PRIMENA METODE DINAMIČKE KRUTOSTI U NUMERIČKOJ ANALIZI SLOBODNIH VIBRACIJA PLOČA SA UKRUĆENJIMA

APPLICATION OF DYNAMIC STIFFNESS METHOD IN NUMERICAL FREE VIBRATION ANALYSIS OF STIFFENED PLATES

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1 UVOD

Primena građevinskih materijala visokih mehaničkih karakteristika, pre svega čelika, dovodi do smanjenja dimenzija (debljine) konstruktivnih pločastih elemenata. Time se postiže slična nosivost za manji utrošak materijala, kao i ušteda u ceni. Međutim, vitki elementi postaju osetljivi na izbočavanje, što uzrokuje upotrebu ploča sa ukrućenjima.

Ploče sa ukrućenjima imaju široku primenu u građevinarstvu i drugim inženjerskim disciplinama, posebno pri projektovanju, na primer, mostova većih raspona i manjih poprečnih preseka, te izradi oplata brodova, konstrukcija aviona. Tokom svog radnog veka, ove konstrukcije često su izložene dinamičkom opterećenju, te je

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1 INTRODUCTION

Application of building materials of high mechanical properties (i.e. steel), leads to the reduction of dimensions (thickness) of structural plate-like elements. This maintains the same structural capacity with reduced material cost, as well the price reduction. However, slender elements become vulnerable to buckling, leading to the use of stiffened plates.

Stiffened plates are widely used in civil engineering and other engineering fields, especially in design of long-span bridges with small cross-sections, construction of ship hulls, aircrafts, etc. During their working life, these structures are usually exposed to dynamic loading, thus the calculation of dynamic structural response is of high

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proračun njihovog dinamičkog odgovora veoma značajan u inženjerskoj praksi. U takvim slučajevima, neophodno je odrediti osnovne dinamičke karakteristike sistema, kao što su sopstvene frekvencije i oblici oscilovanja.

Dinamički odgovor tankih ploča može se odrediti primenom Kirchhoff-ove klasične teorije ploča (*classical plate theory* - CPT). U slučaju debelih ploča, ova teorija ne daje adekvatne rezultate zbog zanemarivanja deformacije smicanja, pa je potrebno primeniti Mindlinovu teoriju (*first-order shear deformation theory* - FSDT), koja uzima u obzir uticaj deformacije smicanja, pretpostavljajući da je klizanje konstantno po debljini ploče. Paralelno sa razvojem različitih teorija ploča, razvijale su se i odgovarajuće analitičke metode [1]. Međutim, one se zasnivaju na tačnom rešenju diferencijalnih jednačina kretanja ploča, sa specijalnim uslovima oslanjanja. Leissa [2] je dao sveobuhvatan pregled analitičkih rešenja slobodnih vibracija ploča različitih oblika, zasnovan na Kirchhoff-ovoj klasičnoj teoriji ploča. Liew i ostali [3] analizirali su slobodne vibracije debelih ploča sa proizvoljnim konturnim uslovima, primenom Raylegh-Ritz-ove metode. Primena pomenutih metoda ograničena je na analizu slobodnih vibracija pojedinačnih ploča i ne može se lako proširiti na analizu složenijih sistema ploča sa različitim geometrijskim i materijalnim karakteristikama, kakvi najčešće jesu u inženjerskoj praksi (npr. ploče sa ukrućenjima, sendvič-paneli, ploče promenljive debljine). U takvim slučajevima, u analizi se primenjuju numeričke metode, kao što je metoda konačnih elemenata (MKE) [4]. Poznata je činjenica da u dinamičkoj analizi primenom MKE veličina konačnog elementa zavisi od dužine talasa najviše frekvencije od interesa za analizu. Prema [5, 6], odnos između talasne dužine najviše frekvencije i veličine konačnog elementa trebalo bi da se kreće u granicama između 10 i 20, kako bi se dobili rezultati zadovoljavajuće tačnosti. Minimalan broj konačnih elemenata direktno je proporcionalan najvišoj razmatranoj frekvenciji, pa u slučaju složenih konstrukcija (gde je pri analizi potrebno imati u vidu i više tonove oscilovanja) potreban broj konačnih elemenata postaje veliki, čime se povećava ukupno trajanje proračuna.

U poslednje vreme, za analizu slobodnih vibracija ploča sve češće se koristi metoda dinamičke krutosti (MDK) [7-15]. MDK kombinuje karakteristike MKE - kao što su fizička diskretizacija i mogućnost povezivanja elemenata u jedinstveni globalni sistem - sa rešenjem polja pomeranja, koje predstavlja tačno rešenje diferencijalne jednačine slobodnih vibracija. Kako interpolacione funkcije kojima se opisuje polje pomeranja u MDK predstavljaju tačno rešenje diferencijalne jednačine kretanja u frekventnom domenu, greške usled aproksimacije su eliminisane. Podela ploče na manje dinamičke elemente neophodna je samo ukoliko unutar ploče postoji neki geometrijski i/ili fizički diskontinuitet. Time se smanjuju broj elemenata u analizi, broj stepeni slobode, kao i vreme potrebno za rad i mogućnost javljanja greške, u poređenju s MKE. Za razliku of MKE - gde je masa koncentrisana u čvorovima konačnih elemenata, u MDK masa je kontinualno raspodeljena. Takođe, materijalno prigušenje može se na jednostavan način uključiti u analizu putem kompleksnog modula elastičnosti $E_c = E\,(1+2iz)$, gde je z koeficijent relativnog prigušenja.

importance in engineering practice. In these cases, it is necessary to derive fundamental dynamic properties of the system, such as natural frequencies and mode shapes.

Dynamic response of thin plates can be obtained using Kirchhoff classical plate theory (CPT). In the case of thick plates, this theory fails to provide adequate results because the transverse shear deformation is neglected. Therefore, it is necessary to use Mindlin plate theory (first-order shear deformation theory - FSDT), which accounts for the constant transverse shear deformation through the plate thickness. Along with the development of different plate theories, the adequate analytical methods have been derived, too [1]. However, these methods are based on the exact solution of differential equations of motion of plates having special boundary conditions. Leissa [2] provided the comprehensive review of analytical solutions of the free vibration problem for plates of different shape, based on the Kirchhoff classical plate theory. Liew et al. [3] studied the free vibration of thick plates with arbitrary boundary conditions using the Raylegh-Ritz method. The application of the above mentioned methods is restricted to the free vibration analysis of single plates and cannot be easily extended to the analysis of more complex plate assemblies with different geometry and material properties, which are mostly used in engineering practice (stiffened plates, sandwich panels, stepped plates, etc.). In such cases, numerical methods are applied, i.e. the finite element method (FEM) [4]. It is a well-known fact that in the dynamic finite element analysis, the size of the finite element depends on the wavelength of the highest frequency of interest for the analysis. According to [5, 6], the ratio between the wavelength of the highest frequency and the finite element size should be between 10 and 20, in order to obtain the results of satisfactory accuracy. Minimum number of finite elements is directly proportional to the highest considered frequency, thus for the complex structures (where higher modes of vibration must be taken into account), the number of finite elements becomes very high, increasing the overall computational time.

Lately, the dynamic stiffness method (DSM) [7-15] is increasingly used in the free vibration analysis of plates. DSM combines the properties of the FEM (such as physical discretization and the possibility of assembly of single elements into the unique global system) with the solution of the displacement field in the form of exact solution of differential equation of the free vibration problem. Having in mind that interpolation functions which describe the displacement field in the DSM represent the exact solution of the differential equation of motion in the frequency domain, the approximation errors are eliminated. The plate discretization is necessary only if some geometrical and/or physical discontinuity is present. This reduces both the numbers of elements in the analysis and degrees of freedom, as well as computational time and error possibility, in comparison with the FEM. In contrary to the FEM where the mass is lumped in nodes of finite elements, in the DSM the mass is continuously distributed. In addition, material damping can be easily included in the analysis using the complex elasticity modulus $E_c = E(1+2iz)$, where z is the relative damping coefficient. Using this

Na taj način, omogućeno je da različiti elementi konstrukcije imaju različito prigušenje, što je još jedna od prednosti MDK u poređenju s MKE.

U okviru ovog rada, prikazan je numerički model za analizu slobodnih neprigušenih vibracija Mindlin-ovih ploča sa ukrućenjima sa proizvoljnim graničnim uslovima, primenom MDK. Na osnovu dinamičkih matrica krutosti za analizu slobodnih poprečnih vibracija i vibracija u ravni, izvedena je matrica rotacije za različite položaje ploča koje su pod pravim uglom u odnosu na referentnu ravan [16]. Primenjen je sličan postupak kao u MKE za formiranje globalne dinamičke matrice krutosti ploče sa ukrućenjima i razvijen je računarski program u MATLAB-u [17] za analizu slobodnih vibracija sistema ploča. Prikazani postupak verifikovan je upoređivanjem tih rezultata sa rezultatima dobijenim primenom programskog paketa Abaqus [18].

2 POSTUPAK FORMIRANJA DINAMIČKE MATRICE KRUTOSTI PRAVOUGAONE PLOČE

Postupak formiranja dinamičke matrice krutosti pravougaonog elementa ploče za poprečne i vibracije u ravni detaljno je prikazan u radovima [13, 15], dok će ovde biti prikazani osnovni koraci u postupku formiranja dinamičke matrice krutosti. Polaznu tačku predstavljaju jednačine kretanja elementa Mindlin-ove ploče u vremenskom domenu:

approach, it is possible that different structural members have different damping values, which is one of the advantages of the DSM in comparison with the FEM.

In the paper, the numerical model for free undamped vibration analysis of stiffened Mindlin plates with arbitrary boundary conditions, based on the DSM, has been presented. Starting from the dynamic stiffness matrices for the free vibration analysis of transverse and in-plane vibrations, rotation matrix for different plate positions (perpendicular to the reference plane), has been derived [16]. Similar procedure as in the FEM has been applied for the development of the global dynamic stiffness matrix of stiffened plate assembly. Computer program in MATLAB [17] has been developed for the free vibration analysis of plate assemblies. The validation of the presented procedure has been performed by comparison of the obtained results with those obtained by using the software package Abaqus [18].

2 PROCEDURE FOR DEVELOPMENT OF DYNAMIC STIFFNESS MATRIX OF RECTANGULAR PLATE

Development of the dynamic stiffness matrix for rectangular plate element undergoing transverse and in-plane vibrations has been given in detail in [13, 15]. In the paper, only the basic steps in the procedure will be presented. The procedure starts from with equations of motion of Mindlin plate element in the time domain:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2}+a_1\frac{\partial^2 u}{\partial y^2}+a_2\frac{\partial^2 v}{\partial x\partial y}=\frac{r h}{D_1}\frac{d^2 u}{dt^2}\\ &\frac{\partial^2 v}{\partial y^2}+a_1\frac{\partial^2 v}{\partial x^2}+a_2\frac{\partial^2 u}{\partial x\partial y}=\frac{r h}{D_1}\frac{d^2 v}{dt^2}\end{aligned} \tag{1a}$$

$$\begin{aligned} &kGh\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial f_y}{\partial x}-\frac{\partial f_x}{\partial y}\right)=r h\frac{d^2 w}{dt^2}\\ &D\left(\frac{\partial^2 f_x}{\partial y^2}-\frac{1+n}{2}\frac{\partial^2 f_y}{\partial x\partial y}+\frac{1-n}{2}\frac{\partial^2 f_x}{\partial x^2}\right)+kGh\left(\frac{\partial w}{\partial y}-f_x\right)=\frac{r h^3}{12}\frac{d^2 f_x}{dt^2}\\ &D\left(\frac{\partial^2 f_y}{\partial x^2}-\frac{1+n}{2}\frac{\partial^2 f_x}{\partial x\partial y}+\frac{1-n}{2}\frac{\partial^2 f_y}{\partial y^2}\right)-kGh\left(\frac{\partial w}{\partial x}+f_y\right)=\frac{r h^3}{12}\frac{d^2 f_y}{dt^2}\end{aligned} \tag{1b}$$

gde u, v, w, f_x, f_y predstavljaju komponentalna pomeranja i rotacije, h je debljina ploče, ρ je gustina, E je modul elastičnosti, G je modul smicanja, n je Poisson-ov koeficijent, $D=Eh^3/12/(1-n^2)$ je krutost ploče na savijanje, $D_1=Eh/(1-n^2)$ je krutost ploče u ravni, $a_1=(1-n)/2$, $a_2=(1+n)/2$ i $k=5/6$ je faktor korekcije smicanja.

Pretpostavlja se da su pomeranja harmonijske funkcije frekvencije ω, odnosno:

where u, v, w, f_x, f_y are the displacement components, h is the plate thickness, ρ is the mass density, E is the modulus of elasticity, G is the shear modulus, n is the Poisson's coefficient, $D=Eh^3/12/(1-n^2)$ is the flexural plate stiffness, $D_1=Eh/(1-n^2)$ is the in-plane plate stiffness, $a_1=(1-n)/2$, $a_2=(1+n)/2$, while $k=5/6$ is the shear correction factor.

It is assumed that the displacements are the harmonic functions of frequency ω, i.e.:

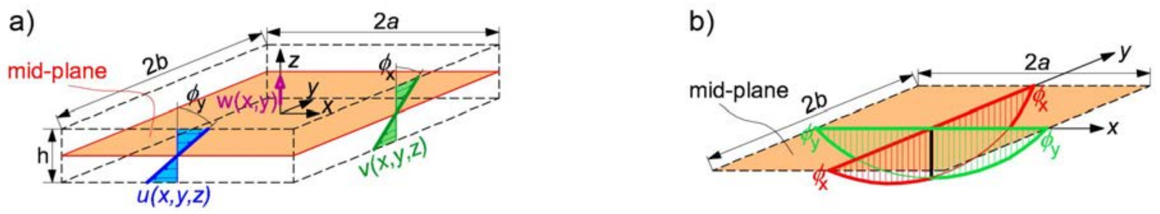
$$\mathbf{u}(x,y,t)=\hat{\mathbf{u}}(x,y,w)e^{iwt} \tag{2}$$

gde $\hat{\mathbf{u}}(x,y,w)$ predstavlja amplitudu polja pomeranja (u, v, w, f_x ili f_y) u frekventnom domenu. Na osnovu ove pretpostavke, jednačine kretanja (1) iz vremenskog domena transformišu se u frekventni domen.

Na slici 1a prikazano je polje pomeranja pravougaone ploče po Mindlin-ovoj teoriji.

where $\hat{\mathbf{u}}(x,y,w)$ is the amplitude of the displacement field (u, v, w, f_x or f_y) in the frequency domain. Using the above assumption, equations of motion (1) are transformed from time to frequency domain.

Figure 1 shows the displacement field of the rectangular plate based on Mindlin plate theory



Slika 1. (a) Polje pomeranja u Mindlin-ovoj teoriji i (b) simetrična deformacija ploče (SS)
Figure 1. (a) Displacement field in Mindlin plate theory and (b) double-symmetric plate deformation (SS)

Proizvoljno polje pomeranja može se prikazati kao superpozicija rešenja za četiri slučaja simetrije u odnosu na x i y koordinatne ose: simetrija-simetrija (SS), simetrija-antimetrija (SA), antimetrija-simetrija (AS) i antimetrija-antimetrija (AA):

Arbitrary displacement field can be represented as a superposition of four symmetry contributions with respect to x and y coordinate axes: symmetric-symmetric (SS), symmetric-anti-symmetric (SA), anti-symmetric-symmetric (AS) and anti-symmetric-anti-symmetric (AA):

$$\hat{\mathbf{u}}(x,y,w) = \hat{\mathbf{u}}^{SS}(x,y,w) + \hat{\mathbf{u}}^{SA}(x,y,w) + \hat{\mathbf{u}}^{AS}(x,y,w) + \hat{\mathbf{u}}^{AA}(x,y,w) \quad (3)$$

Na slici 1b prikazana je simetrična deformacija ploče u odnosu na obe koordinatne ose (SS). Podelom polja pomeranja na četiri slučaja simetrije, moguće je analizirati samo jednu četvrtinu ploče, čime se znatno umanjuje red dinamičkih matrica krutosti i ubrzava proračun. Rešenje jednačina kretanja u frekventnom domenu pretpostavlja se u obliku beskonačnog Fourier-ovog reda u sledećem obliku:

Figure 1b illustrates the double-symmetric plate deformation with respect to both coordinate axes (SS). By dividing the displacement field into four symmetry contributions, it is possible to analyze only a quarter of a plate, which significantly reduces the order of the dynamic stiffness matrices and accelerates the calculation. The solution of the equations of motion in the frequency domain is assumed in the form of an infinite Fourier's series as:

$$\hat{\mathbf{u}}^{ij}(x,y,w) = \sum_{m=0,1}^{\infty} C_m^{ij} f_m^{ij}(x) g_m^{ij}(y) \quad (4)$$

gde su: $f_m^{ij}(x), g_m^{ij}(y)$ bazne trigonometrijske funkcije koje zavise od slučaja simetrije (i,j) i rešenja odgovarajućih jednačina kretanja, C_m^{ij} su integracione konstante, a $i, j = S, A$.

Na osnovu poznatih kinematičkih i konstitutivnih relacija ploče, kao i jednačine (4), vektor sila u preseku u proizvoljnoj tački ploče može se napisati u obliku:

where $f_m^{ij}(x), g_m^{ij}(y)$ are the basis trigonometric functions which depend both on symmetry case (i,j) and the solution of corresponding equations of motion, C_m^{ij} are the integration constants, while $i, j = S, A$.

Based on the well-known kinematic and constitutive relations of the plate, as well as the equation (4), the force vector in an arbitrary point of the plate can be written as:

$$\hat{\mathbf{f}}^{ij}(x,y,w) = \sum_{m=0,1}^{\infty} C_m^{ij} f_m^{f,ij}(x) g_m^{f,ij}(y) \quad (5)$$

gde su: $f_m^{f,ij}(x), g_m^{f,ij}(y)$ izvodi baznih funkcija u zavisnosti od usvojene teorije ploče.

U praktičnoj primeni, beskonačni red u jednačinama (4) i (5) potrebno je prekinuti u tački M (usvojeni broj članova reda), tako da tačnost rešenja praktično zavisi samo od M.

Sledeći korak predstavlja formiranje vektora pomeranja $\hat{\mathbf{q}}^{ij}$, na konturama $\boldsymbol{x = a}$ *i* $\boldsymbol{y = b}$ četvrtine ploče za svaki od četiri slučaja simetrije, koji se dobijaju zamenom koordinata kontura u jednačinu (4):

where $f_m^{f,ij}(x), g_m^{f,ij}(y)$ are derivations of the base functions depending on assumed plate theory.

For practical purposes, the infinite series in equations (4) and (5) is truncated to point M (the number of terms in the series expansion), thus the accuracy of the solution practically depends only on M.

The next step is the definition of the displacement vectors $\hat{\mathbf{q}}^{ij}$ along boundaries $\boldsymbol{x = a}$ *and* $\boldsymbol{y = b}$ of the quarter of the plate, for each symmetry contribution, which are derived by substituting the boundary coordinates in the equation (4):

$$\hat{\mathbf{q}}^{ij} = \begin{bmatrix} \hat{\mathbf{u}}^{ij}(a,y,w) \\ \hat{\mathbf{u}}^{ij}(x,b,w) \end{bmatrix} \quad (6)$$

Na sličan način, vektor sila $\hat{\mathbf{Q}}^{ij}$ na konturama ploče dobija se zamenom koordinata kontura u jednačinu (5):

In a similar manner, the force vector $\hat{\mathbf{Q}}^{ij}$ on the plate boundaries is obtained by replacing the coordinates of the boundaries in equation (5):

$$\hat{\mathbf{Q}}^{ij} = \begin{bmatrix} \hat{\mathbf{f}}^{ij}(a,y,w) \\ \hat{\mathbf{f}}^{ij}(x,b,w) \end{bmatrix} \tag{7}$$

S obzirom da su komponente vektora pomeranja i sila na konturama ploče funkcije prostornih koordinata x i y, nije moguće direktno uspostaviti vezu između tih vektora - sa jedne strane i vektora integracionih konstanti **C** - sa druge strane. Ovaj problem može se rešiti pomoću metode projekcije koja se bazira na predstavljanju funkcija pomeranja i sila na konturi ploče u vidu Fourier-ovog reda:

Since the components of the displacement and force vectors along plate boundaries are functions of spatial coordinates x and y, it is impossible to establish a direct relation between these vectors on one side, and the vector of integration constants **C** on the other side. This problem can be solved by using the projection method, which is based on the representation of the displacement and force functions along the plate boundary in the Fourier series form:

$$\begin{aligned} \tilde{\mathbf{q}}^{ij} &= \frac{2}{L}\int_s \mathbf{H}^{ij}\hat{\mathbf{q}}^{ij}\,ds = \mathbf{D}^{ij}\mathbf{C}^{ij} \\ \tilde{\mathbf{Q}}^{ij} &= \frac{2}{L}\int_s \mathbf{H}^{ij}\hat{\mathbf{Q}}^{ij}\,ds = \mathbf{F}^{ij}\mathbf{C}^{ij} \end{aligned} \tag{8}$$

gde je $\mathbf{H}^{ij}$ matrica baznih funkcija za odgovarajući slučaj simetrije, $L=a$ za konturu paralelnu sa x-osom, dok je $L=b$ za konturu paralelnu sa y-osom. Eliminacijom vektora integracionih konstanti iz jednačina (8) dobija se dinamička matrica krutosti četvrtine ploče $\tilde{\mathbf{K}}_D^{ij}$ za svaki od četiri slučaja simetrije:

where $\mathbf{H}^{ij}$ is the matrix of the base functions for the symmetry contribution, $L=a$ for the boundary parallel with x-axis, while $L=b$ for the boundary parallel with y-axis. By eliminating the vector of integration constants from equations (8), the dynamic stiffness matrix of the quarter of the plate $\tilde{\mathbf{K}}_D^{ij}$ is obtained for all four symmetry contributions:

$$\tilde{\mathbf{Q}}^{ij} = \mathbf{F}^{ij}\left(\mathbf{D}^{ij}\right)^{-1}\tilde{\mathbf{q}}^{ij} = \tilde{\mathbf{K}}_D^{ij}\tilde{\mathbf{q}}^{ij} \tag{9}$$

Dinamička matrica krutosti čitave ploče $\tilde{\mathbf{K}}_D$ može se dobiti primenom transfer matrice, kao što je pokazano u radovima [13, 15].

Dynamic stiffness matrix of the entire plate $\tilde{\mathbf{K}}_D$ can be derived using the transfer matrix, as given in Refs. [13, 15].

3 PLOČE SA UKRUĆENJIMA

3 STIFFENED PLATES

Poprečne vibracije i vibracije u ravni za jednu izotropnu ploču predstavljaju dva nezavisna stanja. Stoga, dinamička matrica krutosti ploče može se napisati kao:

Transverse and in-plane vibrations of a single plate represent two independent states. Therefore, the dynamic stiffness matrix of the single plate can be written as:

$$\tilde{\mathbf{K}}_D = \begin{bmatrix} \tilde{\mathbf{K}}_{Dt} & 0 \\ 0 & \tilde{\mathbf{K}}_{Di} \end{bmatrix} \tag{10}$$

gde je $\tilde{\mathbf{K}}_{Dt}$ dinamička matrica krutosti ploče izložene poprečnim vibracijama, dok je $\tilde{\mathbf{K}}_{Di}$ dinamička matrica krutosti ploče za vibracije u ravni.

Saglasno jednačini (10), vektor projekcija pomeranja i sila na konturi ploče može se prikazati u sledećem obliku:

where $\tilde{\mathbf{K}}_{Dt}$ is the dynamic stiffness matrix of plate element for transverse vibrations, while $\tilde{\mathbf{K}}_{Di}$ is the dynamic stiffness matrix of plate element for in-plane vibrations.

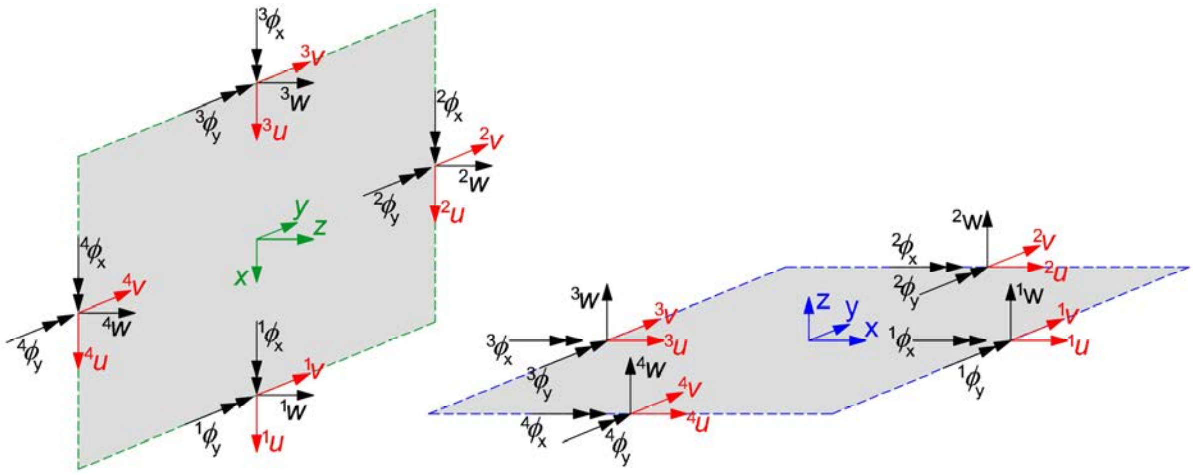
According to equation (10), displacement and force projection vectors on the plate boundary can be written in the following form:

$$\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{\mathbf{q}}_t \\ \tilde{\mathbf{q}}_i \end{bmatrix}, \quad \tilde{\mathbf{Q}} = \begin{bmatrix} \tilde{\mathbf{Q}}_t \\ \tilde{\mathbf{Q}}_i \end{bmatrix} \tag{11}$$

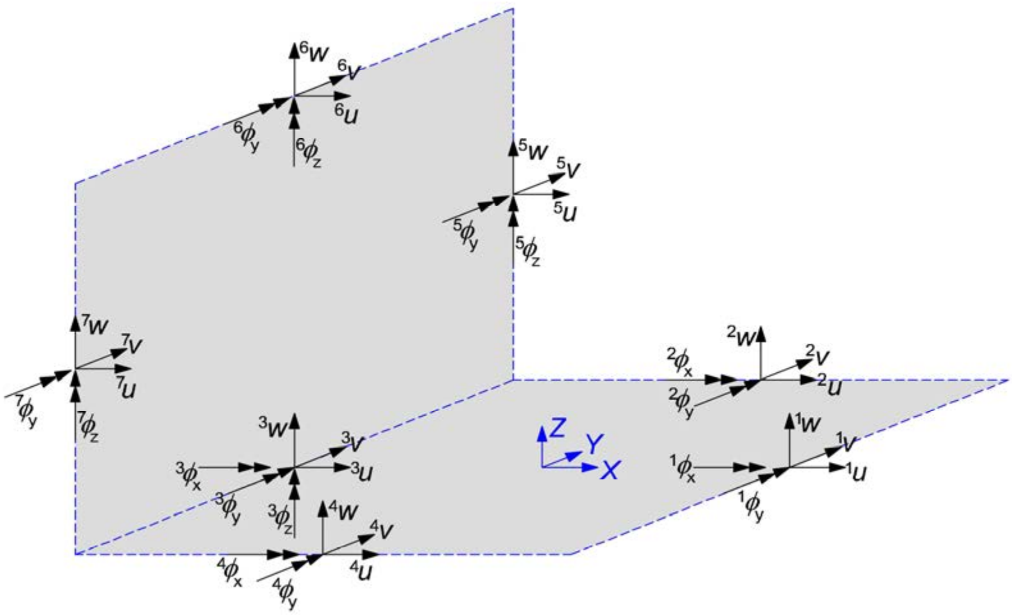
U slučaju ploča sa ukrućenjima, gde su ploče međusobno spojene pod pravim uglom, poprečne vibracije jedne ploče izazivaju vibracije u ravni druge ploče i obrnuto. Zbog toga je potrebno uspostaviti vezu između vektora projekcija pomeranja i sila $\tilde{\mathbf{q}}$ i $\tilde{\mathbf{Q}}$ u lokalnom i vektora $\tilde{\mathbf{q}}^*$ i $\tilde{\mathbf{Q}}^*$ u globalnom koordinatnom sistemu (slike 2 i 3), pomoću matrice rotacije **T**:

For stiffened plates (where the plates are perpendicular to each other), transverse vibrations of a single plate cause the in-plane vibrations of a corresponding perpendicular plate, and vice versa. Consequently, it is necessary to establish the relation between the projection vectors of displacements $\tilde{\mathbf{q}}$ and forces $\tilde{\mathbf{Q}}$ in the local, and the corresponding projection vectors $\tilde{\mathbf{q}}^*$ and $\tilde{\mathbf{Q}}^*$ in the global c. s. (Figures 2 and 3). This is accomplished by using the rotation matrix **T**:

$$\tilde{\mathbf{q}} = \mathbf{T}\tilde{\mathbf{q}}^*, \quad \tilde{\mathbf{Q}} = \mathbf{T}\tilde{\mathbf{Q}}^* \tag{12}$$



Slika 2. Komponente pomeranja na konturama ploča u lokalnom koordinatnom sistemu
Figure 2. Displacement components on the plate boundaries in the local coordinate system



Slika 3. Komponente pomeranja na konturama ploča u globalnom koordinatnom sistemu
Figure 3. Displacement components on the plate boundaries in the global coordinate system

U skladu sa definisanim relacijama projekcija u lokalnom i globalnom koordinatnom sistemu, dinamička matrica krutosti ploče u globalnom koordinatnom sistemu dobija se kao:

According to the established relations between the projection vectors in the local and global coordinate systems, dynamic stiffness matrix of the plate in global coordinate system is derived as:

$$\tilde{\mathbf{K}}_D^* = \mathbf{T}^T \tilde{\mathbf{K}}_D \mathbf{T} \tag{13}$$

Dinamičke matrice krutosti pojedinačnih ploča sabiraju se u globalnu dinamičku matricu krutosti sistema ploča, slično kao u MKE, s tom razlikom što su ploče povezane duž kontura, umesto u čvorovima. U analizi je moguće primeniti proizvoljne granične uslove.

Dinamička matrica krutosti je kvadratna, frekventno zavisna matrica čiji red zavisi od broja članova reda *M* usvojenog rešenja. Sopstvene frekvencije određuju se iz sledeće jednačine:

Dynamic stiffness matrices of individual plates are assembled in the global dynamic stiffness matrix of the plate assembly, using similar assembly procedure as in the conventional FEM. Note that the plates are connected along the boundaries instead at nodes. In the analysis, arbitrary boundary conditions can be applied.

The dynamic stiffness matrix is the square, frequency-dependent matrix whose size depends on the number of terms *M* in the general solution. The natural frequencies can be determined from the following equation:

$$\det\left|\tilde{\mathbf{K}}_{DG,nn}^*(w)\right| = 0 \tag{14}$$

gde je $\tilde{\mathbf{K}}_{DG,nn}^*$ globalna dinamička matrica krutosti sistema uz nepoznata pomeranja.

Kako elementi dinamičke matrice krutosti $\tilde{\mathbf{K}}_{DG,nn}^*$ sadrže transcedentne funkcije, rešenja se mogu dobiti primenom neke od tehnika pretraživanja. Kako bi se izbegle numeričke poteškoće prilikom određivanja nule jednačine (14), sopstvene frekvencije mogu se odrediti kao maksimumi izraza:

where $\tilde{\mathbf{K}}_{DG,nn}^*$ is the global dynamic stiffness submatrix of the plate assembly related to the unknown generalized displacement projections.

Since the elements of the dynamic stiffness matrix $\tilde{\mathbf{K}}_{DG,nn}^*$ contain transcendental functions, the solutions can be obtained applying some of the search methods. To avoid numerical difficulties when calculating the zeroes of equation (14), the natural frequencies can be determined as maxima of the following expression:

$$g(w) = \log \frac{1}{\det\left|\tilde{\mathbf{K}}_{DG,nn}^*(w)\right|} \tag{15}$$

4 NUMERIČKI PRIMERI

Primena metode dinamičke krutosti u analizi slobodnih vibracija ploča sa ukrućenjima ilustrovana je u narednim primerima. Na osnovu izloženog postupka, napisan je program u MATLAB-u [17] za određivanje sopstvenih frekvencija i oblika oscilovanja ploča sa ukrućenjima za različite uslove oslanjanja. Dobijeni rezultati upoređeni su sa rezultatima dobijenim primenom komercijalnog programa Abaqus [18].

U prvom primeru, razmatran je armirano-betonski nosač sandučastog poprečnog preseka, (*E* = 31.5 *GPa*, *n* = 0.2 i *r* = 2500 kg/m^3), čija su geometrija i granični uslovi prikazani na slici 4a. Ploča je podeljena na devet elemenata, što predstavlja minimalni broj elemenata u ovom slučaju, sa obzirom na geometriju nosača. Ukupan broj kontura je 26. Konture paralelne sa *x*- i *z*-osom slobodno su oslonjene (S_x, S_z), dok su konture paralelne sa *y*-osom slobodne (F) ili uklještene (C). Granični uslovi na konturama zadati su na sledeći način:

- slobodno oslonjena ivica paralelna sa *x*-osom (S_x): $u=v=w=f_y=f_z=0$,
- slobodno oslonjena ivica paralelna sa *z*-osom (S_z): $u=v=w=f_x=f_y=0$,

4 NUMERICAL EXAMPLES

Application of the dynamic stiffness method in the free vibration analysis of stiffened plates is illustrated in the following examples. In order to validate the presented method based on the DSM, a computer code in MATLAB [17] was developed for the free vibration analysis of stiffened plates with arbitrary boundary conditions. The results were compared with the results obtained using Abaqus [18].

The first example is concerned with a reinforced concrete girder of a box cross-section, (*E* = 31.5*GPa*, *n* = 0.2 and *r* = 2500kg/m^3). The geometry and boundary conditions are presented in Figure 4a. In this case, the number of dynamic stiffness elements is equal to 9 (which is minimal number of elements), while the number of boundary lines is 26. Boundary lines parallel to *x*- and *z*-axis are simply supported (S_x, S_z), while the boundary lines parallel to *y*-axis are either free (F) or clamped (C). Boundary conditions are assigned in the following way:

- simply supported edge parallel to *x*-axis (S_x): $u=v=w=f_y=f_z=0$,
- simply supported edge parallel to *z*-axis (S_z): $u=v=w=f_x=f_y=0$,

• uklještena ivica paralelna sa *y*-osom (C): $u=v=w=f_x=f_y=f_z=0$,

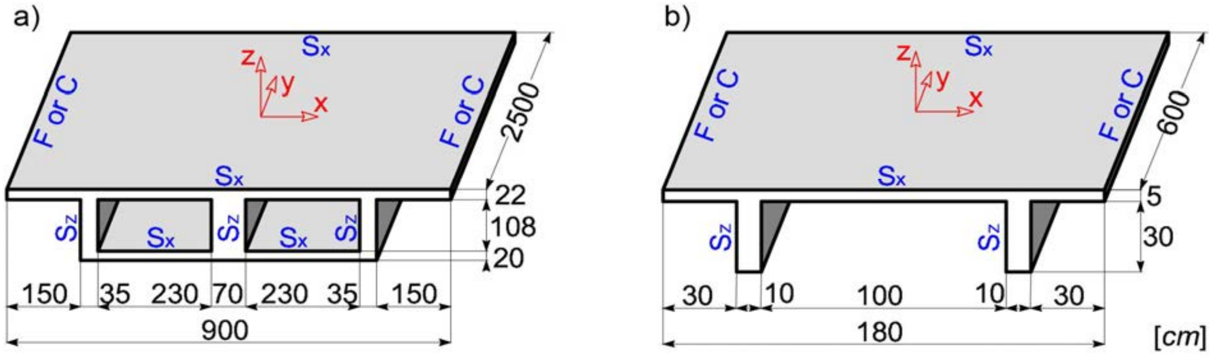
• slobodna ivica paralelna sa *y*-osom (F): $f_z=0$.

Prvih deset sopstvenih frekvencija $f = w/2p$ sračunato je primenom različitog broja članova reda, kako bi se utvrdila konvergencija rešenja. Rezultati su prikazani u Tabeli 1 i upoređeni sa numeričkim rešenjem dobijenim primenom 46250 konačnih elemenata tipa S4R u Abaqus-u (dimenzija elementa 0.1 *m*). Za rešenje s *M*=7 članova reda, sračunato je odstupanje D od rešenja u Abaqus-u.

• clamped edge parallel to *y*-axis (C): $u=v=w=f_x=f_y=f_z=0$,

• free edge parallel to *y*-axis (F): $f_z=0$.

In order to check the convergence of the proposed method, the first ten natural frequencies $f = w/2p$ have been calculated using different number of terms in the general solution. The results are presented in Table 1 and compared with the numerical solutions obtained using 46250 Abaqus S4R type plate elements (element size 0.1 *m*). For the solutions with *M* = 7 terms in the series expansion, the differences D from the solutions obtained in Abaqus are calculated.



Slika 4. Geometrija i granični uslovi: (a) armirano-betonskog nosača sandučastog preseka i (b) armirano-betonske korube

Figure 4. Geometry and boundary conditions of: (a) reinforced concrete girder with box cross-section and (b) reinforced concrete ribbed slab

Prvih deset sopstvenih frekvencija odlično se poklapaju sa rešenjem dobijenim primenom MKE (prosečno odstupanje je 0.33% za slučaj S-F-S-F, odnosno 2.67% za slučaj S-C-S-C), što potvrđuje izuzetne mogućnosti primene izvedenih dinamičkih matrica krutosti u analizi vibracija armirano-betonskih ploča, čak i kada se uzme u obzir mali broj članova reda (*M*=7). Diskretizacija modela svedena je na minimum, čime se smanjuje ukupno trajanje proračuna u poređenju sa klasičnom metodom konačnih elemenata.

The first ten natural frequencies computed using the proposed method are in excellent agreement with the results obtained using the FEM (the average difference is 0.33% for S-F-S-F and 2.67% for S-C-S-C case), which confirms the exceptional possibilities of the application of the dynamic stiffness method in the free vibration analysis of reinforced concrete plates, even for a small number of terms in the general solution (*M*=7). Discretization of the model is reduced to a minimum, thus the computational time have been significantly decreased in comparison with the FEM.

Tabela 1. Sopstvene frekvencije [Hz] armirano-betonskog nosača sandučastog preseka

Table 1. Dimensionless natural frequencies [Hz] of reinforced concrete girder of a box cross-section

n	S – F – S – F						S – C – S – C					
	DSM				Abaqus	D [%]	DSM				Abaqus	D [%]
	M=1	*M*=3	*M*=5	*M*=7			*M*=1	*M*=3	*M*=5	*M*=7		
1	7.8	10.0	9.9	9.9	9.9	0.35	7.8	17.6	17.5	17.5	16.8	4.06
2	8.6	16.8	16.7	16.6	16.6	0.29	8.6	31.0	30.8	30.7	29.2	5.31
3	9.7	24.3	23.9	23.8	23.7	0.52	9.6	33.3	33.2	33.2	32.9	1.01
4	10.2	27.7	27.4	27.3	27.2	0.45	12.1	34.0	39.2	39.1	38.8	0.85
5	12.1	27.9	27.8	27.7	27.7	-0.05	13.3	36.3	42.6	42.5	41.5	2.35
6	13.3	31.0	29.1	29.0	28.9	0.40	14.0	39.2	47.7	47.5	45.5	4.32
7	13.6	33.8	37.9	37.8	37.6	0.59	16.0	43.0	50.8	50.7	50.5	0.35
8	14.0	34.0	38.1	38.0	37.9	0.24	17.9	43.2	52.6	52.4	51.2	2.33
9	16.0	36.3	39.0	38.9	38.7	0.41	18.3	45.4	57.0	56.9	55.4	2.79
10	17.6	38.4	43.7	43.7	43.7	0.04	20.2	48.3	64.5	64.0	61.9	3.32

U drugom primeru, razmatrane su slobodne vibracije armirano-betonske korube (*E* = 31.5 *GPa*, n = 0.2 i r = 2500 *kg/m*^{3}), čija su geometrija i granični uslovi prikazani na slici 4b. Ploča je diskretizovana sa pet elemenata (16 kontura). Prvih deset sopstvenih frekvencija *f* = *w/2p* sračunate su primenom različitog broja članova reda. Rezultati su prikazani u Tabeli 2 i upoređeni sa MKE rešenjem dobijenim primenom 23520 konačnih elemenata tipa S4R u Abaqus-u (dimenzija elementa 0.025 m). Za rešenje sa *M*=7 članova reda, sračunato je odstupanje D od rešenja u Abaqus-u. Kao i u prethodnom primeru, sračunate frekvencije primenom prikazanog modela se odlično poklapaju sa numeričkim rešenjem (prosečno Δ = 0.22% za slučaj S-F-S-F, odnosno 0.14% za slučaj S-C-S-C).

The second example is concerned with the reinforced concrete ribbed slab, (*E* = 31.5*GPa*, n = 0.2 and r = 2500*kg/m*^{3}), whose geometry and boundary conditions are presented in Figure 4b. In this case the number of dynamic stiffness elements is equal to 5, while the number of boundary lines is equal to 16. The first ten natural frequencies *f* = *w/2p* have been calculated using different number of terms in the general solution. The results are presented in Table 2 and compared with the FEM numerical solutions obtained using 23520 Abaqus S4R type plate element (element size 0.025 *m*). For the solutions with *M* = 7 terms in the series expansion, the differences D from the solutions obtained using Abaqus are calculated. As in the previous example, the natural frequencies computed using the proposed method are in excellent agreement with the FEM (the average difference is 0.22% for S-F-S-F and 0.14% for S-C-S-C case).

Tabela 2. Sopstvene frekvencije [Hz] armirano-betonske korube
Table 2. Dimensionless natural frequencies [Hz] of reinforced concrete ribbed slab

n	S – F – S – F						S – C – S – C					
	DSM				Abaqus	D [%]	DSM				Abaqus	D [%]
	M=1	*M*=3	*M*=5	*M*=7			*M*=1	*M*=3	*M*=5	*M*=7		
1	33.9	33.0	32.9	32.9	32.8	0.34	34.0	57.7	57.5	57.5	57.5	0.03
2	36.4	35.5	35.4	35.3	35.3	0.14	40.7	85.7	84.9	84.7	84.5	0.20
3	40.7	55.4	58.0	57.9	57.9	-0.06	52.3	110.8	110.7	110.7	110.7	-0.02
4	52.2	77.8	76.9	76.7	76.5	0.32	52.7	116.6	115.7	115.4	115.1	0.22
5	52.6	87.2	86.2	85.9	85.5	0.43	58.4	129.8	141.6	141.2	141.0	0.13
6	60.0	91.7	90.5	90.1	89.8	0.29	59.7	134.9	145.7	145.4	145.1	0.21
7	67.1	96.9	96.3	96.1	95.8	0.27	67.3	137.3	155.1	164.6	164.5	0.03
8	78.4	109.7	108.6	108.3	108.3	0.05	78.8	144.4	189.0	188.9	189.0	-0.04
9	79.4	129.3	133.1	132.7	132.3	0.29	79.6	146.7	200.3	199.6	199.1	0.27
10	82.8	133.9	136.1	135.6	135.6	0.01	82.7	155.8	201.6	201.6	202.0	-0.22

Na slici 5 izvršeno je upoređivanje pojedinih oblika oscilovanja i odgovarajućih sopstvenih frekvencija (tonovi 2, 4, 7 i 8) armirano-betonske korube, za kombinaciju konturnih uslova S-C-S-C i *M*=7 članova reda. Oblici oscilovanja sračunati su primenom originalnog programa u MATLAB-u – zasnovanog na metodi dinamičke krutosti, kao i primenom komercijalnog programa Abaqus. Dobijeno je odlično poklapanje oblika oscilovanja.

Kako bi se ilustrovala prednost MDK u odnosu na MKE - u pogledu potrebnog vremena za proračun, uporediće se broj stepeni slobode u numeričkom modelu sandučastog nosača, formulisanog prema MDK i prema MKE. Treba imati u vidu i to da broj stepeni slobode u MDK zavisi od broja kontura i broja članova reda *M*, dok u MKE zavisi od broja čvorova i tipa elementa (u razmatranom primeru, konačni element ima 6 stepeni slobode u čvoru). Broj stepeni slobode u MDK (za *M*=7) i MKE je prema gorenavedenom:

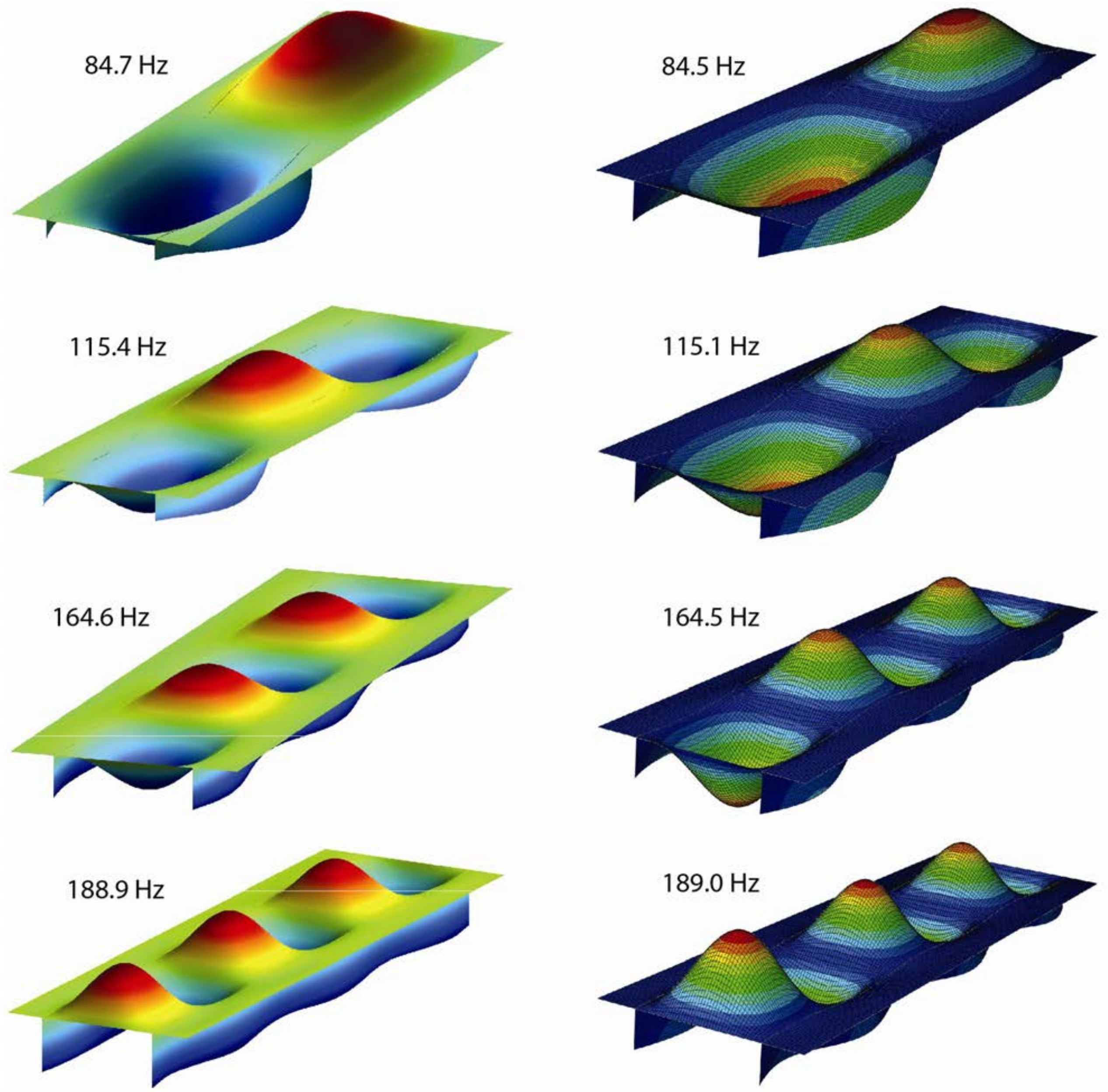
Figure 5 illustrates the comparison of some mode shapes and the corresponding natural frequencies (modes 2, 4, 7 and 8) of the reinforced-concrete ribbed slab, for the combination of boundary conditions S-C-S-C and *M*=7 terms in the series expansion. Mode shapes have been calculated using the original MATLAB code based on the dynamic stiffness method, as well as by using the commercial software Abaqus. Excellent agreement has been obtained.

To illustrate the advantage of the DSM in comparison with the FEM by means of the necessary computational time, number of degrees of freedom in the numerical model of the girder with the box cross-section, formulated using the DSM or the FEM, will be compared. It is worth mentioning that the number of degrees of freedom in the DSM depends on the number of contours and number of terms in series expansion *M*, while in the FEM it depends on the number of nodes and the element type (in the considered example, the finite element has 6 nodal degrees of freedom). Number of degrees of freedom in the DSM (for *M*=7)and the FEM is therefore:

$$\begin{aligned} NDOF_{DSM} &= N_{contours} \cdot (9M+18) = 26 \cdot (9 \cdot 7 + 18) = 1924 \\ NDOF_{FEM} &= N_{nodes} \cdot 6 = 46250 \cdot 6 = 277500 \end{aligned} \quad (16)$$

što pokazuje potencijal prikazane metode u smislu potrebnog vremena proračuna.

that confirms the potential of the presented method by means of the necessary computational time.



Slika 5. Oblici oscilovanja i odgovarajuće sopstvene frekvencije armirano-betonske korube (slučaj S-C-S-C, M=7): (a) metoda dinamičke krutosti i (b) Abaqus

Figure 5. Mode shapes and corresponding natural frequencies of reinforced-concrete ribbed slab (S-C-S-C case, M=7): (a) dynamic stiffness method and (b) Abaqus

5 ZAKLJUČAK

U ovom radu prikazana je primena metode dinamičke krutosti u analizi slobodnih vibracija ploča sa ukrućenjima. Takođe, prikazan je postupak dobijanja dinamičke matrice krutosti ploče, kao i matrica rotacije za različite položaje ploča koje su pod pravim uglom u odnosu na referentnu ravan. Primenjen je sličan postupak kao u MKE za formiranje globalne dinamičke matrice krutosti ploče sa ukrućenjima. Razvijen je računarski program u MATLAB-u za analizu slobodnih vibracija sistema ploča. Sopstvene frekvencije i oblici oscilovanja određeni su za različite kombinacije konturnih uslova.

5 CONCLUSION

This paper deals with the application of the dynamic stiffness method in the free vibration analysis of stiffened plates. The procedure for the development of dynamic stiffness matrix of rectangular plate, as well as rotation matrices for different plate orientations and perpendicularly oriented with respect to the reference plane, has been provided. The similar procedure as in the FEM for the development of the global dynamic stiffness matrix of the stiffened plate, has been applied. Computer program in MATLAB has been developed for the free vibration analysis of plate assemblies. The natural

Dobijeni rezultati verifikovani su upoređivanjem sa rezultatima dobijenim primenom programskog paketa Abaqus zasnovanog na MKE. Tačnost rezultata dobijenih primenom MDK zavisi isključivo od broja članova reda usvojenog rešenja. Uočena je brza konvergencija rezultata dobijenih po MDK. Već sa tri člana do pet članova reda, dobijaju se rezultati visoke tačnosti. Međutim, za više tonove oscilovanja - povećava se broj potrebnih članova reda.

Na osnovu dobijenih rezultata, može se zaključiti da MDK poseduje veliki potencijal u dinamičkoj analizi konstrukcija, koji se može proširiti na analizu ploča i ljuski zasnovane na teoriji višeg reda, ploče spojene pod proizvoljnim uglom, kao i konstruktivne elemente sačinjene od modernih kompozitnih materijala.

frequencies and mode shapes were calculated for various combinations of boundary conditions. Verification of the results was performed by comparing the results obtained using Abaqus software package based on the FEM. The accuracy of the results obtained by the DSM depends solely on the number of terms in the series expansion of the adopted solution. Rapid convergence of the results obtained by the DSM has been detected. By using only three to five terms, the results of high accuracy have been obtained. However, for the higher modes of vibration, the number of required members in the series expansion should be increased.

Based on these results, it can be concluded that the DSM has great potential in the dynamic analysis of structures that can be extended to the analysis of plates and shells using higher-order theory, the plates connected with arbitrary angles, as well as for the structural elements made of modern composite materials.

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REZIME

PRIMENA METODE DINAMIČKE KRUTOSTI U NUMERIČKOJ ANALIZI SLOBODNIH VIBRACIJA PLOČA SA UKRUĆENJIMA

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U okviru ovog rada, analizirane su slobodne vibracije ploča sa ukrućenjima, primenom metode dinamičke krutosti. Formulisan je pravougaoni element Mindlin-ove ploče, zasnovan na metodi dinamičke krutosti. Primenom matrica rotacije, formirane su dinamičke matrice krutosti pojedinačnih ploča u globalnom koordinatnom sistemu. Korišćenjem sličnog postupka kao u metodi konačnih elemenata, izvedena je globalna dinamička matrica krutosti sistema ploča. Određene su sopstvene frekvencije ploča sa ukrućenjima za različite konturne uslove i upoređene sa vrednostima dobijenim u komercijalnom programskom paketu Abaqus. Dobijeni su rezultati visoke tačnosti.

Ključne reči: slobodne vibracije, dinamička matrica krutosti, ploče sa ukrućenjima

SUMMARY

APPLICATION OF DYNAMIC STIFFNESS METHOD IN NUMERICAL FREE VIBRATION ANALYSIS OF STIFFENED PLATES

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The free vibration analysis of stiffened plate assemblies has been performed in this paper by using the dynamic stiffness method. Rectangular Mindlin plate dynamic stiffness element has been formulated. Using the rotation matrices, dynamic stiffness matrices of single plates have been derived in global coordinate system. The global dynamic stiffness matrix of plate assembly has been derived by using similar assembly procedure as in the finite element method. The natural frequencies of stiffened plate assemblies with different boundary conditions have been computed and validated against the results obtained by using the commercial software package Abaqus. High accuracy of the results has been demonstrated.

Key words: free vibrations, dynamic stiffness matrix, stiffened plates