cdots \times R_n$.

Theorem 3.1 *Let R be a finite commutative ring with identity and $|R| \geq 5$. Then $\mathcal{C}\Gamma(R)$ is not planar.*

Proof. As noted above, $R = R_1 \times \cdots \times R_n$, where each R_i is a finite local ring.

1. $n \geq 3$: In $\mathcal{C}\Gamma(R)$, let $A = \{a_1, a_2, a_3\}$, where $a_1 = (1, 1, 0, 0, \ldots, 0)$, $a_2 = (0, 1, 1, 0, \ldots, 0)$ and $a_3 = (1, 1, 1, 0, \ldots, 0)$. We choose the set of vertices $B = \{b_1, b_2, b_3\}$: $b_1 = (0, 0, 0, 0, \ldots, 0)$, $b_2 = (-1, 0, 0, 0, \ldots, 0)$

and $b_3 = (0, -1, 0, 0, \dots, 0)$. So, every vertex in A is adjacent to every vertex in B. Therefore, $K^{3,3}$ is a subgraph of $\mathcal{C}\Gamma(R)$ and we conclude that the graph $\mathcal{C}\Gamma(R)$ is not planar.

2. $n = 2$:

   (a) $R = R_1 \times R_2$, where $\mathrm{char}(R_i) \neq 2$. Then the subsets of vertices $A = \{(0,0), (-1,0), (0,-1)\}$ and $B = \{(0,1), (1,0), (1,1)\}$ show that $K^{3,3}$ is a subgraph of $\mathcal{C}\Gamma(R)$.

   (b) $R = R_1 \times R_2$, where one of the rings, say R_1, has characteristic 2 and the other does not. In this case vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ form a clique. Let us look at the vertex $(1,-1)$. This vertex is adjacent to vertices $(0,1)$, $(0,1)$ and $(1,1)$ while there is a path $(1,-1)$—$(1,2)$—$(0,-1)$—$(1,0)$ from $(1,-1)$ to $(1,0)$. Therefore a subdivision of K^5 is contained in $\mathcal{C}\Gamma(R)$; so this graph is not planar.

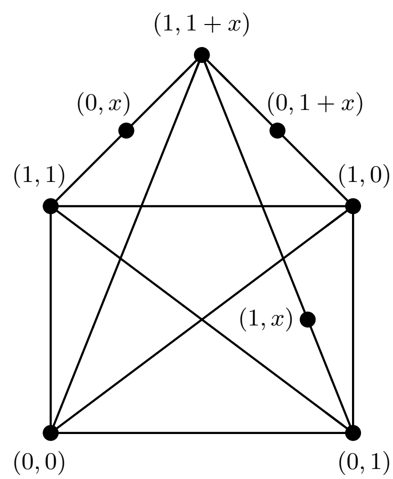


Figure 2.

   (c) $R = R_1 \times R_2$, where both rings have characteristic 2. Since $|R| \geq 5$, at least one of them, say R_2 has more than 2 elements. Let $\mathfrak{m}$ be the maximal ideal in R_2. If $\mathfrak{m} = \{0\}$ then R_2 is a field of characteristic 2. If $a \in R_2 \backslash \{0, 1\}$, then the subsets of vertices $A = \{(0,0), (0,1), (0,a)\}$ and $B = \{(1,1), (1,a), (1,0)\}$ show that $K^{3,3}$ is a subgraph of $\mathcal{C}\Gamma(R)$. If $\mathfrak{m} \neq 0$ let us choose $x \in \mathfrak{m}$, $x \neq 0$. The element $1+x$ is invertible in R_2, therefore $(1, 1+x)$ is invertible in R. We prove that $\mathcal{C}\Gamma(R)$ has as a subgraph a subdivision of K^5 (see Fig. 2). It is clear that $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ form a clique. The vertex $(1, 1+x)$ is adjacent to $(0,0)$ and we have paths: $(1, 1+x)$—$(1, x)$—$(0, 1)$, $(1, 1+x)$—$(0, 1+x)$—$(1, 0)$, $(1, 1+x)$—$(0, x)$—$(1, 1)$.

3. $n = 1$: In this case, R is a finite local ring with maximal ideal $\mathfrak{m}$. It is known that a finite commutative ring with n non-zero zero-divisors has at most $(n+1)^2$ elements (see [8]), i.e., $|R| \leq |Z(R)|^2$. Therefore, $\mathfrak{m}$ must contain at least 2 non-zero zero-divisors m_1, m_2 (otherwise $|R| \leq 4$). The subsets $A = \{1, 1-m_1, 1-m_2\}$ and $B = \{0, m_1, m_2\}$ show that $\mathcal{C}\Gamma(R)$ contains $K^{3,3}$ as its subgraph, therefore it is not planar.

□

Theorem 3.2 *Let R be a finite commutative ring with identity. Then $\mathcal{C}\Gamma(R)$ is planar iff R is isomorphic to one of the following rings: $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{Z}_4$, $\mathbb{F}_2 \times \mathbb{F}_2$, or $\mathbb{F}_2[X]/(X^2)$.*

Proof. From the previous theorem, it directly follows that the finite commutative rings with planar clean graph are actually all finite commutative rings of cardinality ≤ 4. □

In order to go beyond the questions of planarity of our graph, we need a few additional results.

Theorem 3.3 *Let G be a graph with n vertices and $\gamma(G) = g$. Then*

$$\delta(G) \leq 6 + \frac{12g - 12}{n},$$

where $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. In particular, if $\gamma(g) = 1$, we have $\delta(G) \leq 6$, where the equality is attained if and only if G is a 1-skeleton of the 6-regular triangulation of a torus.

This is a known theorem [21, Proposition 2.1].

Theorem 3.4

$$\gamma(K^n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad \gamma(K^{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer greater or equal to x.

One can find the proofs in [20, p. 58, p. 152]. Therefore, the complete toroidal graphs are K^5, K^6, and K^7, while the complete bipartite toroidal graphs are $K^{3,3}$, $K^{4,3}$, $K^{5,3}$, $K^{6,3}$, and $K^{4,4}$. We emphasize that the graphs K^8, $K^{3,7}$ and $K^{5,4}$ have genus 2.

Lemma 3.1 *Let R_1 and R_2 be commutative rings. If $K^{m,n} \subseteq \mathcal{C}\Gamma(R_1)$, then $K^{m,n} \subseteq \mathcal{C}\Gamma(R_1 \times R_2)$.*

Proof. Let $K^{m,n}$ inside $\mathcal{C}\Gamma(R_1)$ be determined by the subsets of vertices $V_1 = \{a_1, \ldots, a_m\}$ and $V_2 = \{b_1, \ldots, b_n\}$. The graph $K^{m,n}$ inside $\mathcal{C}\Gamma(R_1 \times R_2)$ is then determined by subsets of vertices $W_1 = \{(a_1, 1) \ldots, (a_m, 1)\}$ and $W_2 = \{(b_1, 0) \ldots, (b_n, 0)\}$. □

Lemma 3.2 *Let R be a quasi-local commutative ring with identity with maximal ideal $\mathfrak{m}$. Inside $\mathcal{C}\Gamma(R)$, every element in $R \setminus \mathfrak{m}$ is adjacent to every element in $\mathfrak{m}$. The elements m_1, m_2 from $\mathfrak{m}$ are adjacent if and only if $m_1 + m_2 = 0$.*

Proof. This is obvious since the only idempotents in a quasi-local ring are 0 and 1. □

Lemma 3.3 *Let R be a finite commutative local ring with maximal ideal $\mathfrak{m}$. If $|\mathfrak{m}| \geq 5$, then $\gamma(\mathcal{C}\Gamma(R)) \geq 2$. This also holds in the case $|\mathfrak{m}| \geq 3$ and $|\mathrm{U}(R)| \geq 7$.*

Proof. We can find at least 5 different invertible elements since for every $a \in \mathfrak{m}$, one has $1 - a \in \mathrm{U}(R)$. According to the previous lemma every one of them is adjacent to every vertex in $\mathfrak{m}$. Therefore $K^{5,5} \subseteq \mathcal{C}\Gamma(R)$; so the graph is not of genus 1. In the second case, $K^{3,7} \subseteq \mathcal{C}\Gamma(R)$; therefore $\gamma(\mathcal{C}\Gamma(R)) \geq 2$. □

Lemma 3.4 *Let R be a commutative ring. If $|\mathrm{UI}(R)| \geq 8$, then $\gamma(\mathcal{C}\Gamma(R)) \geq 2$.*

Proof. Let $u_1, \ldots, u_8$ be distinct vertices from $\mathrm{UI}(R)$. Every vertex x is then adjacent to vertices $u_i - x$, $i = 1, \ldots, 8$. In the worst case, we have $x = u_i - x$ for some i; say $x = u_1 - x$ hence $2x = u_1$. Then $2x \neq u_j$ and hence $x \neq u_j - x$ for all $j = 2, \ldots, 8$. We certainly have $\deg(x) \geq 7$. The result then follows from Theorem 3.3. □

In what follows, we assume that $R = R_1 \times \cdots \times R_n$ is a finite commutative ring with identity, where each R_i is a finite local ring.

Theorem 3.5 *Let R be a commutative ring. If $n \geq 3$, then $\gamma(\mathcal{C}\Gamma(R)) \geq 2$.*

Proof. It is enough to concentrate on the case $n = 3$. The result is then a direct consequence of Lemma 3.4, since the ring $R = R_1 \times R_2 \times R_3$ has at least 8 idempotents. □

Therefore, we only need to check the cases $n = 1$ and $n = 2$. Let us first assume that $n = 1$, i.e., R is a finite local ring with maximal ideal $\mathfrak{m}$. It is known that in this case $|R| = p^k$ for some prime p and positive integer k and that $|\mathfrak{m}|$ divides $|R|$.

Theorem 3.6 *Let R be a finite local commutative ring with identity. In this case, $\gamma(\mathcal{C}\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_5$, $\mathbb{F}_7$, $\mathbb{Z}_8$, $\mathbb{F}_2[X]/(X^3)$, $\mathbb{F}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X, X^2)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$.*

Proof. Suppose first that $k = 1$ (in the notation mentioned above), i.e., $|R| = p$. Then $R \cong \mathbb{F}_p$. For $p > 7$, $\gamma(\mathcal{C}\Gamma(R)) \geq 2$, according to Lemma 3.4, while $\mathcal{C}\Gamma(\mathbb{F}_2)$ and $\mathcal{C}\Gamma(\mathbb{F}_3)$ are obviously planar. Since $\mathcal{C}\Gamma(\mathbb{F}_5) = K^5$ and $\mathcal{C}\Gamma(\mathbb{F}_7) = K^7$, these are the only clean graphs of genus 1 in this case.

Let us look at the case $k = 2$; so $|R| = p^2$. According to [6], the only possibilities for R are $\mathbb{F}_{p^2}$, $\mathbb{F}_p[X]/(X^2)$ and $\mathbb{Z}_{p^2}$. Let us prove that they are not toroidal. If $R = \mathbb{F}_{p^2}$ and $p > 2$, then $\mathcal{C}\Gamma(R)$, according to Lemma 3.4, is not of genus 1. On the other hand, if $R = \mathbb{F}_{2^2}$, then $\mathcal{C}\Gamma(R)$ is planar. If $R = \mathbb{F}_p[X]/(X^2)$, then $|\mathfrak{m}| = p$; so for $p \geq 5$, from Lemma 3.3, it follows that

$\mathcal{C}\Gamma(R)$ is not of genus 1. Since $\mathcal{C}\Gamma(\mathbb{F}_2[X]/(X^2))$ is planar we are left with the case $R = \mathbb{F}_3[X]/(X^2)$. It is simple to check that here $\delta(\mathcal{C}\Gamma(R)) = 6$, as well as that $\deg(X) = 7$. According to Theorem 3.3, $\mathcal{C}\Gamma(R)$ is not of genus 1. The case $R = \mathbb{Z}_{p^2}$ is similar to the previous one; namely, here we also have $|\mathfrak{m}| = p$ and $\mathcal{C}\Gamma(\mathbb{Z}_{2^2})$ is planar. So only the case $R = \mathbb{Z}_{3^2}$ remains. In this case, $\delta(\mathcal{C}\Gamma(R)) = 6$, and $\deg(3) = 7$. So among the local rings of cardinality p^2, none is such that its clean graph is of genus 1.

We now deal with the case $k = 3$, $|R| = p^3$. Since $\mathcal{C}\Gamma(\mathbb{F}_{2^3}) = K^8$, it is of genus 2. Clean graphs of other finite fields $\mathbb{F}_{p^3}$, for $p \geq 3$, are obviously not of genus 1 according to Lemma 3.4. Furthermore, if R is not a field, then $\mathfrak{m}$ has at least p elements. If $p \geq 3$ then, since it is clear that there are enough invertible elements, $\gamma(\mathcal{C}\Gamma(R)) \geq 2$ according to Lemma 3.3. Therefore, we are left with the case $|R| = 2^3$, where R is not a field. According to [6, p. 687], the following possibilities may occur: $\mathbb{Z}_{2^3}$, $\mathbb{F}_2[X]/(X^3)$, $\mathbb{F}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X, X^2)$, or $\mathbb{Z}_4[X]/(2X, X^2-2)$. We directly check that the clean graphs of the first three rings are isomorphic to $K^{4,4}$, therefore they are toroidal, as well as that the clean graphs of the remaining two rings are isomorphic and may be embedded into a torus (see Fig. 4a). Therefore they all are of genus 1. In the case $|R| = p^k$, $k \geq 4$, it is easy to see there are no toroidal graphs. $\square$

We now move on to the case $n = 2$, a product of two local rings.

Theorem 3.7 *Let $R = R_1 \times R_2$ be a product of two finite local commutative rings with identity such that at least one of them has more than 4 elements. In this case, $\gamma(\mathcal{C}\Gamma(R_1 \times R_2)) \geq 2$.*

Proof. Suppose that $|R_2| \geq 5$.

First of all, if R_2 is a field and 1, a, b, c its different units, then $\mathcal{C}\Gamma(R)$ contains $K^{4,5}$ as a subgraph, so $\gamma(\mathcal{C}\Gamma(R)) \geq 2$. This can be seen if we look at $V_1 = \{(1,1),(1,a),(1,b),(1,c)\}$ and $V_2 = \{(0,0),(0,1),(0,a),(0,b),(0,c)\}$.

If R_2 is not a field, we have $|R_2| = p^k$ and $|\mathfrak{m}| \geq p$. Suppose firstly that $|p| \geq 5$, and let 0, a, b, c, d be different elements from $\mathfrak{m}$. The choice of subsets $V_1 = \{0,a,b,c\}$ and $V_2 = \{1, 1-a, 1-b, 1-c, 1-d\}$ shows that $K^{4,5}$ is a subgraph of $\mathcal{C}\Gamma(R_2)$. According to Lemma 3.1, we have $K^{4,5} \subseteq \mathcal{C}\Gamma(R_1 \times R_2)$. Therefore $\gamma(\mathcal{C}\Gamma(R)) \geq 2$. So, we only have to deal with the cases $p = 2$ and $p = 3$.

Let $p = 3$, $|R_2| = 3^k$, and $|\mathfrak{m}| \geq 3$. It is enough to consider the minimal case when $\mathfrak{m} = \{0, a, -a\}$. Let $V_1 = \{(0,0),(0,a),(0,-a)\}$ and $V_2 = \{(1,1+a),(1,1-a),(1,-1-a),(1,a-1),(1,1),(1,-1),(1,0)\}$. Every vertex in V_2 is adjacent to the vertices in V_1 except for the vertex $(1,0)$ which is not adjacent to $(0,a)$ and $(0,-a)$. The vertex $(1,0)$ is connected to $(0,a)$ via the vertex $(0,1-a)$, and to the vertex $(0,-a)$ via $(0,1+a)$. In this way, we get that a subdivision of $K^{3,7}$ is contained in $\mathcal{C}\Gamma(R)$. So in this case, the graph cannot be of genus 1.

We now have the case $|R_2| = 2^k$. Obviously, $\gamma(\mathcal{C}\Gamma(R_1 \times R_2)) \geq 2$ for $k \geq 4$. For $k = 3$, possibilities for R_2 are, according to Theorem 3.6, $\mathbb{Z}_{2^3}$, $\mathbb{F}_2[X]/(X^3)$, $\mathbb{F}_2[X,Y]/(X,Y)^2$, $\mathbb{Z}_4[X]/(2X,X^2)$, or $\mathbb{Z}_4[X]/(2X,X^2-2)$. In all of these cases, $|\mathfrak{m}| = 4$. Let $\mathfrak{m} = \{0, a, b, a+b\}$.

If $\mathrm{char}(R_1) \neq 2$, the choice of subsets $V_1 = \{(0,a),(0,b),(0,a+b)\}$ and $V_2 = \{(-1,1-a),(-1,1-b),(1,1-a),(1,1-b),(1,1),(-1,1),(1,1-a-b)\}$ gives us that $K^{3,7} \subseteq \mathcal{C}\Gamma(R_1 \times R_2)$. Let $\mathrm{char}(R_1) = 2$ and $V_1 = \{(1,1-a),(1,1-b),(1,1-a-b),(1,1)\}$, $V_2 = \{(0,0),(0,a),(0,b),(0,a+b),(0,1)\}$ (Fig. 3). All vertices of the set V_2 are adjacent to the vertices from V_1 except for the vertex $(0,1)$. However, the vertex $(0,1)$ is connected to the other vertices from V_1 in the following way: to the vertex $(1,1-a)$ via $(1,a)$; to $(1,1-b)$ via $(1,b)$; to $(1,1-a-b)$ via $(1,a+b)$; to $(1,1)$ via $(1,0)$. In this way, we get a subdivision of $K^{4,5}$ inside $\mathcal{C}\Gamma(R)$; so we also have $\gamma(\mathcal{C}\Gamma(R_1 \times R_2)) \geq 2$. $\square$

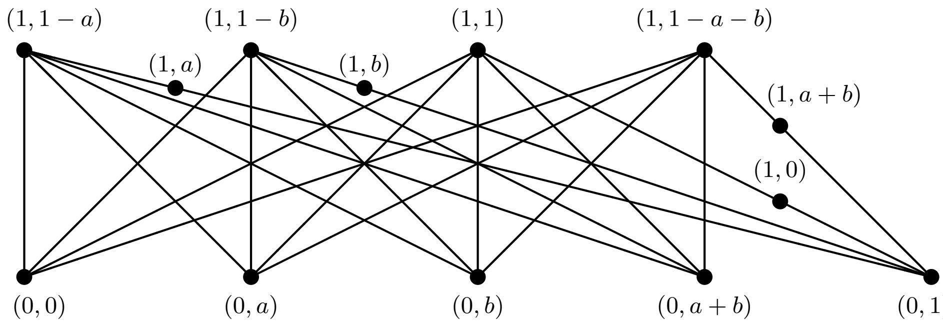


Figure 3.

Theorem 3.8 *Let $R = R_1 \times R_2$, where R_1 and R_2 are finite local commutative rings with identity. Then $\gamma(\mathcal{C}\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{F}_2 \times \mathbb{F}_3$, $\mathbb{F}_2 \times \mathbb{Z}_4$, $\mathbb{F}_2 \times \mathbb{F}_4$, or $\mathbb{F}_2 \times \mathbb{F}_2[X]/(X^2)$.*

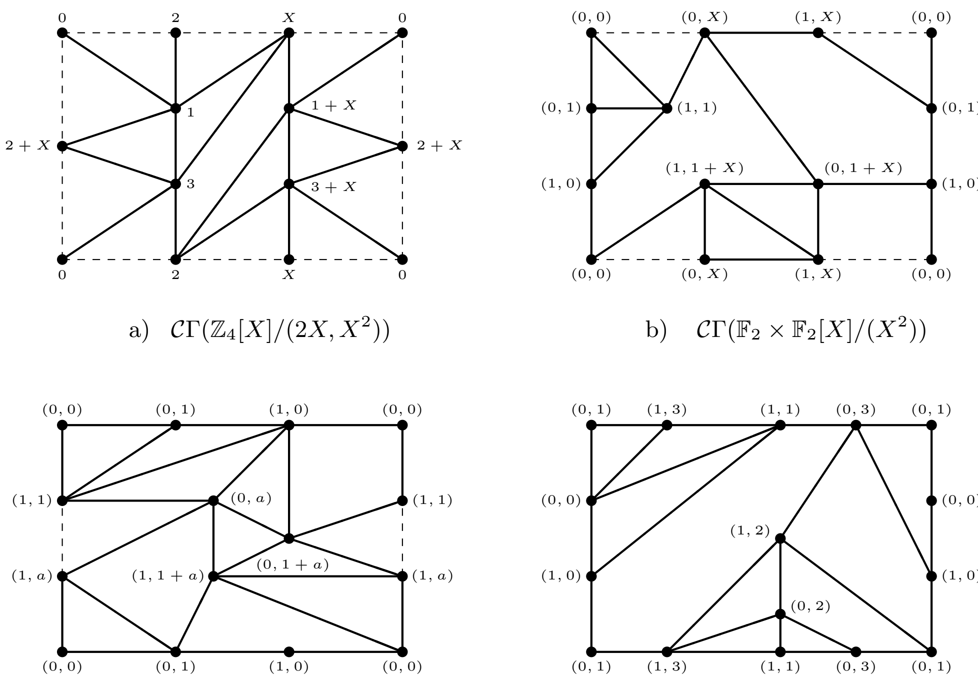
Proof. According to Theorem 3.7 it is enough to check the graphs of the following rings: $\mathbb{F}_2\times\mathbb{F}_2$, $\mathbb{F}_2\times\mathbb{F}_3$, $\mathbb{F}_2\times\mathbb{Z}_4$, $\mathbb{F}_2\times\mathbb{F}_4$, $\mathbb{F}_2\times\mathbb{F}_2[X]/(X^2)$, $\mathbb{F}_3\times\mathbb{F}_3$, $\mathbb{F}_3\times\mathbb{F}_4$, $\mathbb{F}_3\times\mathbb{Z}_4$, $\mathbb{F}_3\times\mathbb{F}_2[X]/(X^2)$, $\mathbb{F}_4\times\mathbb{Z}_4$, $\mathbb{F}_4\times\mathbb{F}_4$, $\mathbb{F}_4\times\mathbb{F}_2[X]/(X^2)$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_4\times\mathbb{F}_2[X]/(X^2)$, or $\mathbb{F}_2[X]/(X^2)\times\mathbb{F}_2[X]/(X^2)$.

If R is one of the rings $\mathbb{F}_3\times\mathbb{F}_4$, $\mathbb{F}_4\times\mathbb{Z}_4$, $\mathbb{F}_4\times\mathbb{F}_4$ or $\mathbb{F}_4\times\mathbb{F}_2[X]/(X^2)$, then $\gamma(\mathcal{C}\Gamma(R)) \geq 2$ according to Lemma 3.4, since one can directly check that for every such ring $|\mathrm{UI}(R)| = 9$.

IF R is one of the rings $\mathbb{F}_3\times\mathbb{F}_3$, $\mathbb{F}_3\times\mathbb{Z}_4$, $\mathbb{F}_3\times\mathbb{F}_2[X]/(X^2)$, $\mathbb{Z}_4\times\mathbb{Z}_4$, or $\mathbb{Z}_4\times\mathbb{F}_2[X]/(X^2)$, then $\gamma(\mathcal{C}\Gamma(R)) \geq 2$ according to Theorem 3.3, because in all those rings we have $\delta(\mathcal{C}\Gamma(R)) = 6$ and there is always a vertex of degree 7. In $\mathbb{F}_3\times\mathbb{F}_3$ and $\mathbb{F}_3\times\mathbb{Z}_4$ for example, $\deg(0,1) = 7$; in $\mathbb{F}_3\times\mathbb{F}_2[X]/(X^2)$, one has $\deg(1,x) = 7$; in $\mathbb{Z}_4\times\mathbb{Z}_4$, one has $\deg(0,3) = 7$; while in $\mathbb{Z}_4\times\mathbb{F}_2[X]/(X^2)$, $\deg(1,0) = 7$.

For the ring $R = \mathbb{F}_2[X]/(X^2)\times\mathbb{F}_2[X]/(X^2)$, one also has $\delta(C(\Gamma(R))) = 6$, but in this case every vertex has degree 6. According to Theorem 3.3, the graph is toroidal if and only if $\mathcal{C}\Gamma(R)$ is a 1-skeleton of the 6-regular triangulation of a torus. In this case, we do not have the requested triangulation. For example, the vertices $(0,x)$ and $(1,1)$ are adjacent, but the edge which connects them is not an edge of any triangle since there are no vertices adjacent to $(0, x)$ and $(1, 1)$.

Because of the cardinality of the set of vertices, the clean graph of the ring $\mathbb{F}_2 \times \mathbb{F}_2$ is planar, while the one for $\mathbb{F}_2 \times \mathbb{F}_3$ is toroidal. Finally, the clean graphs of rings $\mathbb{F}_2 \times \mathbb{F}_2[X]/(X^2)$, $\mathbb{F}_2 \times \mathbb{F}_4$, and $\mathbb{F}_2 \times \mathbb{Z}_4$ are embeddable into a torus (figures 4b, 4c, 4d). $\square$



a) $\mathcal{C}\Gamma(\mathbb{Z}_4[X]/(2X, X^2))$ b) $\mathcal{C}\Gamma(\mathbb{F}_2 \times \mathbb{F}_2[X]/(X^2))$

c) $\mathcal{C}\Gamma(\mathbb{F}_2 \times \mathbb{F}_4)$ d) $\mathcal{C}\Gamma(\mathbb{F}_2 \times \mathbb{Z}_4)$

Figure 4.

Acknowledgements

The authors would like to thank Ivana Božić for her input in previous versions of this paper.

The first author was partially supported by Ministry of Education, Science and Technological Development of Republic of Serbia Project #174032.

References

[1] D. D. Anderson and V. P. Camilo, Commutative rings whose elements are a sum of a unit and idempotent, *Comm. Algebra* **30** (2002) 3327–3336.

[2] D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra* **320** (2008) 2706–2719.

[3] D. F. Anderson and J. D. LaGrange, Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph, *J. Pure Appl. Algebra* **216** (2012) 1626–1636.

[4] N. Bloomfield, The Zero Divisor Graphs of Commutative Local Rings of Order p^4 and p^5, *Comm. Algebra* **42** (2013) 765–775.

[5] V. P. Camillo and H. P. Yu, Exchange rings, units and idempotents, *Comm. Algebra* **22** (1994) 4737–4749.

[6] B. Corbas and G. D. Williams, Rings of order p^5. I. Nonlocal rings *J. Algebra* **231** (2000) 677–690.

[7] B. Corbas and G.D. Williams, Rings of order p^5. II. Local rings, *J. Algebra* **231** (2000) 691–704.

[8] N. Ganesan, Properties of rings with a finite number of zero divisors, *Math. Ann.* **157** (1964) 215–218.

[9] J. Han and W.K. Nicholson, Extensions of clean rings, *Comm. Algebra* **29** (2001) 2589–2595.

[10] M. Henriksen, Two classes of rings generated by their units, *J. Algebra* **31** (1974) 182–193.

[11] I. Kaplansky, *Commutative rings*, Revised Edition, (University of Chicago Press, Chicago, 1974).

[12] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund Math.* **15** (1930) 271–283.

[13] T. Y. Lam, *A First Course in Noncommutative rings*, (Springer, New York, 2001).

[14] W. K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.* **229** (1977) 269–278.

[15] W. K. Nicholson and Y. Zhou, Clean general rings, *J. Algebra* **291** (2005) 297–311.

[16] Z. S. Pucanović and Z. Z. Petrović, Toroidality of intersection graphs of ideals of commutative rings, *Graphs Combin.* Online First (2013) DOI 10.1007/s00373-013-1292-1

[17] R. Raphael, Rings which are generated by their units, *J. Algebra* **28** (1974) 199–205.

[18] P. Vámos, 2-Good rings, *Quart. J. Math (Oxford)* **56** (2005) 417–430.

[19] D. B. West, *Introduction to graph theory*, (Prentice Hall, Inc., Upper Saddle River, NJ, 1996).

[20] T. White, *Graphs, Groups and Surfaces*, North-Holland Math. Studies **188** (Amsterdam, 1984).

[21] C. Wickham, Classification of rings with genus one zero-divisor graphs, *Comm. Algebra* **36** (2008) 1–21.

[22] C. Wickham, Rings whose zero-divisor graphs have positive genus, *J. Algebra* **321**(2) (2009) 377–383.

[23] G. S. Xiao and W. T. Tong, n-Clean rings and weakly unit stable rings, *Comm. Algebra* **33** (2005) 1501–1517.

[24] Y. Q. Ye, Semiclean rings, *Comm. Algebra* **31** (2003) 5609–5625.