THE LATTICE-BOLTZMANN METHOD IN COMPUTATIONAL FLUID MECHANICS

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Summary: *The lattice-Boltzmann method (LBM) is a new method in computational fluid mechanics. While traditional numerical methods directly discretize and solve the macroscopic equations of fluid mechanics, the LBM solves a discrete kinetic equation that reproduces the equations of fluid mechanics in the macroscopic sense. This paper presents in some detail the concept of the distribution function which is essential for the LBM, the Boltzmann equation and the Chapman-Enskog expansion used to reproduce the macroscopic equations of fluid mechanics from the Boltzmann equation. The paper also presents the procedure used for discretizing the Bolzmann equation in velocity and physical space in order to obtain the lattice-Boltzmann equation.*

Keywords: *lattice-Boltzmann method, Chapman-Enskog expansion, computational fluid mechanics*

1. INTRODUCTION

The transport of mass, momentum and heat are usually modelled using one of the two approaches, the continuum approach where differential equations are derived using the conservation of mass, momentum and energy for an infinitesimal control volume. Since there in no general solution of the fluid flow equations, the approximate solution is obtained by finite difference, finite volume, finite element or a similar approach. In order to solve the governing equations they are discretized in elements, grid or a volume, depending on the selected approach.

Another approach is to consider that the medium is made of small particles colliding with each other. Opposite to the first approach where we had the macroscopic scale, in

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this case we are dealing with particles on the microscopic scale, hence it is necessary to identify the forces acting among the particles and solve the Newton's second law equation. It is required at each time step to define the location and velocity of all particles. This is the molecular dynamic simulation. At this level there is no definition of pressure, temperature, viscosity and similar properties.
As a solution for the gap between the micro and macro scale models, the lattice-Boltzmann method is proposed. The lattice-Boltzmann method was developed from the cellular automata and the lattice-gas automata [5]. The idea is that instead of considering the behavior of each of the particles alone, one can consider the behavior of a collection of particles as a unit, where the property of the collection is presented via a distribution function. That is, the distribution function acts as a representative for the collection of particles. This is called the meso-scale [1,2]. Since the Boltzmann equation considers the motion of fluids in meso-scale, it relies on statistical mechanics to determine how particles of the microscopic scale determine the macroscopic properties of fluids.
This paper presents a short survey of the lattice-Boltzmann method in order to enable a more thorough understanding.

2. KINETIC THEORY OF GASES

The distribution function $f(\vec{x},\vec{c},t)$ indicates the density of particles with position $\vec{x}$ and velocity $\vec{c}$ at time *t* and is used to find the macroscopic properties of the fluid such as density, velocity and internal energy. These properties are found as moments of the distribution function that is weighted with a function of $\vec{c}$ and integrated over the velocity-space. By integrating over all velocities we obtain the mass density ρ as

$$\rho(\vec{x},t) = \int f(\vec{x},\vec{c},t)\, d\vec{c}\,. \tag{1}$$

The momentum density is obtained by weighting the distribution function with the velocity $\vec{c}$ and integrating over the velocity field,

$$\rho\vec{u}(\vec{x},t) = \int \vec{c}\, f(\vec{x},\vec{c},t)\, d\vec{c}\,. \tag{2}$$

In Eq.(2), $\vec{u}$ denotes the average particle velocity that corresponds to the fluid velocity. In order to allow better understanding of the forthcoming mathematical operation, in addition to Eq.(1) and (2) it is necessary to define the connection of the particle velocity $\vec{c}$ and the fluid velocity $\vec{u}$,

$$\vec{c} = \vec{u} + \vec{v} \tag{3}$$

where $\vec{v}$ is the peculiar velocity.

2.1. THE MAXWELL-BOLTZMANN DISTRIBUTION

The collision of two particles will cause a change of their velocities which will depend on the position and velocities before the collision as well as the intermolecular forces during the collision. We will assume that the collision causes the particles' velocities to be evenly distributed around $\vec{u}$. This leads to the conclusion that the distribution function depends only on the peculiar velocity $\vec{v}$. Therefore, the equilibrium distribution function can be denoted as $f^{(0)}\left(|\vec{v}|\right)$, and can also be presented through its coordinates $f^{(0)}\left(|\vec{v}|\right) = f_x^{(0)}\left(v_x\right) f_y^{(0)}\left(v_y\right) f_z^{(0)}\left(v_z\right)$. The two functions, that have this property, are the logarithmic function and the exponential function. For a constant velocity $|\vec{v}|^2$, the function $f^{(0)}\left(|\vec{v}|\right)$ is constant, and the following expression applies, $\ln f_x^{(0)}\left(v_x\right) + \ln f_y^{(0)}\left(v_y\right) + \ln f_z^{(0)}\left(v_z\right) = const.$ The last equation can only be solved by presenting the equilibrium distribution function for different directions as shown in Eq.(4), where *a* and *b* mark constants that are independent of the peculiar velocity and are determined through the moments of the distribution function.

$$\ln f_x^{(0)}\left(v_x\right) = a - b v_x \quad \Rightarrow \quad f_x^{(0)}\left(v_x\right) = e^a e^{-b v_x} \quad \Rightarrow \quad f^{(0)}\left(|\vec{v}|\right) = e^{3a} e^{-b|\vec{v}|^2}. \tag{4}$$

Using the expression for the equilibrium distribution function in Eq.(4), one can find the constants *a* and *b*. Afres some algebra, this yields the equilibrium distribution function in the following form

$$f^{(0)}\left(|\vec{v}|\right) = \rho\left(\frac{3}{4\pi e}\right)^{3/2} \exp\left(\frac{-3|\vec{v}|^2}{4e}\right). \tag{5}$$

The equilibrium distribution given by Eq.(5) is called the Maxwell-Boltzmann distribution [3].

2.2. THE BOLTZMANN EQUATION

The distribution function is a function of $\vec{x}$, $\vec{c}$ and ***t***, therefore its total differential can be given as

$$\frac{df}{dt} = \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial f}{\partial c_i}\frac{dc_i}{dt} + \frac{\partial f}{\partial t}. \tag{6}$$

where dx_i/dt presents the particles' velocity c_i, dc_i/dt is the particles' acceleration that is defined as the body force density by the Newton's second law, $dc_i/dt = F_i/\rho$. Additionally, if the left side of Eq.(6) is zero, the equation becomes an advection equation that describes the propagation of the particle distribution f with velocity $\vec{c}$ without a collision. Generally, df/dt represents a source term that indicates the

change of the distribution function due to collisions that are causing particles to change their directions. Writing Eq.(6) in vector form, using notation $\Omega(f)$ for the collision operator, we attain the Boltzmann equation

$$\Omega(f) = \frac{\partial f}{\partial t} + \vec{c} \cdot \nabla f + \frac{\vec{F}}{\rho} \cdot \nabla_c f \ . \tag{7}$$

Examining Eqs.(6) and (7), one can deduce that the collision operator is, by its' definition, equal to the total derivative of the distribution function, i.e. df/dt. The collision operator itself can take many forms as long as it fulfils the conditions given with Eq.(8). It is indeed essential that the collision operator conserves mass, momentum and energy when integrated over the whole velocity space.

$$\begin{aligned} &\text{conservation of mass:} && \int \Omega(f)\, d\vec{c} = 0, \\ &\text{conservation of momentum:} && \int \vec{c}\, \Omega(f)\, d\vec{c} = 0, \\ &\text{conservation of energy:} && \int |\vec{c}|^2\, \Omega(f)\, d\vec{c} = 0. \end{aligned} \tag{8}$$

The collision operator must also ensure that the distribution function evolves towards the equilibrium. The operator that is most often used by authors is the BGK collision operator [1,2,5]

$$\Omega(f) = -\frac{1}{\tau}\left(f - f^{(0)}\right), \tag{9}$$

proposed by Bhatnagar, Gross and Krook where τ marks the relaxation time.

3. MACROSCOPIC CONSERVATION EQUATIONS

Taking the zeroth moment of all terms in the Boltzmann equation (7), the general form of the solution is obtained by

$$\frac{\partial}{\partial t}\int f\, d\vec{c} + \frac{\partial}{\partial x_i}\int c_i\, f\, d\vec{c} + \frac{F_i}{\rho}\int \frac{\partial f}{\partial c_i} d\vec{c} = \int \Omega(f)\, d\vec{c} \ . \tag{10}$$

The integrals on the left side of Eq.(10) are: the density moment from Eq.(1), the momentum moment Eq.(2), while the third term is zero (the multidimensional integration by parts was used for deriving this equation). The right side of Eq.(10) is zero from the mass conservation constraint given by Eq.(8). Finally Eq.(10) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 . \tag{11}$$

The momentum equation is derived similarly, by taking the first moment of the Boltzmann equation,

$$\frac{\partial}{\partial t}\int c_i \; f \, d\vec{c} + \frac{\partial}{\partial x_j}\int c_i \, c_j \; f \, d\vec{c} + \frac{F_j}{\rho}\int c_i \frac{\partial f}{\partial c_j} d\vec{c} = \int c_i \, \Omega(f) \, d\vec{c} \tag{12}$$

where the first term on the left-hand side can be transformed into momentum density. The second term on the left-hand side of Eq.(12) is

$$\int \left(u_i \, u_j + u_i \, v_j + v_i \, u_j + v_i \, v_j \right) f \, d\vec{c} = \rho u_i \, u_j - \left(-\int v_i \, v_j \; f \, d\vec{c} \right) = \rho u_i \, u_j - \sigma_{ij} , \tag{13}$$

where the first term denotes the macroscopic flow of momentum and the second term is the diffusion of momentum. The third term on the left-hand side of Eq.(12) translates to negative density using multidimensional integration by parts, while the right-hand side of Eq.(13) becomes zero using the momentum conservation criteria. Hence, the momentum equation is as follows

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i \, u_j}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + F_i . \tag{14}$$

where $\sigma_{ij} = -\int v_i \, v_j \; f \, d\vec{c}$ marks the stress tensor. It should be noted that the stress tensor defined here does not account for the turbulent stress in the fluid, as it will be shown later on.

3.1. THE CHAPMAN-ENSKOG EXPANSION

The distribution function can be expanded around equilibrium with terms in increasing order of Knudsen number *Kn*,

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots, \tag{15}$$

where ε indicates that $f^{(1)} / f^{(0)} = O(Kn)$, $f^{(2)} / f^{(0)} = O\left(Kn^2\right)$. The Boltzmann equation (7) using the collision operator Eq.(9) can be written as

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + \frac{F_i}{\rho}\frac{\partial f}{\partial c_i} = -\frac{1}{\tau}\left(f - f^{(0)} \right). \tag{16}$$

Substituting Eq.(15) in Eq.(16) gives

$$\left(\frac{\partial}{\partial t}+c_i\frac{\partial}{\partial x_i}+\frac{F_i}{\rho}\frac{\partial}{\partial c_i}\right)\left(f^{(0)}+\varepsilon f^{(1)}+\varepsilon^2 f^{(2)}+\cdots\right)=-\frac{1}{\varepsilon\tau}\left(\varepsilon f^{(1)}+\varepsilon^2 f^{(2)}+\cdots\right), \tag{17}$$

where division with ε has the sole purpose of expressing the order of magnitude of the right to left side terms ratio. Introducing the assumption that terms of different order of *Kn* are semi-independent, Eq. (17) can be thought of as hierarchy of equations, one equation at $O\left(Kn^0\right)$, one at $O\left(Kn^1\right)$ etc. The presented expansion technique is called the Chapman-Enskog expansion.

Since f and $f^{(0)}$ have the same moments of density, momentum and energy, it can be assumed that the contribution of the higher order terms to these moments are zero. Therefore, at the $O\left(Kn^0\right)$ the expanded Boltzmann equation (17) becomes

$$\frac{\partial f^{(0)}}{\partial t}+\vec{c}\cdot\nabla f^{(0)}+\frac{\vec{F}}{\rho}\cdot\nabla_c f^{(0)}=-\frac{f^{(1)}}{\tau} \tag{18}$$

By dividing Eq.(18) with $f^{(0)}$, rearanging it and using Eq.(3) and the using the rule for differentiating the normal logarithm the following equation is attained

$$\frac{f^{(1)}}{f^{(0)}}=-\tau\left[\frac{\partial\left(\ln f^{(0)}\right)}{\partial t}+\left(u_i+v_i\right)\frac{\partial\left(\ln f^{(0)}\right)}{\partial x_i}+\frac{F_i}{\rho}\cdot\frac{\partial\left(\ln f^{(0)}\right)}{\partial c_i}\right] \tag{19}$$

The logarithm of the equilibrium distribution function Eq.(5) is

$$\ln f^{(0)}=\frac{3}{2}\ln\left(\frac{3}{4\pi}\right)+\ln\rho-\frac{3}{2}\ln e-\left(\frac{3}{4e}\right)\left|\vec{c}-\vec{u}\right|^2. \tag{20}$$

The derivatives in Eq.(19) are determined throughout the use of the chain rule for the time and space derivatives, by keeping in mind that the distribution function is uniquely defined by the conserved quantities of density, momentum and energy.

$$\frac{f^{(1)}}{f^{(0)}}=-\tau\left[\begin{array}{l}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+\left(u_i+v_i\right)\frac{\partial\rho}{\partial x_i}\right)+\frac{3v_j}{2e}\left(\frac{\partial u_j}{\partial t}+\left(u_i+v_i\right)\frac{\partial u_j}{\partial x_i}\right)+\\ +\frac{1}{e}\left(\frac{3\left|\vec{v}\right|^2}{4e}-\frac{3}{2}\right)\left(\frac{\partial e}{\partial t}+\left(u_i+v_i\right)\frac{\partial e}{\partial x_i}\right)-\frac{3}{2\rho e}F_i v_i\end{array}\right] \tag{21}$$

The time derivatives in Eq.(21) are replaced using the conservation equations that apply on level $O\left(Kn^0\right)$ and after some rearranging, Eq.(22) is derived.

$$f^{(1)} = -\tau f^{(0)}\left(\frac{1}{e}\frac{\partial e}{\partial x_i}\left(\frac{3|\vec{v}|^2}{4e}-\frac{5}{2}\right)v_i + \frac{3}{2e}v_i v_j\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)-\frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right]\right) \tag{22}$$

In order to derive the Navier-Stokes equations, the first order moment perturbation $\vec{\sigma}^{(1)}$ must be determined. The stress tensor was defined earlier as $\sigma_{ij} = -\int v_i\, v_j\, f\, d\vec{c}$. Using the previously derived Eq.(22) the first order perturbation of the stress tensor becomes

$$\sigma_{ij}^{(1)} = -\int v_i\, v_j\, f^{(1)}\, d\vec{c} = p\,\tau\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}-\frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right), \tag{23}$$

where p denotes the pressure. Finally, by letting $\varepsilon Kn \to Kn$, the approximation $\vec{\sigma} \approx \vec{\sigma}^{(0)} + \vec{\sigma}^{(1)}$ can be inserted in Eq.(14) to acquire the Navier-Stokes mass conservation equation

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\delta_{ij}\, p + \sigma_{ij}^{dev}\right) + F_i, \tag{24}$$

where σ_{ij}^{dev} is given with Eq.(23).

4. THE LATTICE-BOLTZMANN METHOD

Since it is impossible to find an analytical solution of the Boltzmann equation for practical problems, it is solved numerically. Contrary to the case of standard transport equations, which are discretized in the physical space and time, the Boltzmann equation requires additional discretization in the velocity space. The first step is to transform the continuous velocity space into discrete velocities $\vec{c}_\alpha$, consequently followed by the standard time and space discretization.

The first step is to transform Eq.(5) using the pressure and density relation for an isothermal ideal gas $p = c_0^2\,\rho$, where c_0 marks the speed of sound, and using the Taylor expansion to the order $O\left(u^2\right)$ to obtain

$$\begin{aligned} f^{(0)}\left(\vec{x},\vec{c},t\right) &= \frac{\rho}{\left(2\pi c_0^2\right)^{3/2}}\exp\left(-\frac{c_i c_i - 2c_i u_i + u_i u_i}{2c_0^2}\right) \approx \\ &\approx \frac{\rho}{\left(2\pi c_0^2\right)^{3/2}}\left(1+\frac{c_i u_i}{c_0^2}+\frac{\left(c_i u_i\right)^2}{2c_0^4}-\frac{u_i u_i}{2c_0^2}\right)\exp\left(-\frac{c_i c_j}{2c_0^2}\right). \end{aligned} \tag{25}$$

The next step is to discretize the velocity space that results in a finite number of velocities $\vec{c}_\alpha$. Accordingly, the distribution function formerly presented as $f(\vec{x},\vec{c},t)$ becomes $f_\alpha(\vec{x},t)$ that represents the density of particles with velocity $\vec{c}_\alpha$ at $(\vec{x},t)$. Additionally, the term $\exp\left(-\frac{c_i c_j}{2c_0^2}\right)\Big/\left(2\pi c_0^2\right)^{3/2}$ is replaced with weighting coefficients w_α, resulting in the discrete form of the distribution function

$$f^{(0)}(\vec{x},\vec{c},t)=\rho\, w_\alpha\left(1+\frac{c_i u_i}{c_0^2}+\frac{(c_i u_i)^2}{2c_0^4}-\frac{u_i u_i}{2c_0^2}\right). \tag{26}$$

Using the discrete form of the BGK operator given by Eq.(9) the Boltzmann equation is transformed into the discrete-velocity Boltzmann equation (DVBE),

$$\frac{\partial f_\alpha}{\partial t}+c_{\alpha i}\frac{\partial f_\alpha}{\partial x_i}=-\frac{1}{\tau}\left(f_\alpha-f_\alpha^{(0)}\right). \tag{27}$$

The discrete velocities defined by $\vec{c}_\alpha$ and w_α must satisfy certain constraints in order to ensure that the model reproduces the hydrodynamic equations. The DVBE gives the same equations as the continuous Boltzmann equation (mass and momentum conservation equations) if the zeroth, first, second and third moment of the discrete distribution function $f_\alpha^{(0)}$ has to be same as the appropriate moment of the continuous distribution function $f^{(0)}$.

$$\begin{aligned}
&\sum_\alpha f_\alpha^{(0)}(\vec{x},t)=\rho(\vec{x},t),\\
&\sum_\alpha c_{\alpha i}\, f_\alpha^{(0)}(\vec{x},t)=\rho u_i(\vec{x},t),\\
&\sum_\alpha c_{\alpha i}\, c_{\alpha j}\, f_\alpha^{(0)}(\vec{x},t)=\Pi_{ij}^{(0)}(\vec{x},t),\\
&\sum_\alpha c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k}\, f_\alpha^{(0)}(\vec{x},t)=\Pi_{ijk}^{(0)}(\vec{x},t),
\end{aligned} \tag{28}$$

where $\Pi_{ij}^{(0)}(\vec{x},t)=\int c_i\, c_j\, f^{(0)}(\vec{x},t)\,d\vec{c}$, and $\Pi_{ijk}^{(0)}(\vec{x},t)=\int c_i\, c_j\, c_k\, f^{(0)}(\vec{x},t)\,d\vec{c}$. Using Eqs.(28) we can attain the constraints on the velocity vectors and weighting coefficients given with Eqs.(29) that are the symmetry properties of $\vec{c}_\alpha$ and w_α.

$$\begin{aligned}
&\sum_{\alpha} w_{\alpha} = 1, \\
&\sum_{\alpha} w_{\alpha} c_{\alpha i} = 0, \\
&\sum_{\alpha} w_{\alpha} c_{\alpha i}\, c_{\alpha j} = c_0^2 \delta_{ij}, \\
&\sum_{\alpha} w_{\alpha} c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k} = 0, \\
&\sum_{\alpha} w_{\alpha} c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k}\, c_{\alpha l} = c_0^4 \left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right), \\
&\sum_{\alpha} w_{\alpha} c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k}\, c_{\alpha l}\, c_{\alpha m} = 0.
\end{aligned} \tag{29}$$

Keeping higher order terms in Eq.(25) produces additional constraints leading to a larger set of velocities.

The next step is to perform the Chapman-Enskog expansion using the perturbation expansion Eq.(15) as well as a multiple-scale expansion of time in orders of *Kn* so that $t \to t_1 + \varepsilon^{-1} t_2 + \ldots$ using Eq.(30) to expand the derivatives

$$\frac{\partial}{\partial t} \to \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} + \cdots, \quad \frac{\partial}{\partial x_i} \to \varepsilon \frac{\partial}{\partial x_i}. \tag{30}$$

Although the expansion has no clear physical meaning, it can be explained through taking into account that different phenomena affect the time derivative at different orders of *Kn* number. If t_1 is a time scale dealing with fast phenomena and t_2 is a time scale dealing with slower phenomena, this allows the separation of the derivatives into multiple equations corresponding to the order of ε. Using these expansions the DVBE is modified and separated into first and second order ε to give Eqs.(31)

$$\begin{aligned}
&O(\varepsilon): && \left(\frac{\partial}{\partial t_1} + c_{\alpha i} \frac{\partial}{\partial x_i} \right) f_{\alpha}^{(0)} = -\frac{1}{\tau} f_{\alpha}^{(1)}, \\
&O(\varepsilon^2): && \frac{\partial f_{\alpha}^{(0)}}{\partial t_2} + \left(\frac{\partial}{\partial t_1} + c_{\alpha i} \frac{\partial}{\partial x_i} \right) f_{\alpha}^{(1)} = -\frac{1}{\tau} f_{\alpha}^{(2)}.
\end{aligned} \tag{31}$$

As in the case of the continuous Boltzmann equation, the higher order perturbations have no influence on the macroscopic property, so the mass and momentum conservation imply that $\sum_{\alpha} f_{\alpha}^{(n)} = \sum_{\alpha} c_{\alpha} f_{\alpha}^{(n)} = 0,$ for $n \geq 1$, therefore the zeroth, first and second order moments of Eqs.(31) are

$$
\begin{aligned}
O(\varepsilon): \quad & \frac{\partial \rho}{\partial t_1} + \frac{\partial (\rho u_i)}{\partial x_i} = 0, \\
& \frac{\partial (\rho u_i)}{\partial t_1} + \frac{\partial \Pi_{ij}^{(0)}}{\partial x_j} = 0, \\
& \frac{\partial \Pi_{ij}^{(0)}}{\partial t_1} + \frac{\partial \breve{\Pi}_{ijk}^{(0)}}{\partial x_k} = -\frac{1}{\tau} \breve{\Pi}_{ij}^{(1)} \\
O(\varepsilon^2): \quad & \frac{\partial \rho}{\partial t_2} = 0, \\
& \frac{\partial \rho u_i}{\partial t_2} + \frac{\partial \breve{\Pi}_{ij}^{(1)}}{\partial x_j} = 0,
\end{aligned}
\tag{32}
$$

where $\breve{\Pi}_{ijk}^{(0)} = \sum_{\alpha} c_{\alpha i}\, c_{\alpha j}\, c_{\alpha k}\, f_{\alpha}^{(0)}$ and $\breve{\Pi}_{ij}^{(1)} = \sum_{\alpha} c_{\alpha i}\, c_{\alpha j}\, f_{\alpha}^{(1)}$ .By recombining the moment equations at different orders of ε the conservation equation are produced. By multiplying the first equation in Eqs.(32) with ε and adding it to the fourth equation in Eqs.(32) multiplied by ε^2, using the expansion given with Eq.(30) results in the standard mass conservation equation previously presented with Eq.(11). The momentum equation, as presented in Eq.(24) without the force term, is attained by multiplying the second equation in Eqs.(32) with ε and adding it to the fifth equation in Eqs.(32) multiplied by ε^2, after applying Eq.(30) and using the third equation in Eqs.(32).

It can be concluded that the discrete-velocity Boltzmann Eq.(27) with conditions Eqs.(29) reproduces the mass and momentum equations of fluid mechanics with the exception of a $O(u^3)$ term that can be neglected for Mach numbers $Ma^2 << 1$.

Although Eq. (27) is discrete in velocity space, it is still continuous in physical space and time, therefore further discretization is necessary. Since both the Boltzmann equation and DVBE are hyperbolic, they can be discretized by integration along the characteristics. Using this approach, the distribution function can be presented as $f_{\alpha} = f_{\alpha}\left(\vec{x}(a), t(a)\right)$ where a marks the position along the characteristic. The total derivative of f_{α} along a, assuming no external forces leads to

$$
\frac{df_{\alpha}}{da} = \left(\frac{\partial f_{\alpha}}{\partial t}\right)\frac{dt}{da} + \left(\frac{\partial f_{\alpha}}{\partial x_i}\right)\frac{dx_i}{da} = -\frac{1}{\tau}\left(f_{\alpha} - f_{\alpha}^{(0)}\right) \tag{33}
$$

where the equality on the right side holds only if the total differential is the left side of DVBE which is true if $dt/da = 1$ and $dx_i/da = c_{\alpha i}$. By slightly misusing the notation the distribution function is $f_{\alpha}\left(\vec{x}(a), t(a)\right) \approx f_{\alpha}\left(\vec{x} + \vec{c}a, t + a\right)$. After integrating from one time step to another the following equation is derived

$$f_\alpha\left(\vec{x}+\vec{c}\Delta t,t+\Delta t\right)-f_\alpha\left(\vec{x},t\right)=-\frac{1}{\tau}\int_{a=0}^{a=\Delta t}\left[f_\alpha\left(\vec{x}+\vec{c}a,t+a\right)-f_\alpha^{(0)}\left(\vec{x}+\vec{c}a,t+a\right)\right]da\,. \qquad (34)$$

The left side of Eq.(34) is accurate while the right side is solved approximately. If the integral in Eq.(34) is approximated using the rectangle method, the first order lattice Boltzmann equation is developed,

$$f_\alpha\left(\vec{x}+\vec{c}\Delta t,t+\Delta t\right)-f_\alpha\left(\vec{x},t\right)=-\frac{\Delta t}{\tau}\left[f_\alpha\left(\vec{x},t\right)-f_\alpha^{(0)}\left(\vec{x},t\right)\right]. \qquad (35)$$

Equation (35) is fully explicit sine all f_α in the next time step can be determined using f_α from the current time step.

5. CONCLUSION

This paper presents a short introduction to the lattice-Boltzmann method. The essence of the lattice-Boltzmann method is the indirect solution of the fluid motion equations. Namely, one can generate accurate solutions even though simple arithmetic calculations are performed instead of solving the standard fluid flow equation.
As stated in the Introduction, the Boltzmann equation describes the propagation of a distribution function instead of the propagation of each particle in a fluid mass. Hence, the paper starts with a short description of the distribution function and its fundamental properties. The authors gave a brief overview of the properties which the collision operator must satisfy and selected the BGK collision operator for further derivation of the governing equations. After deriving the Boltzmann equation it was discretized using the lattice-Boltzmann method in order to obtain the form of the equations that allows their solution.

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LATTICE-BOLTZMANN МЕТОДА У НУМЕРИЧКОЈ МЕХАНИЦИ ФЛУИДА

Резиме: *Lattice-Boltzmann-ова метода (ЛБМ) је нова метода у нумеричкој механици флуида. Док традиционалне нумеричке методе непосредно дискретизују и решавају макроскопске једначине механике флуида, ЛБМ решава дискретну кинетичку једначину која репродукује једначине механике флуида у макроскопском смислу. Овај рад представља концепт функције расподеле која је неопходна у ЛБМ, Boltzmann-ову једначину и развијање у низ по Chapman-Enskog-м које се користи у циљу репродуковања макроскопских једначина механике флуида полазећи од Boltzmann-ове једначине. У раду је такође представљен поступак дискретизације Boltzmann-ове једначине у простору брзина и физичком простору ради успостављања lattice-Boltzmann-ове једначине.*

Кључне речи: *lattice-Boltzmann метода, развијање у низ по Chapman-Enskog-у, нумеричка механика флуида*