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ON THE FIFTH COEFFICIENTS FOR THE CLASS $\mathcal{U}(\lambda)$

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ABSTRACT. Let class $\mathcal{U}(\lambda)$, $0<\lambda\le 1$, consists of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ that are analytic in the unit disk $\mathbb{D}=\{z:|z|<1\}$ and satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda \qquad (z\in\mathbb{D}).$$

In this paper we prove sharp upper bound of the modulus of the fifth coefficient of f from $\mathcal{U}(\lambda)$ in the case when $\frac{2}{3}\le\lambda\le 1$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D}=\{z:|z|<1\}$ and are normalized such that

$$f(z)=z+a_2z^2+a_3z^3+\cdots,$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

Highly distinguished achievement in the study of univalent functions was the proof of the famous Bieberbach conjecture $|a_n|\le n$ for $n\ge 2$ by Lewis de Branges in 1985 [1]. The conjecture inspired development of new methods and branches in the analysis of complex functions of one complex variable. Although, the conjecture is closed it remains an intriguing question to find upper

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bounds (preferably sharp) of the modulus of the coefficient for functons in various sublasses of univalent functions. One such class, that attracts significant attention in past decades is the class $\mathcal{U}(\lambda)$, $0<\lambda\le 1$,

$$\mathcal{U}(\lambda)=\left\{f\in\mathcal{A}:\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<\lambda,\, z\in\mathbb{D}\right\}.$$

Functions from this class are proven to be univalent but not starlike which makes them interesting since the class of starlike functions is very wide. So, Overview of the most valuable results is given in Chapter 12 from [7].

In [4], the authors conjectured $|a_n|\le 1+\lambda+\lambda^2+\cdots+\lambda^{n-1}$ for the class $\mathcal{U}(\lambda)$ and $n\ge 5$. In the same paper they proved that the conjecture is valid for $n=3$ and $n=4$, while for $n=2$ the proof is given in [5].

In this paper we will prove the conjecture for $n=5$ and $2/3\le\lambda\le 1$.

In the study we will use the following result due to Leverenz ([2, Theorem 4(b)]) for the class $\mathcal{P}$ of Caratheodory functions, that are functions p analytic in $\mathbb{D}$, of form $p(z)=1+p_1z+p_2z^2+\cdots$ with positive real part, i.e., $\operatorname{Re} p(z)>0$ for $z\in\mathbb{D}$.

Lemma 1.1. *Function $p(z)=1+p_1z+p_2z^2+\cdots$ has positive real part on the unit disk, if, and only if,*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j+\sum_{k=1}^{\infty}p_kz_{k+j}\right|^2-\left|\sum_{k=0}^{\infty}p_{k+1}z_{k+j}\right|^2\right\}\ge 0 \tag{1.1}$$

for every sequence $\{z_k\}$ of complex numbers that satisfy $\lim_{k\to\infty}|z_k|^{1/k}<1$.

2. Main result

As a preparation for the proof of the main result, we are going to make the following analysis.

For every function $p(z)=1+p_1z+p_2z^2+\cdots$ from $\mathcal{P}$, there exists a function ω, analytic in $\mathbb{D}$, such that $\omega(0)=0$, $|\omega(z)|<1$ for all $z\in D$ and

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}\left(=1+2\omega(z)+2\omega^2(z)+\cdots\right).$$

If $\omega(z) = c_1 z + c_2 z^2 + \cdots$ then by comparing the coefficients we have

$$\begin{aligned} p_1 &= 2c_1, \\ p_2 &= 2c_2 + 2c_1^2, \\ p_3 &= 2c_3 + 4c_1c_2 + 2c_1^3, \\ p_4 &= 2c_4 + 4c_1c_3 + 2c_2^2 + 6c_1^2c_2 + 2c_1^4. \end{aligned} \tag{2.1}$$

From (1.1), if we choose $z_k = 0$ for $k > 3$, we have

$$\begin{aligned} &|2z_0 + p_1z_1 + p_2z_2 + p_3z_3|^2 - |p_1z_0 + p_2z_1 + p_3z_2 + p_4z_3|^2 \\ &+|2z_1 + p_1z_2 + p_2z_3|^2 - |p_1z_1 + p_2z_2 + p_3z_3|^2 \\ &+|2z_2 + p_1z_3|^2 - |p_1z_2 + p_2z_3|^2 + |2z_3|^2 - |p_1z_3|^2 \geq 0. \end{aligned}$$

From here, for $z_0 = 0$,

$$\begin{aligned} |p_2z_1 + p_3z_2 + p_4z_3|^2 \leq& |2z_1 + p_1z_2 + p_2z_3|^2 + |2z_2 + p_1z_3|^2 \\ &- |p_1z_2 + p_2z_3|^2 + |2z_3|^2 - |p_1z_3|^2. \end{aligned} \tag{2.2}$$

Using (2.1) and (2.2), after some calculations, we have

$$\begin{aligned} &|2(c_2 + c_1^2)z_1 + 2(c_3 + 2c_1c_2 + c_1^3)z_2 + 2(c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4)z_3|^2 \\ \leq& |2z_1 + 2c_1z_2 + 2(c_2 + c_1^2)z_3|^2 + |2z_2 + 2c_1z_3|^2 - |2c_1z_2 + 2(c_2 + c_1^2)z_3|^2 \\ &+ 4|z_3|^2 - 4|c_1z_3|^2, \end{aligned}$$

and after dividing with 4,

$$\begin{aligned} L =:& |(c_2 + c_1^2)z_1 + (c_3 + 2c_1c_2 + c_1^3)z_2 + (c_4 + 2c_1c_3 + c_2^2 + 3c_1^2c_2 + c_1^4)z_3|^2 \\ \leq& |z_1 + c_1z_2 + (c_2 + c_1^2)z_3|^2 + |z_2 + c_1z_3|^2 - |c_1z_2 + (c_2 + c_1^2)z_3|^2 \\ &+ |z_3|^2 - |c_1z_3|^2 := R. \end{aligned} \tag{2.3}$$

On the other hand, if $f(z) = z + a_2^2 + \cdots \in \mathcal{U}(\lambda)$, then (see [3, 5])

$$\frac{f(z)}{z} = \frac{1}{(1-\omega(z))(1-\lambda\omega(z))} = 1 + \sum_{n=1}^{\infty} \frac{1-\lambda^{n+1}}{1-\lambda}\omega^n(z), \tag{2.4}$$

where $\omega(z) = c_1z + c_2z^2 + \cdots$, $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, and $\left.\frac{1-\lambda^{n+1}}{1-\lambda}\right|_{\lambda=1} = n+1$ for $n = 1, 2, \ldots$. From (2.4) we have

$$a_5 = \frac{1-\lambda^2}{1-\lambda}c_4 + 2\frac{1-\lambda^3}{1-\lambda}c_1c_3 + \frac{1-\lambda^3}{1-\lambda}c_2^2 + 3\frac{1-\lambda^4}{1-\lambda}c_1^2c_2 + \frac{1-\lambda^5}{1-\lambda}c_1^4.$$

Now we can formulate and proof the main result.

Theorem 2.1. *Let $f(z) = z + a_2z^2 + \cdots$ belong to the class $\mathcal{U}(\lambda)$, $2/3 \le \lambda \le 1$. Then the following estimate is sharp*

$$|a_5| \le 1 + \lambda + \lambda^2 + \lambda^3 + \lambda^4.$$

Proof. If we choose $z_1 = \lambda^2(3\lambda - 1)c_2$, $z_2 = 2\lambda^2 c_1$ and $z_3 = 1 + \lambda$ in (2.3), then after some calculations we obtain

$$L = |a_5 + \lambda^2(3\lambda - 2)c_2^2 + \lambda^2(1 - \lambda - \lambda^2)c_1^4|^2, \tag{2.5}$$

$$\begin{aligned} R =&|((2\lambda^2 + \lambda + 1) - 3(1 - \lambda)\lambda^2)c_2 + (2\lambda^2 + \lambda + 1)c_1^2|^2 + (2\lambda^2 + \lambda + 1)^2|c_1|^2 \\ &- |(1 + \lambda)c_2 + (2\lambda^2 + \lambda + 1)c_1^2|^2 + (1 + \lambda)^2 - (1 + \lambda)^2|c_1|^2 \\ =&[((2\lambda^2 + \lambda + 1) - 3(1 - \lambda)\lambda^2)^2 - (1 + \lambda)^2]|c_2|^2 + [(2\lambda^2 + \lambda + 1)^2 \\ &- (1 + \lambda)^2]|c_1|^2 + 2\lambda^2(3\lambda - 1)(2\lambda^2 + \lambda + 1)\operatorname{Re}\{c_1^2\overline{c_2}\} + (1 + \lambda)^2 \end{aligned}$$

and using $\operatorname{Re}\{c_1^2\overline{c_2}\} \le |c_1|^2|c_2|$,

$$R \le A|c_2|^2 + B|c_1|^2 + C|c_1|^2|c_2| + (1 + \lambda)^2, \tag{2.6}$$

where

$$\begin{aligned} A &= (2\lambda^2 + \lambda + 1 - 3(1 - \lambda)\lambda^2)^2 - (1 + \lambda)^2 \\ &= \lambda^2(3\lambda - 1)[\lambda^2(3\lambda - 1) + 2\lambda + 2], \\ B &= (2\lambda^2 + \lambda + 1)^2 - (1 + \lambda)^2 = 4\lambda^2(\lambda^2 + \lambda + 1), \\ C &= 2\lambda^2|3\lambda - 1|(2\lambda^2 + \lambda + 1) = 2\lambda^2(3\lambda - 1)(2\lambda^2 + \lambda + 1). \end{aligned} \tag{2.7}$$

First let note that A, B and C are all positive since $\lambda > 1/3$.

Next, using (2.5), (2.6) and (2.7), we get

$$\begin{aligned} |a_5| \le&\lambda^2|3\lambda - 2||c_2|^2 + \lambda^2|1 - \lambda - \lambda^2||c_1|^4 \\ &+ \sqrt{A|c_2|^2 + B|c_1|^2 + C|c_1|^2|c_2| + (1 + \lambda)^2}, \end{aligned}$$

or, since $2/3 \le \lambda \le 1$, $|3\lambda - 2| = 3\lambda - 2$, $|1 - \lambda - \lambda^2| = \lambda^2 + \lambda - 1$, and

$$\begin{aligned} |a_5| \le&\lambda^2(3\lambda - 2)|c_2|^2 + \lambda^2(\lambda^2 + \lambda - 1)|c_1|^4 \\ &+ \sqrt{A|c_2|^2 + B|c_1|^2 + C|c_1|^2|c_2| + (1 + \lambda)^2} = \varphi(|c_1|, |c_2|), \end{aligned}$$

where

$$\varphi(x, y) = \lambda^2(3\lambda - 2)y^2 + \lambda^2(\lambda^2 + \lambda - 1)x^4 + \sqrt{Ay^2 + Bx^2 + Cx^2y + (1 + \lambda)^2}.$$

In [6] (see expression (13) on page 128), $|c_1| \le 1$ and $|c_2| \le 1 - |c_1|^2$, i.e., $0 \le x \le 1$, $0 \le y \le 1 - x^2$. So, it remains to find the maximum of the function φ on the region $\Delta = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1 - x^2\}$.

Since, the system of equations $\partial\varphi/\partial x = 0$ and $\partial\varphi/\partial y = 0$ has no solutions in the interior of Δ (all terms are positive), it remains to find its maximum on its boundary.

For $y = 0$ we have

$$\begin{aligned}
\varphi(x, 0) &= \lambda^2(\lambda^2 + \lambda - 1)x^4 + \sqrt{Bx^2 + (1 + \lambda)^2} \\
&\le \lambda^2(\lambda^2 + \lambda - 1) + \sqrt{B + (1 + \lambda)^2} \\
&\le \lambda^2(\lambda^2 + \lambda - 1) + 2\lambda^2 + \lambda + 1 \\
&= \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1.
\end{aligned}$$

For $x = 0$ we have

$$\begin{aligned}
\varphi(0, y) &= \lambda^2(3\lambda - 2)y^2 + \sqrt{Ay^2 + (1 + \lambda)^2} \\
&\le \lambda^2(3\lambda - 2) + \sqrt{A + (1 + \lambda)^2} \\
&= \lambda^2(3\lambda - 2) + 1 + \lambda + \lambda^2(3\lambda - 1) \\
&= 6\lambda^3 - 3\lambda^2 + 1 + \lambda \\
&\le \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1.
\end{aligned}$$

The last inequality holds since it is equivalent to the inequality $\lambda^4 - 5\lambda^3 + 4\lambda^2 \ge 0$, i.e., $\lambda^2(1 - \lambda)(4 - \lambda) \ge 0$.

The edge $x = 1$ is covered by $\varphi(1, 0) \le \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$.

The last edge, $y = 1 - x^2$ and $0 \le x \le 1$, requires bigger attention:

$$\begin{aligned}
\varphi(x, 1 - x^2) = \lambda^2(3\lambda - 2) - 2\lambda^2(3\lambda - 2)|c_1|^2 + \lambda^2(\lambda^2 + 4\lambda - 3)|c_1|^4 \\
+ \sqrt{A_1 + A_2|c_1|^2 - A_3|c_1|^4},
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
A_1 &= (2\lambda^2 + \lambda + 1 - 3(1 - \lambda)\lambda^2)^2 = (1 + \lambda - \lambda^2 + 3\lambda^3)^2, \\
A_2 &= 2\lambda^2(3 - (3\lambda - 2)^2\lambda^2), \\
A_3 &= (3\lambda - 1)(5 - 3\lambda)\lambda^4.
\end{aligned} \tag{2.9}$$

We want to prove that for $2/3 \le \lambda \le 1$,

$$\varphi(x, 1-x^2) \le 1+\lambda+\lambda^2+\lambda^3+\lambda^4. \tag{2.10}$$

From (2.8) we have that (2.10) is equivalent to

$$B_1 + B_2|c_1|^2 - B_3|c_1|^4 - \sqrt{A_1 + A_2|c_1|^2 - A_3|c_1|^4} \ge 0, \tag{2.11}$$

where

$$\begin{aligned} B_1 &= 1+\lambda+3\lambda^3-2\lambda^3+\lambda^4 = 2\lambda^2+\lambda+1+\lambda^2(1-\lambda)^2,\\ B_2 &= 2\lambda^2(3\lambda-2),\\ B_3 &= \lambda^2(\lambda^2+4\lambda-3). \end{aligned} \tag{2.12}$$

If we put $t = |c_1|^2$, $0 \le t \le 1$, then (2.11) is equivalent to

$$\psi(t) =: (B_1 + B_2 t - B_3 t^2)^2 - (A_1 + A_2 t - A_3 t^2) \ge 0.$$

Using (2.9) and (2.12), after some calculations we obtain that for $0 \le \lambda \le 1$,

$$\begin{aligned} \psi(0) =& B_1^2 - A_1 = (2\lambda^2+\lambda+1+\lambda^2(1-\lambda)^2)^2\\ &- (2\lambda^2+\lambda+1-3(1-\lambda)\lambda^2)^2 \ge 0,\\ \psi(1) =& 0. \end{aligned}$$

Also, since

$$\psi''(t) = 2(B_2^2 - 2B_1B_3 + A_3) - 12B_2B_3t + 12B_3^2t^2, \tag{2.13}$$

$$\begin{aligned} \psi''(0) &= 2(B_2^2 - 2B_1B_3 + A_3)\\ &= 2\lambda^2(6-2\lambda+19\lambda^2-68\lambda^3+43\lambda^4-4\lambda^5-2\lambda^6) < 0 \end{aligned}$$

and

$$\psi''(1) = 2\lambda^2(6-2\lambda+19\lambda^2-68\lambda^3+11\lambda^4+4\lambda^5-6\lambda^6) < 0$$

for $2/3 \le \lambda \le 1$, then we conclude from (2.13) that $\psi''(t) < 0$ in the interval $[2/3, 1]$.

Finally, since ψ is concave function with $\psi(0) \ge 0$ and $\psi(1) = 0$, then also $\psi(t) \ge 0$ for every $t \in [0, 1]$. □

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