SHEAR DEFORMABLE DYNAMIC STIFFNESS ELEMENTS FOR FREE VIBRATION ANALYSIS OF RECTANGULAR ISOTROPIC MULTILAYER PLATES

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Summary: *In this paper, two shear deformable dynamic stiffness elements for the free vibration analysis of rectangular, transversely isotropic, single- and multi-layer plates having arbitrary boundary conditions are presented. Dynamic stifness matrices are developed for the Reddy's higher-order shear deformation theory (HSDT) and the Mindlin-Reissner's first-order shear deformation theory (FSDT). The dynamic stiffness matrices contain both the stiffness and mass properties of the plate and can be assembled similarly as in the conventional finite element method. The influence of face-to-core thickness ratio and face-to-core module ratio of sandwich plate, as well as the influence of the shear deformation on the free vibration characteristics of sandwich plates have been analysed. The results obtained by proposed HSDT and FSDT dynamic stiffness element are validated against the results obtained using the conventional finite element analysis (ABAQUS), as well as the results obtained by 4-node layered rectangular finite element. The proposed model allows accurate prediction of free vibration response of rectangular layered plate assemblies with arbitrary boundary conditions.*

Keywords: *free vibrations od layered plates, dynamic stiffness method, FSDT, HSDT*

1. INTRODUCTION

Multi-layer plates composed of several laminas of different properties are widely used in different areas of engineering. Sandwich panels are usually applied in civil engineering as components of light roofs and walls to provide thermal isolation of buildings. These elements are often placed in a dynamic loading environment, thus the adequate

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computational models capable to predict the dynamic response of such structures are required. The dynamic response is usually predicted using different plate theories, where the transverse shear effects are accounted by means of the shear correction factors, or the higher-order approximation of the displacement field. Theories that consider the multi-layer structure as a single homogeneous layer are referred as equivalent-single-layer (ESL) theories [1-3]. The comparison of different ESL plate theories is given in the Reddy's overview [4] and monographs [5, 6]. To overcome the problems that may arise due to the simplifications associated with the plate kinematics in the ESL theories, the generalized layerwise plate theory (GLPT) of Reddy [7] is used to improve the representation of the kinematics. To obtain the numerical solutions for the dynamic response of plates, finite element methods (FEM) are adopted [8-14].

In the vibration analysis, the dynamic stiffness method (DSM) [15-17] is used to obtain more accurate and reliable results in comparison with the conventional FEM. The DSM uses a unique element matrix (dynamic stiffness matrix) containing both stiffness and mass properties of the structure. The selection of the DSM for solving the free vibration problem is motivated by the fact that only one dynamic stiffness element per structural member with constant material and geometrical properties can be used to accurately represent its dynamic behavior at any frequency. Different applications of the dynamic stiffness method based on the ESL plate theories are given in [18-20]. However, the main lack of the proposed methods is the inapplicability to the plates having arbitrary combinations of boundary conditions. This has been overcome in the authors' investigations [21-23], where the dynamic stiffness matrices for a completely free rectangular isotropic plate based on the Mindlin-Reissner's first-order shear deformation theory (FSDT) and the Reddy's higher-order shear deformation theory (HSDT) were developed. These solutions are free of restrictions regarding the boundary conditions.

In this paper the dynamic stiffness matrix for a completely free rectangular multi-layer plate element based on the HSDT and FSDT is presented. Three coupled Euler-Lagrange equations of motion have been transformed into two uncoupled equations of motion using a boundary layer function [24]. The proposed method enables free transverse vibration analysis of rectangular multi-layer plates with transversely isotropic layers, having arbitrary combinations of boundary conditions. The natural frequencies obtained using different dynamic stiffness multi-layer plate elements have been validated against the solutions from the commercial software Abaqus [25] and the previously verified results [13, 14]. The influence of face-to-core thickness ratio and face-to-core module ratio of sandwich plate, as well as the influence of the shear deformation on the free vibration characteristics of sandwich plates have been discussed. A variety of new results is provided as a benchmark for future investigations.

2. FORMULATION OF THE MULTI-LAYER HSDT DYNAMIC STIFFNESS ELEMENT

The geometry of rectangular multi-layer plate composed of *n* isotropic layers is presented in Figure 1. The assumptions and restrictions introduced in the derivation of the model are: (1) all layers are perfectly bonded together, (2) the material of each layer is homogeneous, transversely isotropic and linearly elastic, (3) small strains and small rotations are assumed and (4) inextensibility of the transverse normal is imposed.

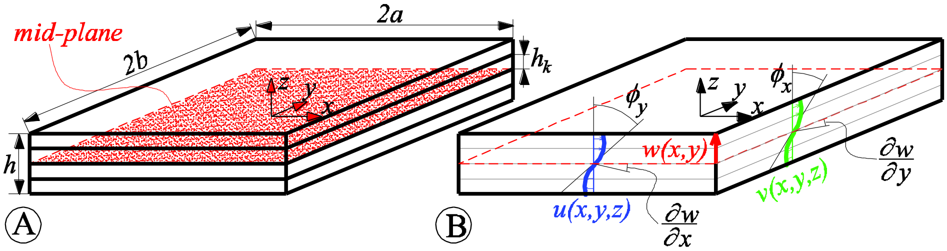

Figure 1. (A) Geometry of multi-layer plate; (B) displacement components of the HSDT

Assuming zero-deformation in the mid-plane of the plate (see Figure 1a), the displacement field of the HSDT at point (*x*,*y*,*z*) of a plate in the arbitrary time instant *t* is:

$$\begin{aligned}
u(x,y,z,t) &= z\phi_y(x,y,t) - c_1\cdot z^3\left(\phi_y(x,y,t)+\frac{\partial w(x,y,t)}{\partial x}\right)\\
v(x,y,z,t) &= -z\phi_x(x,y,t) - c_1\cdot z^3\left(-\phi_x(x,y,t)+\frac{\partial w(x,y,t)}{\partial y}\right)\\
w(x,y,z,t) &= w(x,y,t)
\end{aligned} \quad (1)$$

where ϕ_x and ϕ_y are the rotations about the *x*- and *y*-axis, respectively (Figure 1), while $c_1=4/(3h^2)$. Cross-sectional warping is accounted with a cubic approximation of the displacement field. The Euler-Lagrange equations of motion of the HSDT are derived using the Hamilton's principle [3]:

$$\begin{aligned}
&-\bar{\bar{D}}_{12}\frac{\partial^2\phi_y}{\partial x\partial y}+\bar{\bar{D}}_{11}\frac{\partial^2\phi_x}{\partial y^2}+c_1\bar{F}_{12}\frac{\partial^3 w}{\partial x^2\partial y}+c_1\bar{F}_{11}\frac{\partial^3 w}{\partial y^3}-\bar{\bar{D}}_{66}\left(\frac{\partial^2\phi_y}{\partial x\partial y}-\frac{\partial^2\phi_x}{\partial x^2}\right)+\\
&+2c_1\bar{F}_{66}\frac{\partial^3 w}{\partial x^2\partial y}+\bar{\bar{A}}_{44}\left(\frac{\partial w}{\partial y}-\phi_x\right)-K_2\ddot{\phi}_x-c_1J_4\frac{\partial\ddot{w}}{\partial y}=0\\
&\bar{\bar{D}}_{11}\frac{\partial^2\phi_y}{\partial x^2}-\bar{\bar{D}}_{12}\frac{\partial^2\phi_x}{\partial x\partial y}-c_1\bar{F}_{11}\frac{\partial^3 w}{\partial x^3}-c_1\bar{F}_{12}\frac{\partial^3 w}{\partial x\partial y^2}+\bar{\bar{D}}_{66}\left(\frac{\partial^2\phi_y}{\partial y^2}-\frac{\partial^2\phi_x}{\partial x\partial y}\right)-\\
&-2c_1\bar{F}_{66}\frac{\partial^3 w}{\partial x\partial y^2}-\bar{\bar{A}}_{44}\left(\frac{\partial w}{\partial x}+\phi_y\right)-K_2\ddot{\phi}_y+c_1J_4\frac{\partial\ddot{w}}{\partial x}=0 \qquad (2)\\
&\bar{\bar{A}}_{44}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial x}-\frac{\partial\phi_x}{\partial y}\right)+c_1\bar{F}_{11}\frac{\partial^3\phi_y}{\partial x^3}-c_1\bar{F}_{12}\frac{\partial^3\phi_x}{\partial x^2\partial y}-c_1^2H_{11}\frac{\partial^4 w}{\partial x^4}-\\
&-2c_1^2H_{12}\frac{\partial^4 w}{\partial x^2\partial y^2}+c_1\bar{F}_{12}\frac{\partial^3\phi_y}{\partial x\partial y^2}-c_1\bar{F}_{11}\frac{\partial^3\phi_x}{\partial y^3}-c_1^2H_{11}\frac{\partial^4 w}{\partial y^4}-4c_1^2H_{66}\frac{\partial^4 w}{\partial x^2\partial y^2}+\\
&+2c_1\bar{F}_{66}\left(\frac{\partial^3\phi_y}{\partial x\partial y^2}-\frac{\partial^3\phi_x}{\partial x^2\partial y}\right)-I_0\ddot{w}+c_1^2I_6\left(\frac{\partial^2\ddot{w}}{\partial x^2}+\frac{\partial^2\ddot{w}}{\partial y^2}\right)-c_1J_4\left(\frac{\partial\ddot{\phi}_y}{\partial x}-\frac{\partial\ddot{\phi}_x}{\partial y}\right)=0
\end{aligned}$$

The higher-order stiffness coefficients $\bar{\bar{D}}_{11}, \bar{\bar{D}}_{12}, \bar{\bar{D}}_{66}, \bar{F}_{11}, \bar{F}_{12}, \bar{F}_{66}, \bar{\bar{A}}_{44}$ and mass moments of inertia K_2, J_4 are calculated by the integration of the plane stress stiffness coefficients through the plate thickness, while the above dots denote the differentiation in time. The natural (Neumann) boundary conditions of the HSDT theory are:

$$
\begin{aligned}
&\delta\phi_x: \quad -\overline{M}_{xy}n_x - \overline{M}_y n_y = \overline{M}_s^* \\
&\delta\phi_y: \quad \overline{M}_x n_x + \overline{M}_{xy} n_y = \overline{M}_n^* \\
&\delta w: \quad c_1 \left[\begin{array}{l} \left(\frac{\partial P_x}{\partial x} + 2\frac{\partial P_{xy}}{\partial y} - J_4\ddot{\phi}_y + c_1 I_6 \frac{\partial \ddot{w}}{\partial x} \right) n_x + \\ + \left(2\frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y} + J_4\ddot{\phi}_x + c_1 I_6 \frac{\partial \ddot{w}}{\partial y} \right) n_y \end{array} \right] + \overline{Q}_x n_x + \overline{Q}_y n_y = \overline{V}_n^* \\
&\frac{\partial w}{\partial x}: \quad -c_1\left(P_x n_x + P_{xy} n_y\right) = P_x^* \\
&\frac{\partial w}{\partial y}: \quad -c_1\left(P_{xy} n_x + P_y n_y\right) = P_y^*
\end{aligned}
\tag{3}
$$

The system of three coupled partial differential equations of motion (Eq. (2)) can be split into two uncoupled equations introducing the boundary layer function [24]:

$$
\psi = \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \tag{4}
$$

$$
\begin{aligned}
&\bar{\bar{D}}_{66}\nabla\psi - \bar{\bar{A}}_{44}\psi = K_2\ddot{\psi} \\
&C_1 \cdot \nabla\nabla\nabla w + C_2 \cdot \nabla\nabla w = C_3 \cdot \nabla\nabla\ddot{w} + C_4 \cdot \nabla\ddddot{w} - C_5 \cdot \nabla\ddot{w} + C_6 \cdot \ddddot{w} + C_7 \cdot \ddot{w}
\end{aligned}
\tag{5}
$$

In Eq. (5), $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes the Laplace operator, while constants C_i are:

$$
\begin{aligned}
&C_1 = \frac{c_1^2\left(\bar{\bar{D}}_{11}H_{11} - \bar{F}_{11}^2\right)}{\bar{\bar{A}}_{44}}, \quad C_2 = -D_{11}, \quad C_3 = \frac{c_1^2\left(\bar{\bar{D}}_{11}I_6 + K_2H_{11} - 2J_4\bar{F}_{11}\right)}{\bar{\bar{A}}_{44}}, \\
&C_4 = \frac{c_1^2\left(J_4^2 - K_2I_6\right)}{\bar{\bar{A}}_{44}}, \quad C_5 = I_2 + \frac{\bar{\bar{D}}_{11}I_0}{\bar{\bar{A}}_{44}}, \quad C_6 = \frac{K_2I_0}{\bar{\bar{A}}_{44}}, \quad C_7 = I_0
\end{aligned}
\tag{6}
$$

Introducing a harmonic representation of the transverse displacement and boundary layer function, the Fourier transform of Eq. (5) can be expressed as a function of the amplitudes of transverse displacement ($\hat{w}(x,y,\omega)$) and boundary layer function ($\hat{\psi}(x,y,\omega)$), in the frequency domain (according to the procedure from [22, 23]). The amplitudes of rotations $\hat{\phi}_x(x,y,\omega)$ and $\hat{\phi}_y(x,y,\omega)$ can be expressed in terms of $\hat{w}(x,y,\omega)$ and $\hat{\psi}(x,y,\omega)$ as follows (ω is the angular frequency):

$$\begin{aligned} d_1\hat{\phi}_x + d_2\nabla\hat{\phi}_x &= \overline{\overline{D}}_{66}\frac{\partial}{\partial x}\left(d_6\hat{\psi} + d_7\nabla\hat{\psi}\right) + \frac{\partial}{\partial y}\left(d_5\hat{w} + d_3\nabla\nabla\hat{w} + d_4\nabla\hat{w}\right) \\ d_1\hat{\phi}_y + d_2\nabla\hat{\phi}_y &= \overline{\overline{D}}_{66}\frac{\partial}{\partial y}\left(d_6\hat{\psi} + d_7\nabla\hat{\psi}\right) - \frac{\partial}{\partial x}\left(d_5\hat{w} + d_3\nabla\nabla\hat{w} + d_4\nabla\hat{w}\right) \end{aligned} \tag{7}$$

where the constants d_i are:

$$\begin{aligned} & d_1 = \overline{\overline{A}}_{44} - \omega^2 K_2 - c_1 J_4 - \omega^4\frac{c_1 J_4 K_2}{\overline{\overline{A}}_{44}},\; d_2 = c_1\left(\overline{F}_{11} - \omega^2\frac{\overline{F}_{11}K_2}{\overline{\overline{A}}_{44}}\right), \\ & d_3 = c_1^2\frac{\overline{F}_{11}^2 - \overline{\overline{D}}_{11}H_{11}}{\overline{\overline{A}}_{44}},\; d_4 = 2c_1\overline{F}_{11} + \overline{\overline{D}}_{11} - \omega^2 c_1^2\frac{\overline{\overline{D}}_{11}I_6 - 2J_4\overline{F}_{11}}{\overline{\overline{A}}_{44}}, \\ & d_5 = \overline{\overline{A}}_{44} + \omega^2\left(2c_1 J_4 + \frac{\overline{\overline{D}}_{11}I_0}{\overline{\overline{A}}_{44}} + \omega^2\frac{c_1^2 J_4^2}{\overline{\overline{A}}_{44}}\right),\; d_6 = 1 + \omega^2\frac{c_1 J_4}{\overline{\overline{A}}_{44}},\; d_7 = \frac{c_1\overline{F}_{11}}{\overline{\overline{A}}_{44}} \end{aligned} \tag{8}$$

The displacement field of the FSDT can be easely derived from Eq. (1) by setting the constant c_1 to zero, making the reduction from the HSDT to the FSDT very convenient. This reduction is not discussed here.

3. SOLUTION PROCEDURE

The amplitudes of the transverse displacement, the boundary layer function, as well as the rotations of a rectangular plate element can be presented as a sum of four symmetry contributions: symmetric-symmetric (SS), symmetric - anti-symmetric (SA), anti-symmetric - symmetric (AS) and anti-symmetric - anti-symmetric (AA) [26]. Following the procedure given in [22, 23], the deflections $\hat{w}(x,y,\omega)$, the rotations $\hat{\phi}_y(x,y,\omega)$ and $\hat{\phi}_x(x,y,\omega)$, the forces and moments in all symmetry contributions can be obtained. Then, the corresponding displacement and the force vectors ($\hat{\mathbf{q}}$ and $\hat{\mathbf{Q}}$) that contain displacements and forces on the boundaries $x=a$ and $y=b$ are obtained for all symmetry contributions. Using the Projection method [27, 28] as shown in [21-23], new vectors $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{Q}}$ are introduced, whose components are the coefficients in the Fourier series expansion of the displacements and forces on the boundaries $x=a$ and $y=b$. The relation between the force vector $\tilde{\mathbf{Q}}$ and the displacement vector $\tilde{\mathbf{q}}$ for each symmetry contribution is given as:

$$\tilde{\mathbf{Q}}_{IJ} = \tilde{\mathbf{K}}_D^{IJ}\tilde{\mathbf{q}}_{IJ}, \qquad I,J = S,A \tag{9}$$

where $\tilde{\mathbf{K}}_D^{IJ}$ is the dynamic stiffness matrix for considered symmetry contribution. The details regarding the SS case are givein in [22, 23] . Based on the procedure presented in Refs. [15, 19, 21-23], the dynamic stiffness matrix for completely free HSDT plate

element is obtained by using the following expression: $\tilde{\mathbf{K}}_D^G = \frac{1}{2}\mathbf{T}^T\tilde{\mathbf{K}}_o\mathbf{T}$, where $\mathbf{T}$ is the transformation matrix and $\tilde{\mathbf{K}}_o$ is the dynamic stiffness matrix obtained collecting the dynamic stiffness matrices of the four symmetry contributions [23]. The transformation matrix $\mathbf{T}$ relates the displacement vector $\tilde{\mathbf{q}}_o$ (containing the displacement vectors of all symmetry contribution) and the displacement vector $\tilde{\mathbf{q}}$ (containing the displacements and rotations along the boundary lines for completely free rectangular plate [23]). The transformation matrix is given in [23]. The size of the dynamic stiffness matrix $\tilde{\mathbf{K}}_D^G$ depends on the number of terms in the general solution M and is equal to $32M+12$.
The dynamic stiffness matrices of individual plates are assembled to compute the global dynamic stiffness matrix of plate assembly consisting of several plates. The assembly procedure is carried out in the same manner as in the FEM, except the plates are connected along boundary lines instead at nodes. The procedure was demonstrated in the previous works of Kolarević et al [22, 23]. The boundary conditions are applied to the global dynamic stiffness matrix by removing the rows and columns corresponding to the components of constrained displacement projections. The boundary conditions used in the numerical verification of the model are:

- Simply supported (S): $w = 0$ and $\phi_x = 0$ for the edge parallel to the y-axis and $w = 0$ and $\phi_y = 0$ for the edge parallel to the x-axis;
- Clamped (C): $w = \phi_x = \phi_y = w_{,x} = 0$ for the edge parallel to the y-axis, and $w = \phi_y = \phi_x = w_{,y} = 0$ for the edge parallel to the x-axis;
- Free (F): all displacements (w, ϕ_x, ϕ_y, $w_{,x}$ and $w_{,y}$) are $\neq 0$.

The proposed shear deformable dynamic stiffness elements have been implemented in the original program coded in MATLAB [29] and used for the numerical validation.

4. NUMERICAL VALIDATION AND DISCUSSION

The applicability of the proposed model is illustrated considering square sandwich (3-layer) panels, having the dimensions $2a\times2b = 2.0\times2.0m$ and the total thickness $h = 0.2m$. The face thicknesses are $t_f = 2mm$ ($h/t_f = 100$). The panels are clamped along all edges and composed of two rigid isotropic faces ($E_f = 100GPa$) and core having the Young's modules varying from 0.2-100GPa (where $E_f/E_c = 1$ corresponds to the isotropic plate). The Poisson's ratio and the mass density of both faces and core are constant: $\nu_f = \nu_c = 0.3$ and $\rho_f = \rho_c = 3000kg/m^3$. The plates are analized using four different numerical models: FSDT dynamic stiffness element - FSDT DSM (shear correction factor k=5/6 and 2 elements), HSDT dynamic stiffness element - HSDT DSM (2 elements), 4-node GLPT layered rectangular finite element with reduced integration - GLPT P4R (20×20 elements) (see [21, 22]) and 4-node conventional shell element with reduced integration (S4R), built in the commercial software Abaqus (100×100 elements). In the calculations performed by the dynamic stiffness method, $M = 9$ terms in the series expansion were used to obtain the accurate solution, according to the convergence studies presented in [22, 23]. The first four natural frequencies are illustrated in Figure 2.

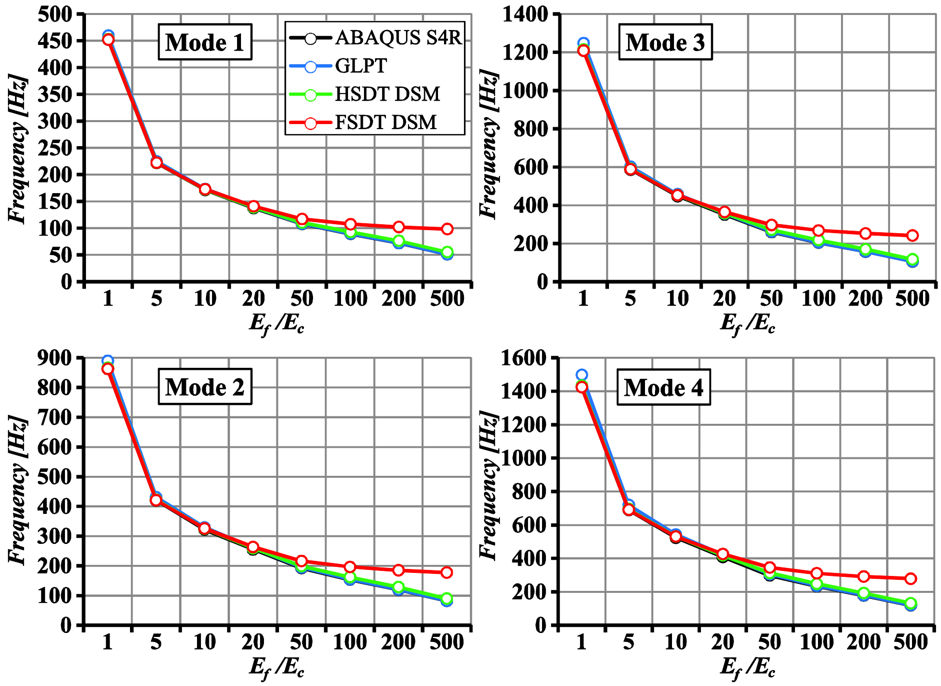

Figure 2. Natural frequencies of sandwich panels with variable E_f/E_c ratios (h/t_f = 100)

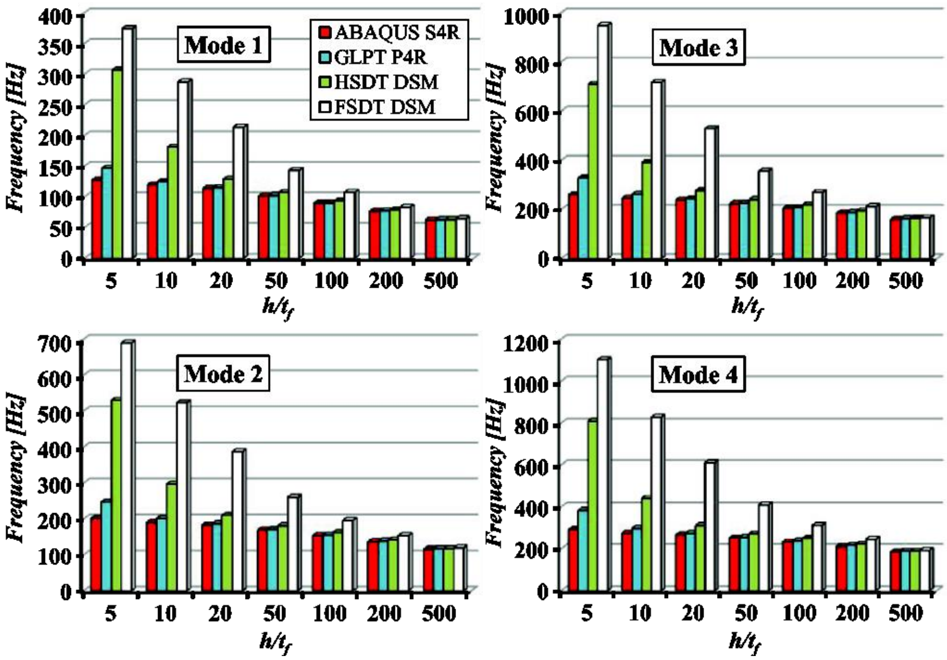

Figure 3. Natural frequencies of sandwich panels with variable h/t_f ratios (E_f/E_c = 100)

In the second part of this benchmark example, the influence of the face thickness on natural frequencies of sandwich panels isanalysed. The Young's modules of faces and core are fixed: E_f = 100GPa, E_c = 1GPa, while the ratios h/t_f are varying: h/t_f = {5, 10, 20, 50, 100, 200 and 500}. The results are presented in Figure 3.

5. CONCLUSIONS

The development of the dynamic stiffness matrix for a completely free rectangular multi-layer plate element based on the HSDT has been presented in this study, implemented in a MATLAB computer code and applied in the free vibration analysis of sandwich panels. The numerical study presented in this paper proves the ability of the proposed HSDT-based model to accurately predict the dynamic behavior of sandwich panels, with some restrictions regarding the h/t_f and E_f/E_c ratios. For h/t_f =100, the model accurately predicts the fundamental frequencies for all considered E_f/E_c ratios, varying from the isotropic plate (E_f/E_c=1) to typical sandwich panel (E_f/E_c=500). The discrepancy in the results is detected when the quality of the core layer decreases (E_f/E_c>20). For all considered cases, the results obtained using the GLPT P4R layered finite elements are in excellent agreement with the finite element solution from Abaqus. Generally, better agreement is obtained for lower modes of vibration. The FSDT dynamic stiffness element exhibits higher stiffness in comparison with other models due to the simplifications regarding the transverse shear deformation. For E_f/E_c =100, the proposed model accurately predicts the fundamental frequencies if h/t_f >50.

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СЛОБОДНЕ ВИБРАЦИЈЕ ПРАВОУГАОНИХ ИЗОТРОПНИХ ВИШЕСЛОЈНИХ ПЛОЧА ПРИМЕНОМ МЕТОДЕ ДИНАМИЧКЕ КРУТОСТИ

Резиме: *У овом раду приказане су динамичке матрице крутости за правоугаону, (трансверзално) изотропну, једнослојну и вишеслојну плочу са произвољним граничним условима, које су примењене у анализи слободних вибрација. Динамичке матрице крутости изведене су за Reddy-еву смичућу теорију плоча вишег реда (HSDT), као и за Mindlin-ову теорију плоча (FSDT). Динамичке матрице крутости садрже параметре крутости и масе разматраних плоча и могу се сабирати на сличан начин као у Методи коначних елемената (МКЕ). Разматран је утицај односа дебљине површинског слоја и језгра, као и утицај односа модула еластичности површинског слоја и језгра код сендвич плоча, као и утицај деформације смицања на слободне вибрације сендвич плоча. Резултати добијени применом динамичких матрица крутости HSDT и FSDT елемената су упоређени са резултатима комерцијалног програмског пакета Abaqus и резултатима заснованим на слојевитом правоугаоном коначном елементу са 4-чвора. Предложени модели омогућавају прецизно одређивање динамичког одговора система правоугаоних плоча са произволјним граничним условима.*

Кључне речи: *слободне вибрације слојевитих плоча, метода динамичке крутости, Mindlin-ова теорија плоча, смичућа теорија плоча вишег реда*

ULTIMATE STRENGTH OF LONGITUDINALLY STIFFENED PLATE GIRDERS UNDER COMPRESSION

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Summary: *In this paper a unified frame for quasi-static and dynamic inelastic buckling and ultimate strength of uniformly compressed longitudinally stiffened plate girders is presented. The finite strip method is used in structural analysis. The nonlinear behavior of the material is modelled using the rheological-dynamical theory. According to this theory, a very complicated nonlinear problem in the inelastic range of strains is solved as a simple linear dynamic one. The orthotropic constitutive relations for inelastic buckling and a new modulus iterative method for the solution of nonlinear equations are derived in previous papers and the extensive numerical application is presented here.*

Keywords: *Finite strip method, rheological-dynamical theory, ultimate strength*

1. THEORETICAL BACKGROUND

The purpose of this paper is to investigate ultimate limit state (*ultimate strength*) of uniformly compressed plate girders with longitudinally stiffeners under quasi-static and dynamic loading. These are the structures which are generally made by joining flat plates at their edges. Some important subsets of these systems are those composed of structures with essentially prismatic form, with or without stiffeners, such as the ones used in column members, stiffened slabs and box girders. Analysis of the behavior of these structures is here performed using the finite strip method (FSM). The FSM approximation of displacement field is based on beam eigenfunctions, which are derived as the solution of the differential equation of beam transverse vibration, and proved to be an efficient tool for analyzing a great deal of structures for which both geometry and

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material properties can be considered as constant along a main direction. This method was pioneered by Cheung [1], who combined the plane elasticity and the Kirchhoff plate theory. Wang and Dawe [2] have applied the elastic geometrically nonlinear FSM to the large deflection and post-overall-buckling analysis of diaphragm-supported plate structures. Also, the FSM is very rapidly increasing in popularity for the analysis of thin-walled structures. Kwon and Hancock [3] developed the spline FSM to handle local, distortional and overall buckling modes in post-buckling range. Interaction of two types of column failure (buckling) in thin-walled structures, local and global (Euler) column buckling, may generate an unstable coupled mode, rendering the structure highly imperfection sensitive. The geometrically nonlinear harmonic coupled finite strip method (HCFSM) [4, 5, 6] is also one of the many procedures that can be applied to analyze the large deflection of folded-plate structures and buckling-mode interaction in thin-walled structures. For these problems, only geometrically nonlinear terms such as square derivatives of transverse displacement w need to be included (von Karman approach). An analysis of the buckling-mode interaction is carried out using the HCFSM in [7], taking into account the visco-elastic behavior of material.

If uniformly compressed plate girders or thin-walled girders undergo inelastic deformation, these structures generally sustain both nonlinearities, geometrically nonlinear effects and a nonlinear behavior of the material caused by inelastic deformation. A mathematical-physical analogy named the rheological-dynamical analogy (RDA) has been proposed in explicit form to predict a range of inelastic and time-dependent problems related to one-dimensional members, such as buckling, fatigue etc. [8, 9]. Consequently, the RDA inelastic theory enables the engineer concerned with materials (*for various quasi-static and dynamic structural problems*) to utilize simple models, expressible in a mathematically closed form, to predict the stress-strain behavior. The main results in the paper [8] are obtained in regard to inelastic buckling in the short to intermediate column range taking into account the governing RDA modulus. However, wide-flange column members or thin-walled girders fail as continua by first developing local or global buckling modes, which may develop into plastic mechanisms and failure, which is why two-dimensional (2D) or three dimensional (3D) analyses must be used.

The proposed approach combines the RDA and damage mechanics [10] to solve the nonlinear problem of plate girders under compression using 2D analysis in the frame of the FSM. The one-dimensional RDA modulus is used to obtain one simple continuous modulus function and a stress-strain curve [11]. When the critical stress exceeds the limit of elasticity, the first iteration of the modulus provides the Hencky loading function and the von Mises yield stress, whereas the next ones involve the strain-hardening of the material through visco-plastic flow. At the end of the iterations the member failure occurs. The key global parameters, such as the creep coefficient, Poisson’s coefficient and the damage variable are functionally related. However, it is a fact that material damage growth is accompanied by an emission of elastic waves which propagate within the bulk of the material [12]. Consequently, a 3D analysis of the propagation of mechanical waves is used in this paper. The elastic properties of steel and aluminum determined on test cylinders and based on longitudinal resonance frequencies [13], are used in the numerical applications. For the analysis of plate girders using the FSM, an inelastic isotropic 2D constitutive matrix is derived starting from the one-dimensional state of stress. Although the quasi-static and dynamic constitutive relations are derived for isotropic materials, different stress components induce orthotropy in the material through the RDA modulus-stress dependence. The nonlinear term is the stiffness matrix, which depends on the inelastic orthotropic constitutive matrix. Because of that, a new modulus iterative method for the solution of nonlinear equations is used. In the case of inelastic buckling of rectangular slabs it has been demonstrated that convergence of the method is fast and that it gives satisfactorily accurate solutions with only several iterations [14]. Presented numerical algorithm is implemented in software package BASS [15] and the exhaustive numerical study is performed. Obtained solutions for known modulus are compared with the ones from CUFSM [16], and they excatly match.

2. NUMERICAL APPLICATION

A theoretical investigation into the effectiveness of a stiffener against the ultimate strength of a stiffened plate girders under thrust is carried out. The transition from the various buckling modes by changing the plate/stiffener proportions for various stiffening configurations is shown. Four models are analyzed for two different materials, steel and aluminum. For each material, plate without stiffeners is analyzed first. Then, plate girders with one, two and three stiffeners are examined and buckling curves are given.
Series of the buckling analyses, the elastic, visco-plastic (VP) and failure (ultimate strength) are performed on the stiffened plate girders under quasi-static and dynamic loading. The panels (*Model 1*, *2* and *3*) were divided into finite strips as shown in Fig. 1.

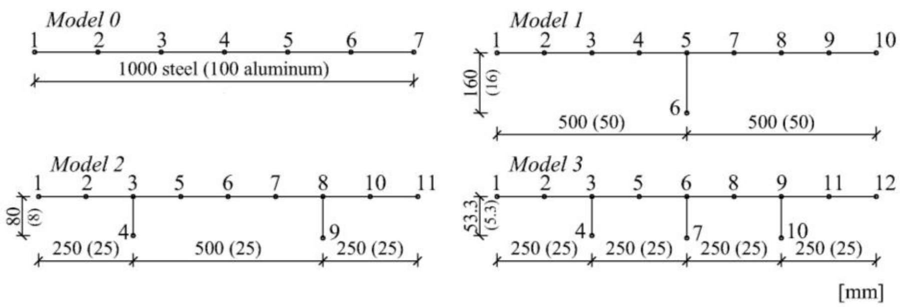


Figure 1. Models of steel and aluminum stiffened panels with nodal lines

Figures 2 and 3 illustrate the buckling curves (critical stress versus length/width ratio, a/b) for steel and aluminum slabs. In order to obtain the inelastic quasi-static critical stresses, the Euler formula for buckling of an isolated plate strip was employed to find the structural-material constant of a plate. The convergence of the failure stresses for all a/b ratios is obtained using only six or seven iterations. The first iteration gives the visco-plastic yield stress. An excellent agreement with the generalized beam theory (GBT) is observed [14], in which the values of both ratios E/E_T and E/E_S (E_T is the unixial tangent modulus and E_S is the secant modulus) depend on the applied stress level and are obtained using the uniaxial stress-strain law which adequatly describes the material behavior along the fundamental path. The GBT theory used the Ramberg-Osgood curve. The dynamic visco-plastic and failure buckling curves of steel and aluminum slabs for relative angular frequencies (RAF) 1 and 10 are also presented. All dynamic stresses are below the elastic critical stresses. The reason for that is the cyclic stress variation in the material under which the visco-plastic effects like viscous damping are developed.

Figures 4 to 9 present the quasi-static elastic, visco-plastic and failure buckling curves for three panels (*Model 1*, *2* and *3*) made of steel and aluminum.

For buckling of a stiffened plate (panel), it is well known that there exists a minimum stiffness ratio of a stiffener to the plate, $(EIs/bD)B_{min}$, which gives the maximum limiting value of the buckling strength. Considering the ultimate strength it was confirmed that there exists a significant stiffness ratio of a stiffener to the plate, $(EIs/bD)U_{min}$, similar to $(EIs/bD)B_{min}$ for the buckling strength.

The effect of various parameters like panel geometry, stiffening scheme, stiffener size and position are considered in quasi-static and dynamic stability analysis of stiffened panels. Figure 10 presents typical buckling modes.

The proposed appoach for quasi-static and dynamic inelastic buckling and for global failure analysis combines the FSM linear stability analysis and the RDA inelastic theory. The RDA inelastic theory is a new theory for the simulation of inelastic material behavior, alternative to other theory based on nonlinear fracture mechanics, plasticicity theory or damage mechanics previously published in the literature.

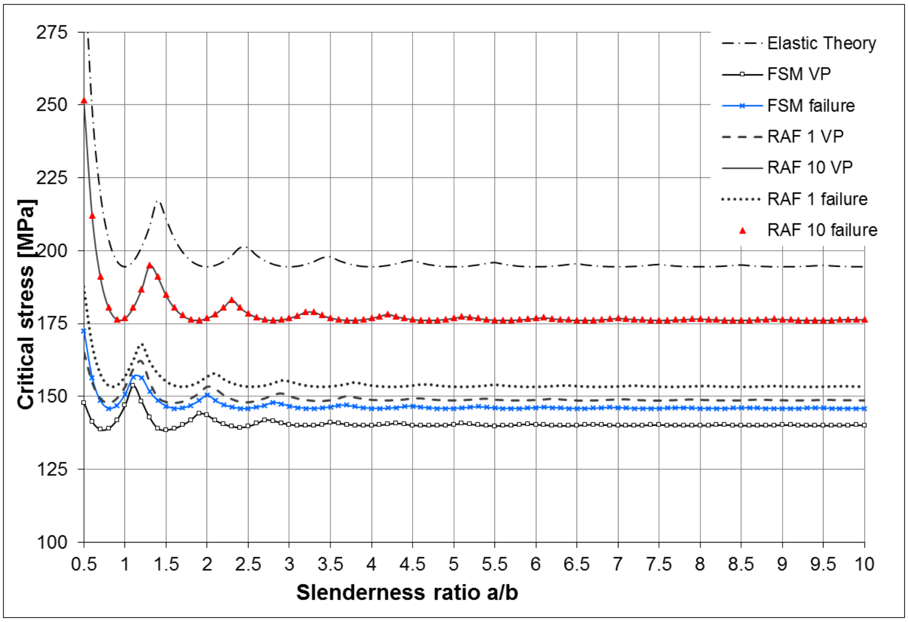


Figure 2. Quasi-static and dynamic elastic, visco-plastic and failure buckling curves for a steel slab - Model 0

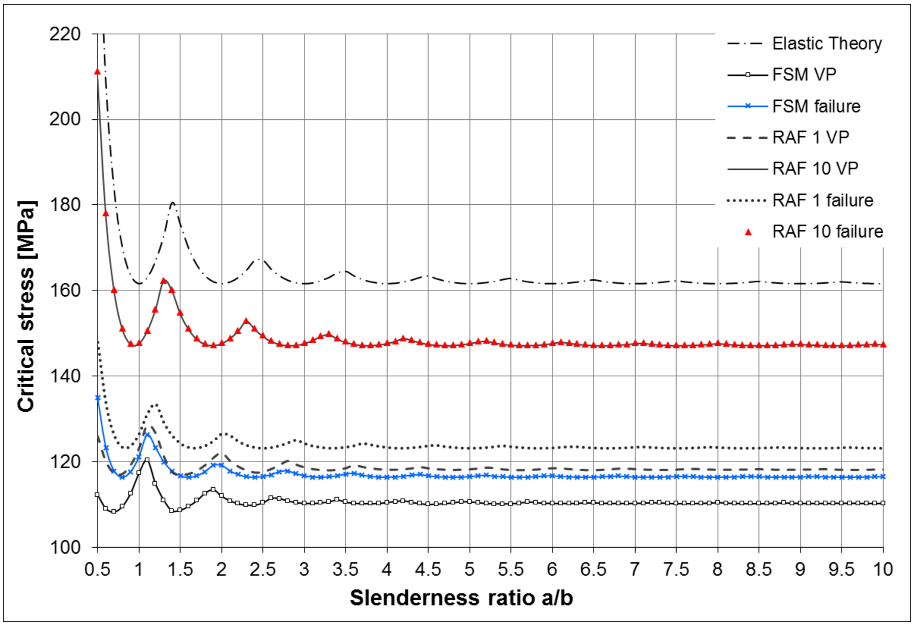


Figure 3. Quasi-static and dynamic buckling curves for a aluminum slab - Model 0

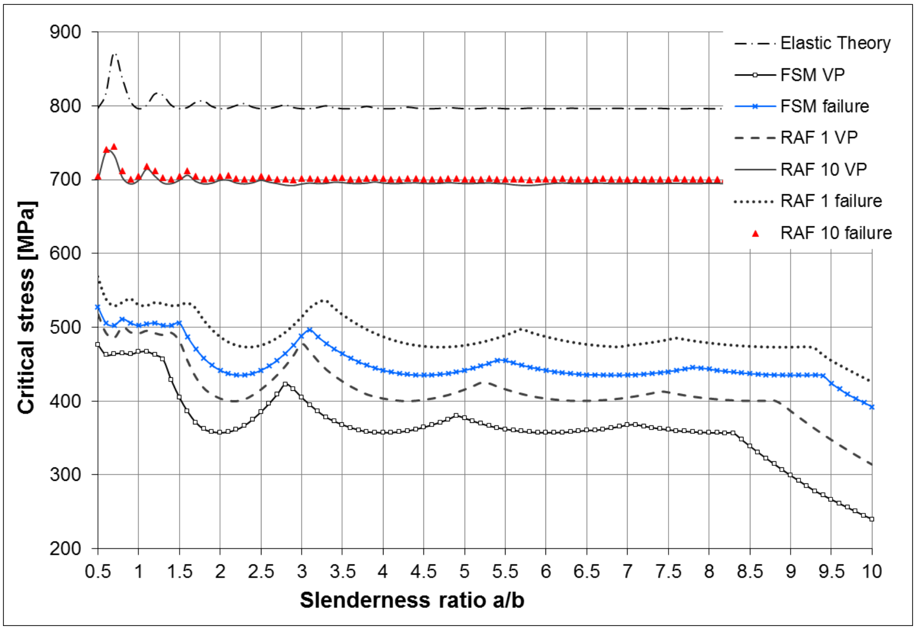


Figure 4. Quasi-static and dynamic buckling curves for a steel panels - Model 1

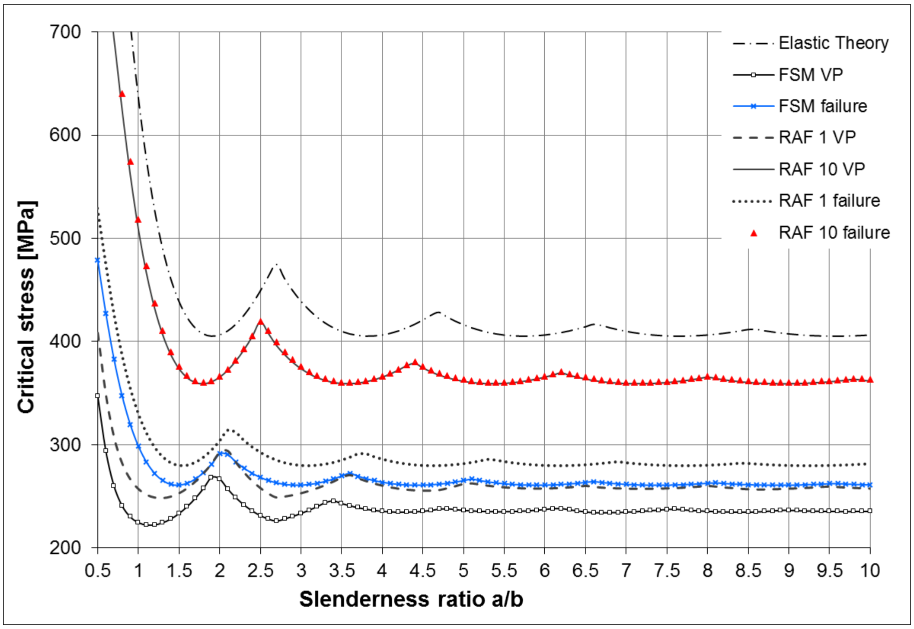


Figure 5. Quasi-static and dynamic buckling curves for a steel panels - Model 2

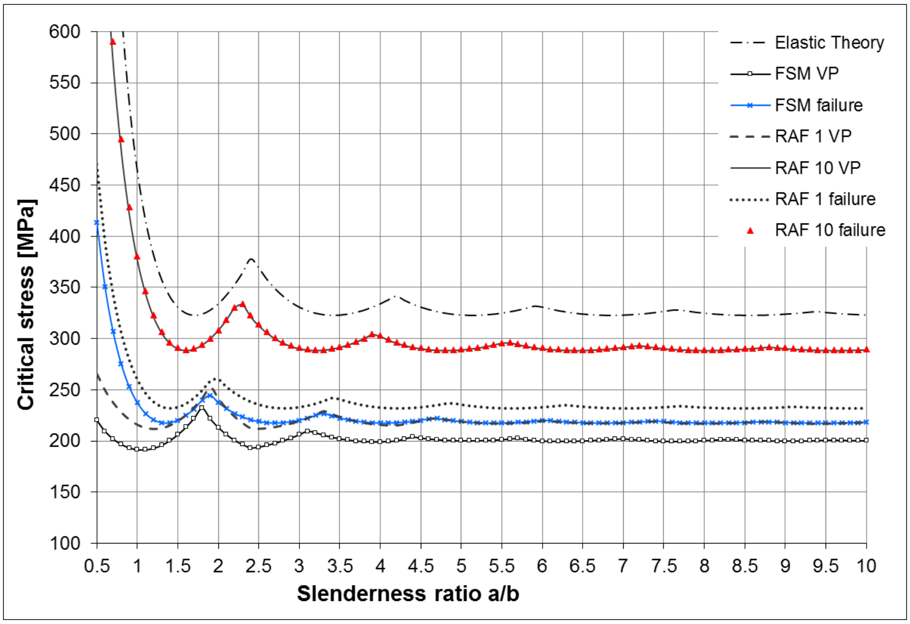


Figure 6. Quasi-static and dynamic buckling curves for a steel panels - Model 3

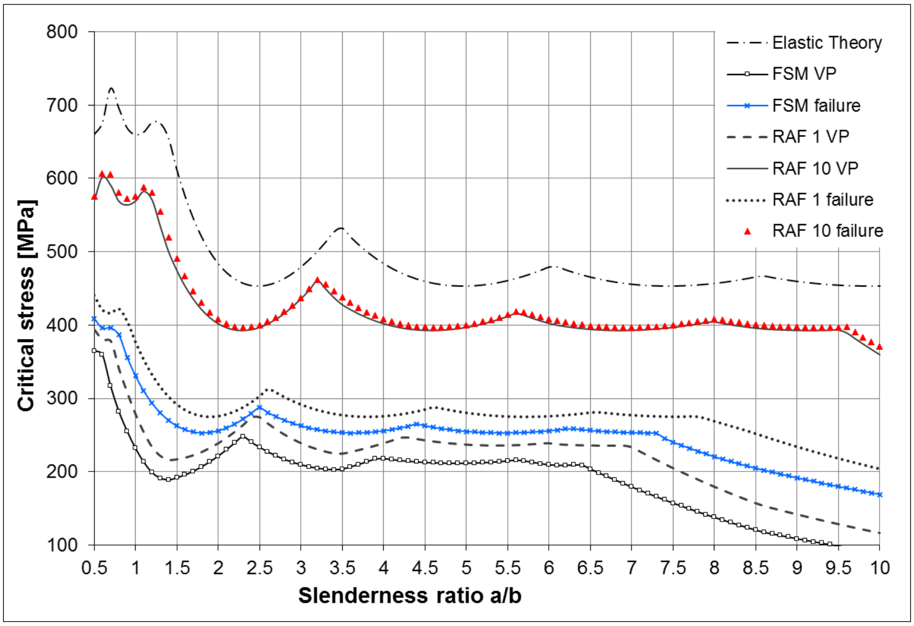


Figure 7. Quasi-static and dynamic buckling curves for a aluminum panels - Model 1

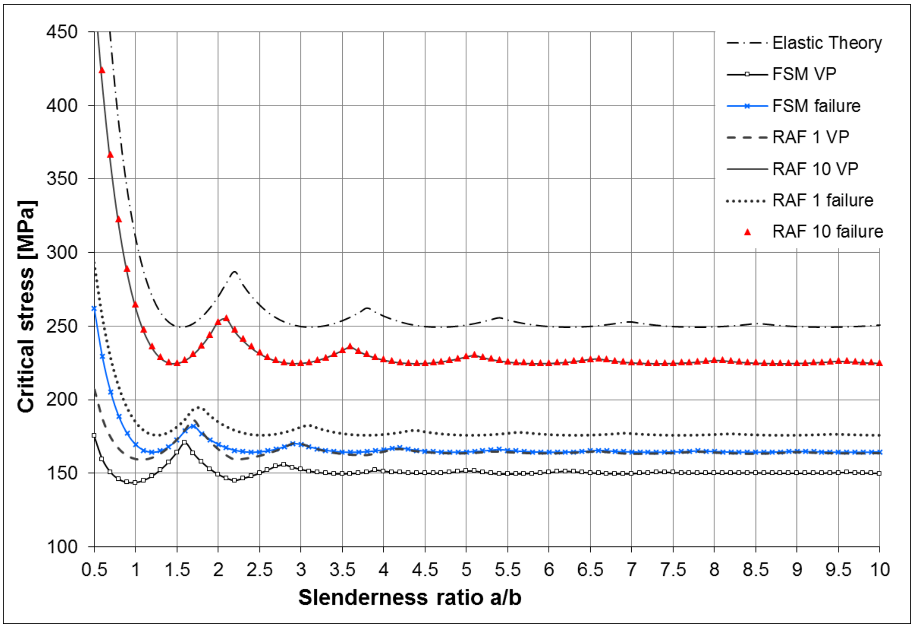


Figure 8. Quasi-static and dynamic buckling curves for a aluminum panels - Model 2

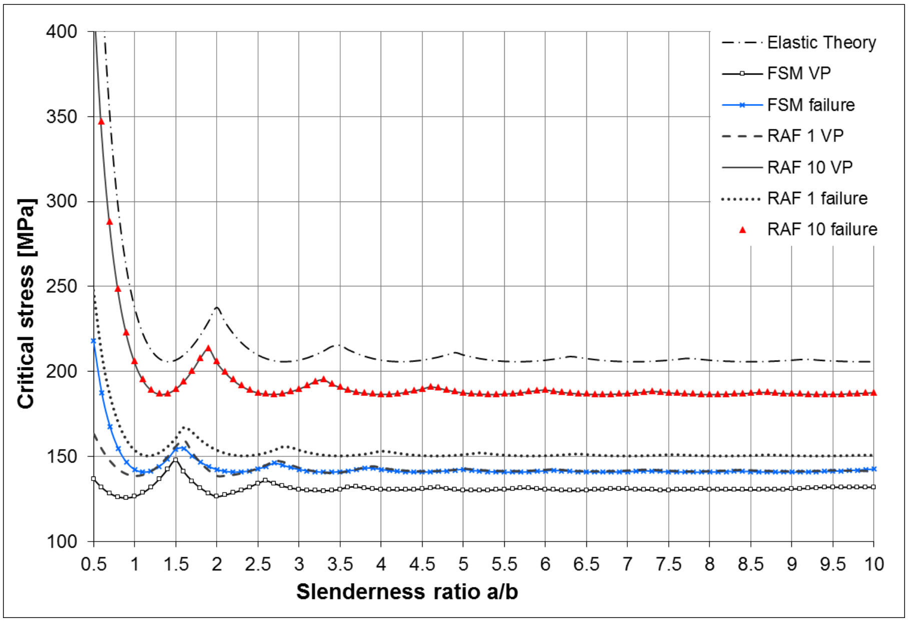
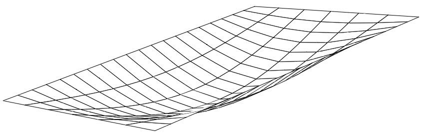
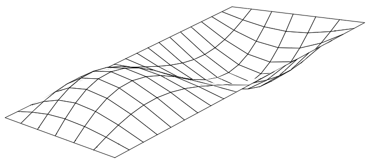


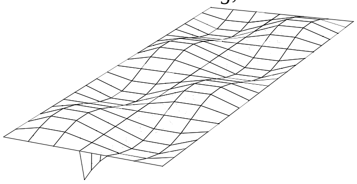
Figure 9. Quasi-static and dynamic buckling curves for a aluminum panels - Model 3



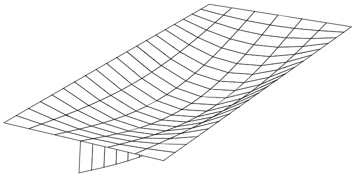
Steel elastic buckling, l= 1.4m



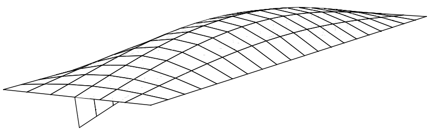
Steel failure buckling



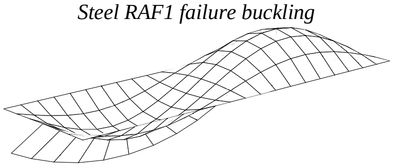
Steel elastic and RAF10 failure, l= 2m



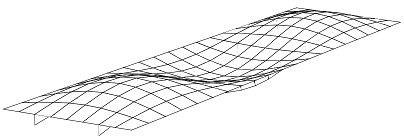
Steel RAF1 failure buckling



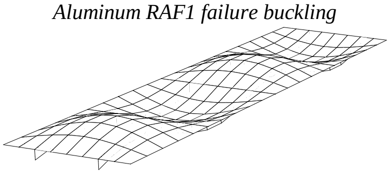
Aluminum elastic and RAF10 failure, l= 0.3m



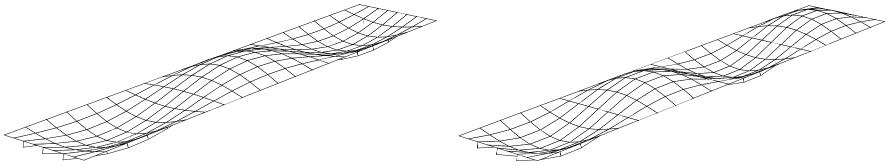
Aluminum RAF1 failure buckling



Steel elastic and RAF10 failure, l= 5.5m



Steel RAF1 failure buckling



Steel elastic and RAF10 failure, l= 5m *Steel RAF1 failure buckling, l= 5m*

Figure 10. Typical buckling modes of stiffened panel

3. CONCLUSION

Derived theory is composed into algorithm and implemented into software package. Intensive numerical study is performed and the infulence of various parameters is examined. The most important results obtained in this paper are the failure stresses by which the failure buckling curves are determined. The failure stress present the ultimate strength of longitudinally stiffened plate girder under compression.

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ГРАНИЧНА ЧВРСТОЋА ПРИТИСНУТИХ ПОДУЖНО УКРУЋЕНИХ ПЛОЧАСТИХ НОСАЧА

Резиме: *У раду је дат заједнички оквир за квазистатичко и динамичко нееластично извијање и граничну чврстоћу подужно укрућених плочастих носача при једнако расподијељеном притиску на крајевима носача. Конструкције су моделиране примјеном метода коначних трака. Материјална нелинеарност је укључена реолошко-динамичком теоријом. Према овој теорији, компликован нелинеаран проблем у подручју нееластичних деформација је рјешен као једноставан линеаран динамички проблем. У претходним радовима су изведене ортотропне конститутивне релације за нееластично извијање као и поступак за итеративно рјешавање нелинеарних једначина, док је овдје приказана исцрпна нумеричка анализа.*

Кључне речи: *Метод коначних трака, реолошко-динамичка теорија, гранична чврстоћа*