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FREE VIBRATION ANALYSIS OF CURVED BERNOULLI-EULER BEAM USING ISOGEOMETRIC APPROACH

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Abstract:

Isogeometric analysis (IGA) is based on a concept that uses the same base functions for representing both the model geometry and the solution space. The most common base functions used in the IGA are NURBS (Non-Uniform Rational B-Splines) functions for their capability to analytically represent various geometries. In this paper, the IGA is applied in the free vibration analysis of rotation-free plane curved Bernoulli-Euler beam. The stiffness and mass matrices have been developed using basic concepts of continuum mechanics and the principle of virtual work. Geometry of the undeformed and deformed beam is defined using convective coordinates and cross section basis vectors. Results of the free vibration analysis for beam with arbitrary curvature are compared with the results obtained from the conventional finite element method (FEM) software. The significant advantages of the IGA approach over the FEM are shown and discussed.

Key words: isogeometric analysis, free vibration, rotation-free Bernoulli-Euler beam

1. Introduction

One of the first steps in structural analysis is to define a model geometry. Nowadays, structural geometry is modeled in the computer aided design (CAD) software, while the analysis is mostly conducted using the finite element method (FEM). The FEM based software uses the CAD models to subdivide the geometry model into smaller domains called finite elements (FE), whereby forming the finite element mesh. Generally, as the size of FE is decreased, the results obtained using the FEM software are more accurate. In order to carry out the FEM refinements, geometry of the analysis model has to be generated from the CAD geometry model. On the contrary, in the isogeometric analysis (IGA), the geometry of the analysis model has to be generated only once from the CAD geometry model, while the refinements of the IGA analysis model are carried out in the parametric domain. This property of the IGA is possible by using the same basis functions for the geometry description and the solution space. In recent years, the IGA has been center of research of many authors. Cottrell, Hughes and Bazilevs have defined basic concepts of the IGA in their book [1]. Lately, more attention has been paid to the analysis of one-dimensional elements (beams) using the IGA. Radenković [2] has defined one-dimensional element based on Bernoulli-Euler and Timoshenko theory for static load using the IGA. In his work the nonlinear analysis is has been taken into account. In addition, Bauer [3] has defined the spatial Bernoulli-Euler beam for static load, and Marino [4] has defined a spatial Timoshenko beam for nonlinear analysis using the IGA. Lee [5] studied free vibration analysis of straight Timoshenko beam using the IGA.

In this paper free vibration analysis of rotation-free plane curved Bernoulli-Euler beam has been carried out using the IGA. The beam element has been derived using basic principles of the continuum mechanics and implemented in the program coded in MATLAB [6], which has been used to compute the free vibration characteristics of beams with arbitrary curvature. The obtained results are compared with the results from the conventional FEM software Abaqus [7] and SAP2000 [8].

2. NURBS basis functions

The major property of the IGA is the usage of the same formulation for structural geometry description and for definition of the solution space. The geometry of the structure is defined by control points and NURBS basis functions. The control points represent the discrete parameters of the structural geometry given in the Cartesian coordinates. In general, the IGA uses NURBS basis functions for their capability to describe exactly the geometry of conic sections like parabola, hyperbola and elliptic configurations. NURBS basis functions are rational polynomial functions constructed of polynomial B-spline functions. Non-negativity, partition of unity, interpolatory property at boundary domain, recursive formulation are the key properties of the NURBS basis functions. Parameterized representation of the structural geometry $\mathbf{C}(\xi)$ using NURBS basis functions is given as:

$$\mathbf{C}(\xi)=\sum_{i=1}^{n} R_{i,p}(\xi)\cdot\mathbf{P}_i=\sum_{i=1}^{n}\frac{N_{i,p}(\xi)\cdot w_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\cdot w_j}\cdot\mathbf{P}_i \tag{1}$$

where ξ represents the parametric coordinate, $\mathbf{P}_i$ is the vector of control points, $R_{i,p}(\xi)$ is the NURBS basis function, $N_{i,p}(\xi)$ is the B-spline basis function, w_i is the weight and p is the polynomial degree of the basis function.

The B-spline basis functions are obtained using Cox de Boor algorithm, where for p=0 basis function is given as:

$$N_{i,0}(\xi)=\begin{cases}1, & \text{if } \xi\in[\xi_i,\xi_{i+1}[\\ 0, & \text{otherwise}\end{cases} \tag{2}$$

while for p>0 is given as:

$$N_{i,p}(\xi)=\begin{cases}\dfrac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\dfrac{\xi_{i+p+1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi), & \text{if } \xi\in[\xi_i,\xi_{i+1}[\\ 0, & \text{otherwise}\end{cases} \tag{3}$$

In equations (2) and (3) terms ξ_i, called knots, divide the parametric space into subspaces forming the knot spans. The nonzero knot span represents the isogeometric element i.e. the number of nonzero knot spans is equal to the number of the isogeometric elements. The refinement in the IGA can be accomplished by knot insertion in the parametric domain.

Fig. 1a presents a quadratic NURBS curve generated using four NUBRS basis functions (Fig. 1b). More about the NURBS basis functions, B-spline basis functions, their properties and implementation can be found in [9].

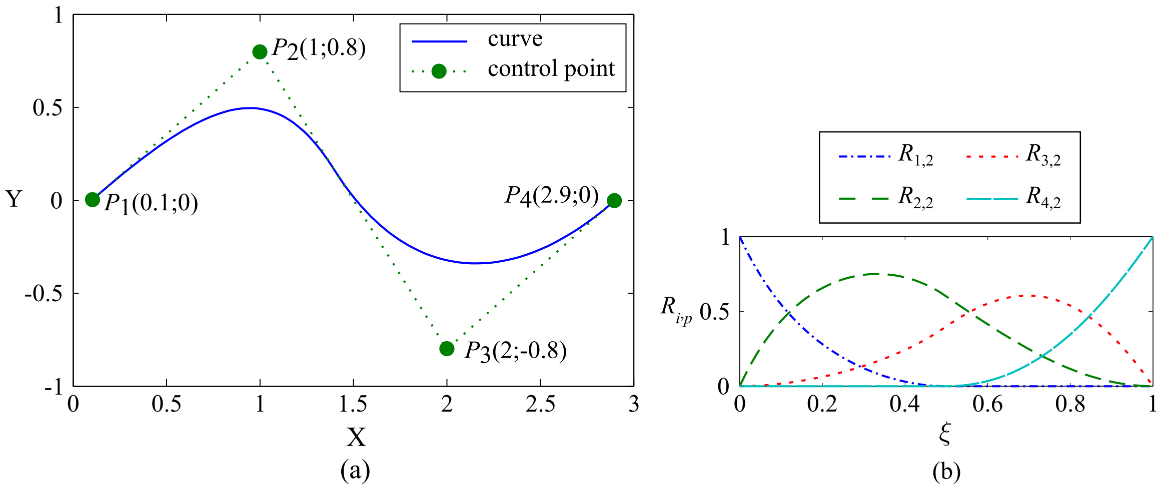

Fig. 1. (a) NURBS curve, (b) NURBS basis functions

3. Bernoulli-Euler isogeometric element

The main goal of this paper is to define Bernoulli-Euler beam with arbitrary shape in plane using IGA. For this reason, the basic principles of the continuum mechanics are used.

For a given undeformed beam in the x-y plane, Fig. 2, the position of the cross section is defined using the position vector:

$$\mathbf{r} = x^{\alpha} \cdot \mathbf{i}_{\alpha}, \quad \alpha = 1,2 \tag{4}$$

where $\mathbf{i}_{\alpha}$ are the basis vectors of the Cartesian coordinate system, and x_{α} are the coordinates of the position vector in the Cartesian coordinate system.

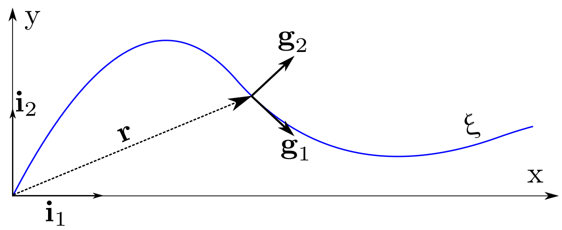

Fig. 2. Geometry of undeformed beam

The position of the undeformed beam is defined using the IGA approach as:

$$\mathbf{r} = \sum_{i=1}^{n} R_{i,p}(\xi) \cdot x_i^{\alpha} \cdot \mathbf{i}_{\alpha}, \quad \alpha = 1,2 \tag{5}$$

where x_i^{α} represents the coordinates of the control points in the Cartesian coordinate system.

When the position vector is defined, the basis vectors denoted $\mathbf{g}_1$ and $\mathbf{g}_2$ are given as:

$$\mathbf{g}_1 = \frac{d\mathbf{r}}{d\xi} = \sum_{i=1}^{n} \left[R_{i,p}(\xi) \right]_{,1} \cdot x_i^{\alpha} \cdot \mathbf{i}_{\alpha} = \hat{x}_{,1}^{\alpha} \cdot \mathbf{i}_{\alpha}, \quad \alpha = 1,2 \tag{6}$$

$$\mathbf{g}_2 = \mathbf{n} = \frac{1}{K} \cdot \frac{d^2\mathbf{r}}{d\xi^2} \frac{d^2\xi}{ds^2}, \quad \alpha = 1,2 \tag{7}$$

where K represents the beam curvature. For a general case the beam curvature is a function of the parametric coordinate.

The properties of the cross section basis vectors are:

$$\mathbf{g}_1 \cdot \mathbf{g}_1 = g \neq 1 \qquad \mathbf{g}_2 \cdot \mathbf{g}_2 = 1 \qquad \mathbf{g}_1 \cdot \mathbf{g}_2 = 0 \tag{8}$$

With this description it is relatively easy to define the beam curvature as:

$$K = \frac{\sqrt{\left(\hat{x}^m_{,11} - \Gamma^1_{11}\hat{x}^m_{,1}\right)\hat{x}_{m,11}}}{g} \tag{9}$$

where Γ^1_{11} represents the Christoffel symbol.

For a given geometric and material properties of the structure the goal of the analysis is to find the deformed geometry of the structure. The geometry of deformed beam in plane, presented in Fig. 3, is given via the position vector $\mathbf{r}^*$ and the relation between the position vector of undeformed and deformed beam is:

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u} \tag{10}$$

where $\mathbf{u}$ is the displacement vector and can be presented as:

$$\mathbf{u} = u^\alpha \cdot \mathbf{i}_\alpha = \sum_{i=1}^{n} R_{i,p}(\xi) \cdot u_i^\alpha \cdot \mathbf{i}_\alpha \tag{11}$$

Term u_i^α represents the α-th displacement of the i-th control point. Displacements of the control points are basic unknowns in the IGA.

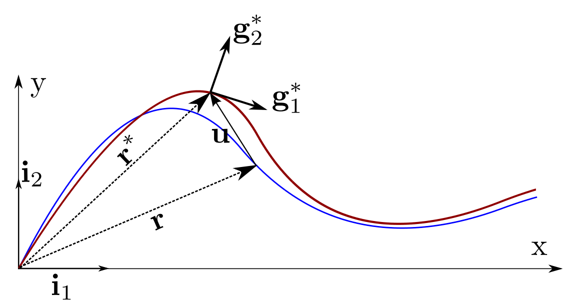

Fig. 3. Geometry of deformed beam

The representation of the undeformed position vector i.e. geometry of the undeformed beam and the representation of the displacement vector is given in the same form, equation (1). As mentioned before, this is the main property of the IGA analysis.

Using convective coordinates, the basis vectors after deformation, denoted as $\mathbf{g}_1^*$ and $\mathbf{g}_2^*$, are not the same basis vectors of the cross section for undeformed beam. The first basis vector of the deformed beam is given as:

$$\mathbf{g}_1^* = \frac{d\mathbf{r}^*}{d\xi} = \frac{\left(d\hat{x}^m + du^m\right) \cdot \mathbf{i}_m}{d\xi} = \left(\hat{x}^m_{,1} + u^m_{,1}\right) \cdot \mathbf{i}_m \tag{12}$$

With the defined basis vector $\mathbf{g}_1^*$ normal strain of the beam centroidal axis is given as:

$$\varepsilon_{11} = \frac{1}{2}\left(g_1^* \cdot g_1^* - g_1 \cdot g_1\right) = \hat{x}_{,1}^m \cdot u_{,1}^m \tag{13}$$

In order to define the change of the curvature κ of the deformed beam, the difference between the curvature of the deformed beam K^* and the curvature of the undeformed beam has to be found [2]:

$$\kappa = K^* - K = \hat{x}_{,2}^m \left(u_{m,11} - \Gamma_{11}^1 u_{m,1}\right) \tag{14}$$

Terms ε_{11} and κ form the deformation vector **E** of the Bernoulli-Euler beam.

After defining the normal strain of an arbitrary fiber $\varepsilon_{11}(\eta,\zeta)$, where η and ζ are the cross sectional coordinates, the energetic conjugate forces **S** can be found [2]. Using the principle of virtual work [10], the deformation vector **E**, force vector **S** and vector of volume forces can be combined as:

$$\int_s \mathbf{S} : \delta\mathbf{E} dx + \int_s \rho\mathbf{B} : \delta\mathbf{u} dx = 0 \tag{15}$$

where ρ represents the mass density while δ represents virtual terms of the defined vectors.
From equation (15), by using known procedures, the equation of motion for the free vibration is obtained as:

$$\mathbf{K} \cdot \mathbf{q} + \mathbf{M} \cdot \ddot{\mathbf{q}} = \mathbf{0} \tag{16}$$

where **K** is the stiffness matrix, **M** is the mass matrix and **q** is the vector of unknown terms u_i^α defined by equation (11). The Bernoulli-Euler beam element defined in this procedure has no rotation degree of freedom i.e. displacement is the only considered degree of freedom. For this reason it is called the rotation-free element.

4. Numerical example

In order to validate the IGA approach, numerical examples are presented in the following. After the free vibration analysis of beam with constant curvature the free vibration analysis of beam with parabolic shape will be presented. The results obtained by using the IGA will be validated with the results obtained from the conventional FEM software. In the IGA, the geometry of the beams and the solution spaces are described using the NURBS functions of order two.

4.1 Quarter circle beam

Free vibration analysis of a quarter circle curved beam with simply supported boundary conditions, Fig. 4, is presented here. Radius of the quarter circle is R = 3.75m, the dimension of the cross section in plane of the beam is h = 0.6m, the dimension of the cross section out of plane of the beam is b = 0.3m, the elastic modulus is E = 31.5 GPa, mass density ρ = 2500 kg/m^3, the angle between boundary cross sections $\phi = 90°$.

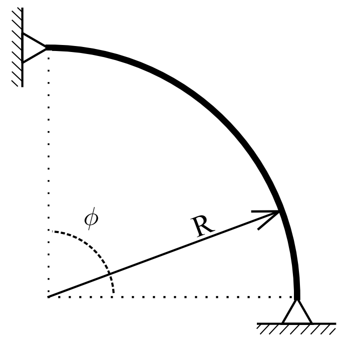

Fig. 4. Quarter circle beam

In order to illustrate the convergence and the accuracy of the IGA approach, the first five natural frequencies are computed and compared with the FEM solutions using Abaqus [7]. The results are summarized in Table 1. Abaqus is software based on the FEM, able to model curved beams with constant curvature. The convergence and the accuracy of the results obtained using the IGA is investigated using twenty isogeometric elements. The results have shown excellent agreement with the result from Abaqus obtained using 196 finite elements. The last row in Table 1 represents the relative error between the converged results obtained using the IGA and the results obtained from Abaqus. The maximum relative error of 1.51 % has been encountered for the fifth mode. Finally, the first five mode shapes of the quarter circle beam obtained using the IGA are presented in Fig. 5.

Modes	1	2	3	4	5
5 IGA elements	103.44	134.14	289.60	328.98	579.37
10 IGA elements	96.67	133.05	253.70	326.05	468.92
15 IGA elements	95.59	132.86	248.56	325.26	452.70
20 IGA elements	95.30	132.77	246.50	325.03	447.45
FEM	94.36	132.98	244.51	324.62	440.79
[%]	0.99	0.16	0.81	0.13	**1.51**

Table 1. Natural frequencies [Hz] of quarter circle beam

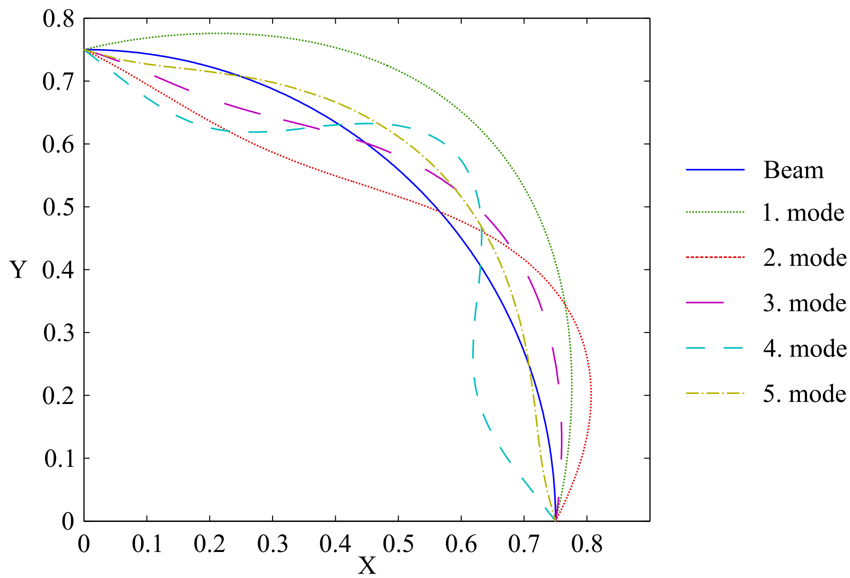

Fig. 5. Mode shapes of quarter circle beam

4.2 Beam with parabolic shape

The geometry of parabolic beam with simply supported boundary conditions is presented in Fig. 6. In this example the arch rise is $f = 1.125m$ and the span length is $L = 3.75m$, the dimension of the cross section in plane of the beam is $h = 0.6m$, the dimension of the cross section out of plane of the beam is $b = 0.3m$, the elastic modulus is $E = 31.5\ GPa$, material density $\rho = 2500\ kg/m^3$.

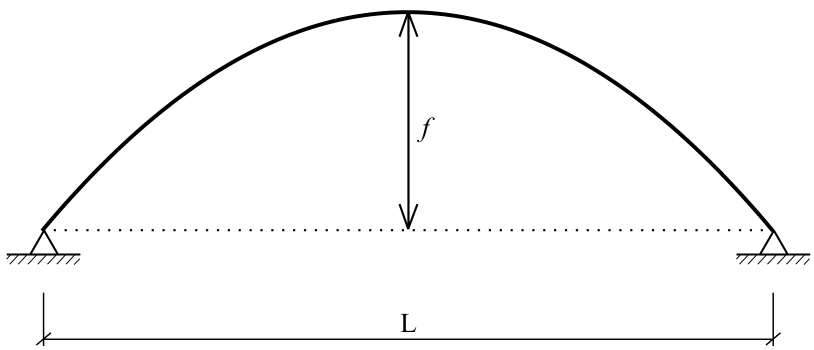

Fig. 6. Beam with parabolic shape

The natural frequencies of the parabolic beam calculated using the IGA and SAP2000 [8] are given in Table 2. Using twenty isogeometric elements the results have shown excellent agreements with the converged results obtained using 30 finite elements in the software SAP2000 [8]. In Table 2 the last row presents the relative error between the results obtained using 20 isogeometric elements and the converged results obtained from SAP2000. The maximum relative error is encountered for fifth mode 2.92%. In Fig. 7 the mode shapes of the parabolic beam obtained using IGA are presented.

Modes	1	2	3	4	5
5 IGA elements	160.47	228.97	438.08	443.92	823.60
10 IGA elements	152.52	228.84	408.05	434.93	771.56
15 IGA elements	150.92	228.83	399.89	434.41	745.64
20 IGA elements	150.46	228.82	396.97	434.26	736.91
FEM 30 FE	147.97	231.12	388.89	438.31	716.01
[%]	1.68	0.99	2.08	0.92	**2.92**

Table 2. Natural frequencies [Hz] of parabolic beam

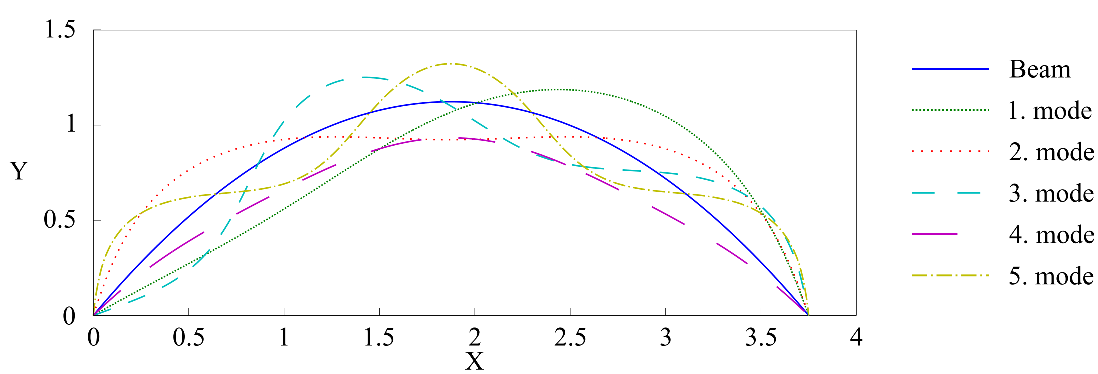

Fig. 7. Mode shapes of parabolic beam

As mentioned before, in the FEM-based analysis the geometry of the model should be taken from the CAD model, whenever the refinement should be carried out. The refinement procedure for the parabolic beam using the FEM-based analysis is presented in Fig 8.

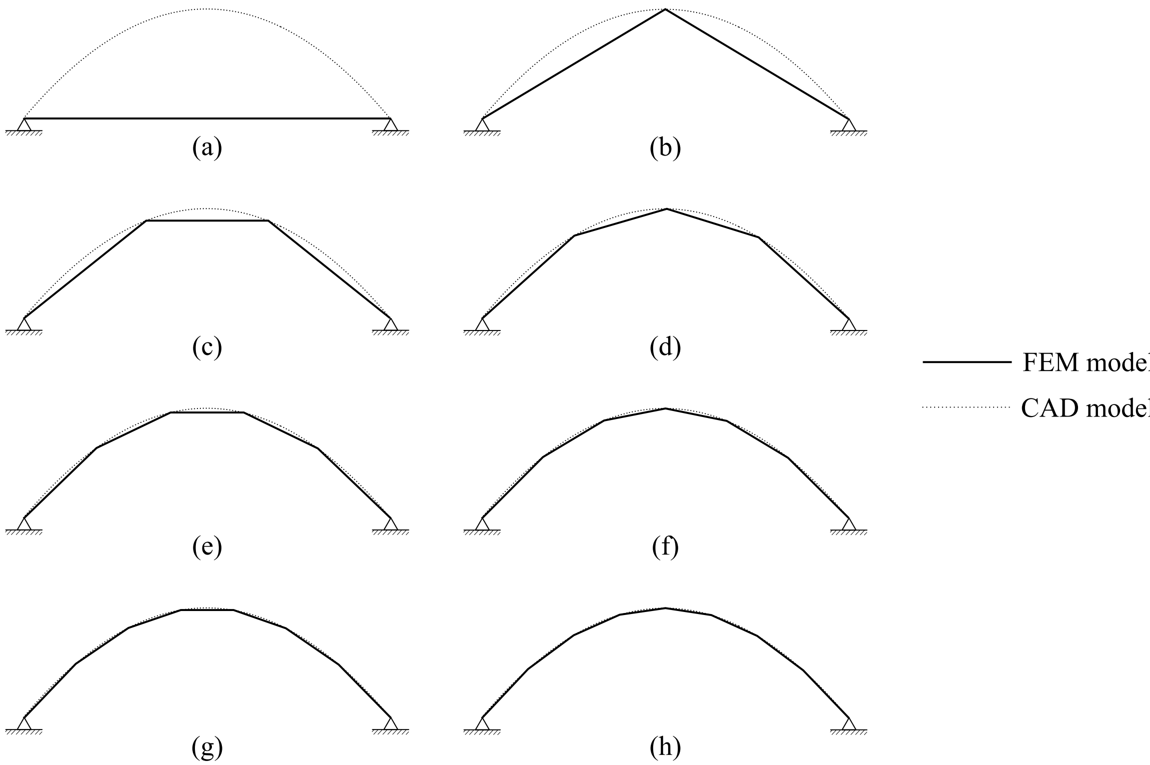

Fig. 8. FEM model of the parabolic beam with: (a) 1FE, (b) 2FE (c) 3FE, (d) 4FE, (e) 5FE, (f) 6FE, (g) 7FE, (h) 8FE

On the contrary, in the IGA the refinement is carried out in the parametric domain. The refinement only has the influence on the control points and not on the model geometry i.e. the model geometry needs to be taken only once from the CAD model, Fig. 9. This represents one of the main advantages of the IGA over the FEM-based analysis.

5. Conclusion

The application of the isogeometric approach in the free vibration analysis of plane curved Bernoulli-Euler beams is presented in this paper. The NURBS basis functions have been used for description of the structural geometry and the solution space. NURBS basis functions are used for its capability to describe various geometric shapes of beams. The stiffness and mass matrices have been developed for the rotation-free Bernoulli-Euler curved beam. In order to derive these matrices, the convective coordinates are used, and the change of the basis vectors of the cross sections is analysed. Using the principle of virtual work the equation of motion is derived. Free vibration analysis is conducted for two curved beams, quarter circle and the parabolic beam. The results are compared with the results obtained using the FEM software. It can be noticed that the results have shown good performance and accuracy of the approach. In addition, the numerical example for parabolic beam has shown that in the FEM software in order to perform the refinement, geometry of the structure should be taken from the CAD model. In the IGA the geometry of the model should be taken only once, and the model geometry is not influenced by the refinement.

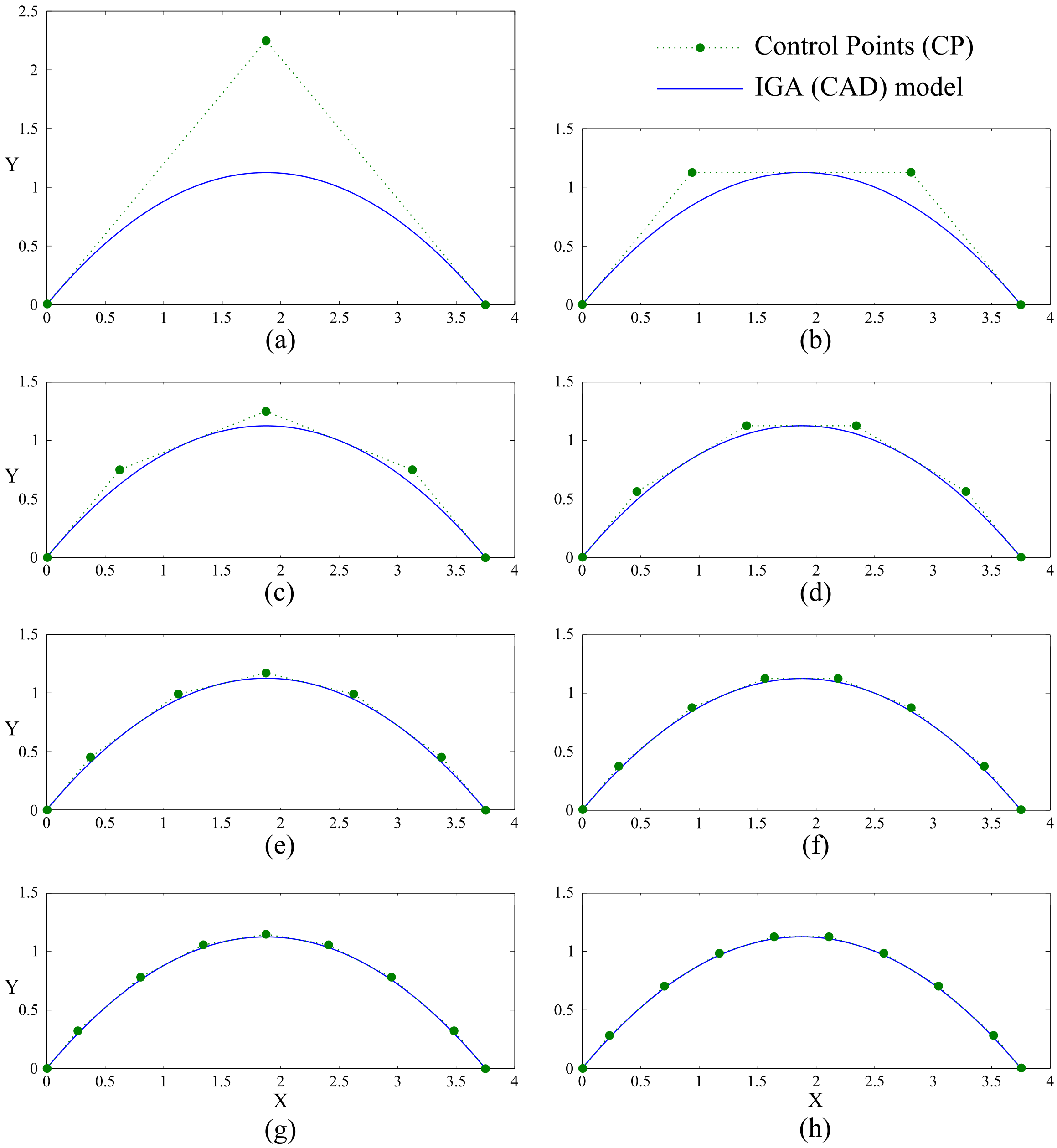

Fig. 9. Isogeometric model of parabolic beam defined with: (a) 3 control points (CP), (b) 4CP, (c) 5CP, (d) 6CP, (e) 7CP, (f) 8CP, (g) 9CP, (h) 10CP

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