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THE LIFE AND WORK OF _MILUTIN _MILANKOVIĆ: PAST, PRESENT, FUTURE

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MATHEMATICAL LECTURES OF MILUTIN MILANKOVIĆ

Abstract | In 1909 Milutin Milanković accepted the invitation of the Faculty of Philosophy of the University of Belgrade and became a professor at the Department for Applied Mathematics. *"The Applied Mathematics Chair is my great desire. There I could find all the conditions to develop my abilities and repay my people. I always appreciated mathematics as a wonderful tool for solving the problems we face when studying nature and space"*, he explained his decision. Unfortunately, very little is known about the teaching work of one of the greatest scientists of all time. In this paper we will try to look at his lectures, based on the preserved notes of one of his students, B. Pujić.

Keywords | Milutin Milanković, mathematics, education.

1. INTRODUCTION

"Attention needs to be drawn, the way to the solution should be shown. A few attempts and a student can love mathematics. Students should be interested in solving the problem, then the student will seek a solution..."
These words (see [1]) Milutin Milanković addressed the director of one of Belgrade's high schools approximately around 1930. His lectures in mathematics were based precisely on these principles. We find it interesting to see how one such brilliant scientist taught his students.

2. MILANKOVIĆ'S EDUCATION

The one of the most important scientist in the world Milutin Milanković was born on May 28 1879, in the village of Dalj near the Danube. As his health was fickle, Milutin received his elementary education at home and completed the grammar school in Osijek in 1896. His teacher of mathematics, Vladimir Varićak, later a member of the Yugoslav Academy of Sciences, noticed his exceptional abilities and versed him in mathematical sciences, remaining his lifelong friend and adviser. In his memoirs (see [2]) Milutin wrote:

"He taught me to think exactly and speak clearly. He developed my mathematical ingenuity and developed its elegance in me."

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He was awarded a bachelor's degree in Civil Engineering at the Vienna University of Technology in 1902. In Vienna, he met Professor Czuber, later examination committee member for Milutin's PhD thesis, who left a special impression on him. On this professor mentorship Milanković's talent has fully developed. The knowledge that he got from the professor Czuber was crucially important for his later development (see [2]).

"Professor Czuber was teaching us mathematics. His every sentence was the masterpiece of strict logic, without any extra word, without any error."

Milanković was the first Serb to be awarded doctorate at the Vienna Grand School of Engineering in 1904. with PhD thesis *"Contribution to the Theory of Pressure Curves"*. Milutin's thesis was noted for its original approach and thesis was published in the eminent German scientific, non-technical, journal *"Zeitschrift fur Mathematik und Physik"* in 1907. During the period 1905. to 1910. he worked as an design engineer in the construction company of Baron Pittel. At the same time his six patents were officially recognized which brought him enormous professional reputation. His ability to solve civil engineering problems mathematically was evident in his early articles and in his four patents granted in Austria and Hungary. Milanković wrote (see [2]):

"At that time my inventor's career finished. It is necessary to have commercial skills for the exploitation of patents and be prepared for real conflicts with crafty individuals. Having realized that, I decided to use my inventing abilities in a wider field where their results could be protected and that was the field of science. What is created there remains inviolable property for ages...!"

3. SCIENTIFIC CAREER AT THE UNIVERSITY OF BELGRADE

At the recommendation of Mihailo Petrović Alas, the dean, Jovan Cvijić, associated dean, and Bogdan Gavrilović, Milanković was elected as associate professor of applied mathematics at the Faculty of Philosophy in Belgrade in 1909. The Royal Decree of his appointment was signed on September 9 in the same year (see [3]):

"We, Petar I, by God's grace and people's will King of Serbia, at the proposal of the Ministry of Education and Church Affairs, appoint at the University as associated professor under contract at the Faculty of Philosophy for Applied Mathematics, Milutin Milanković, a doctor of technical science."

His real scientific career begins from that moment. This combination of topics, which he had to teach, helped him to start his theory of the insolation of the earth and other planets. In his memoirs [2], Milanković wrote:

"The Applied Mathematics chair is my great desire. There I could find all the conditions to develop my abilities and repay my people. That would combine those three branches of exact science which I had studied the most and, taken together, they would offer me wide possibilities for such scientific work that I wanted and there I thought I could create great works. I was enchanted with the very title of the Chair for Applied Mathematics. I always appreciated mathematics as a wonderful tool for solving the problems we face when studyng nature and space, the extraordinary nature of wich is best reflected in Celestial Mechanics and Theoretical Physics. And these two scienece were integral parts of my Chair."

In 1919. Milanković became a full professor and corresponding member of the Serbian Royal Academy. Next year, he became a full memeber of the Serbian Royal Academy and member of the Yugoslav Academy of Sciences and Arts.

3.1. Belgrade 's school of mathematics

Fig. 1 *Belgrade's school of mathematics*

There is a photo from 1926. titled *"Belgrade's school of mathematics"* in which the brightest minds of a generation of Serbia are presented. Mika Petrović Alas was under the science spotlite of the time, but there are also Jovan Karamata, Milos Radojčić, Tadija Pejović who gave enormeus contribution to the theory of mathematics. Milanković had his own place in this universe with his modest and simple yet convivial approach.

It can be said that one of the results of this school's work is the astronomical theory of the Ice Age, which is based exclusively on mathematics. His wide mathematical knowledge helped him with solving practical tasks in the astronomy and geology areas. The mathematics he uses covers the area nowadays studied in university courses of Analysis I and II and it was enough for Milanković to determine the quantity of heat that arrives to some celestial body by mathematical tools. Milanković did not make an abstract mathematical theory out of problems he treated, which would exist in mathematics independently of those problems. Milanković was the advisor and committee member of 26 doctoral dissertations, most of which are kept by the University Library in Belgrade.

In addition to his scientific work, Milanković always showed great interest in the historical development of science, so he dedicated his latest memories to the beginnings of exact science. In *"Through the Realm of Science"* he wrote about lives of the biggest mathematicians Pythagoras, Archimedes, Isaac Newton.

In 1927, he became an honorary member of the Matica srpska, prestigious Serbian literary and cultural society. Three years before his death he was elected for a fellow of the Academy of Natural Sciences of Halle.

4. MATHEMATICAL LECTURES

Milutin Milanković was a very responsible teacher. Soon after his arrival at the University of Belgrade, Milanković had to face the problem of organizing classes because there was no adequate literature, so students could not hear anything about modern science. He was devoted to the idea to prepare his lectures in the best possible manner. Just like his professor Czuber,

Milutin's lectures were precise, so he tried to make mathematical formulas and performances as clear as possible. His student Miodrag Tomić in [1] wrote:

"I later learned that every lecture was prepared in advance. Classes were always followed by the historical development of science. He was able to emphasize the path to scientific discoveries, often indicate difficulties on the way to resolving. Sometimes he pointed to the literature and to the latest works - it was the highest-level course."

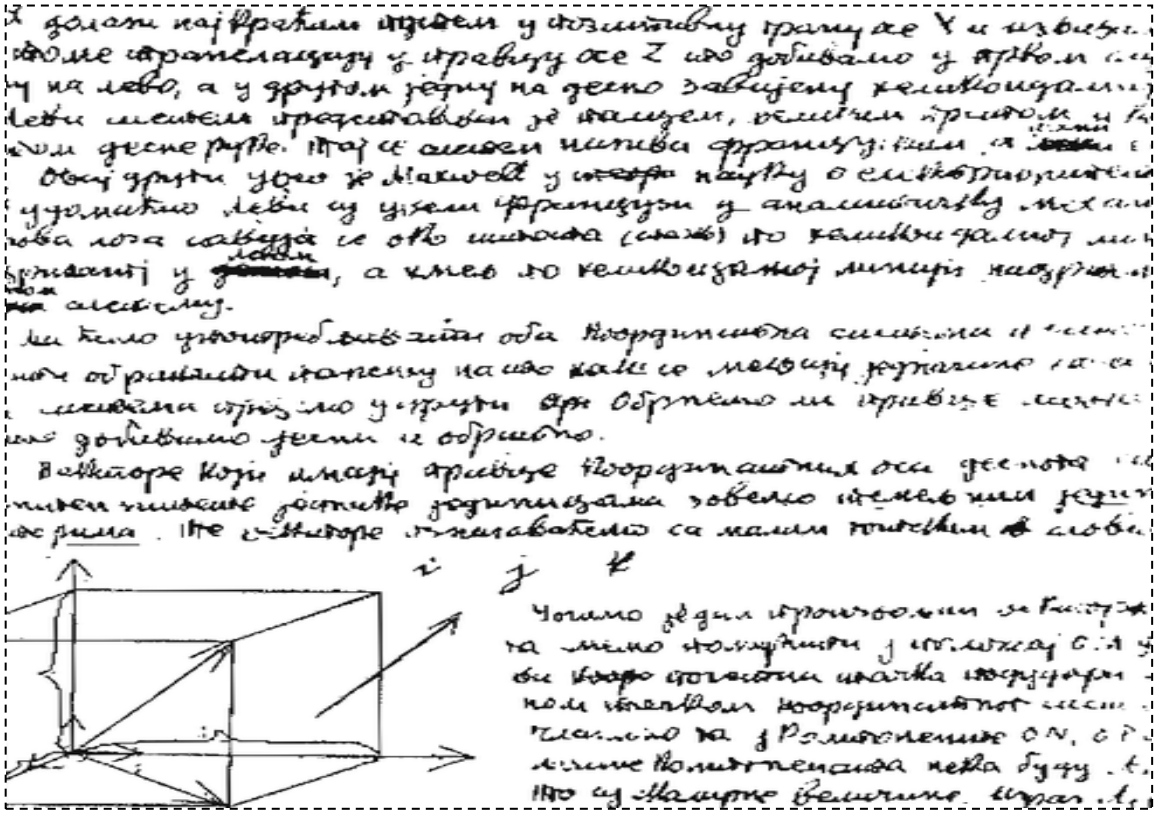


Fig. 2 *Original Milanković notes* [4]

Milanković gave lectures by cycles, which took 6 semesters, on Tuesdays and Wednesdays. In that 6 semesters Rational Mechanics was taught 2 semesters, Theoretical Physics about 3 semesters and Celestial Mechanics not quite a semester. Thereafter he would return, through a short course of Higher Dynamics, to the Rational Mechanics. He was the first professor in Serbia to teach the modern course in Einstein's theory of relativity. About twenty students attended classes.

Milanković deeply believed that mathematics is the best language of physical reality and that all physical phenomena could be described in the language of mathematics, which had a crucial impact on his lectures. His effort is best to be seen in the fact that he used geometrical reasoning in rational mechanics, so that he can prepare his students for the course of vectors analysis. Based on the preserved notes of one of his students Borivoj Pujić, we will try to look at his lectures of the course of vectors analysis and rational mechanics (see [5]).

The underlying concepts and principles in theoretical physics and mechanics have a mathematical basis. Already at the introductory lecture in rational mechanics Milanković points to the importance of mathematics. The first lecture begins with a sentence (see[6]):

"*Apart from time, other physical quantities that the Mechanics operate, such as length, area, angles, etc. were taken from the Geometry from which the Mechanics originated…*"

Throughout the whole course of theoretical physics persistent attention to its mathematical aspect is presented. Milanković used vector calculus as the main tool to solve various problems in rational and celestial mechanics. Therefore, his course in applied mathematics is mainly a course on the unitary three-dimensional vector space $\mathbf{R}^3$.

It should be said that this was not a classical course in linear algebra. At the beginning of the 20th century many mathematicians paid more attention to vector analysis, as vectors in $\mathbf{R}^3$ proved to be basic mathematical elements which connects physics and mathematics. In his lectures Milanković omitted the concepts of vector spaces, linear operators and matrices.

Instead, more attention is given to each new definition being illustrated by an appropriate example. The first part operates with a vector algebra and the other with vector analysis. Each lesson is carefully prepared and followed by physical examples. Several of them are illustrated in the following pictures (see [5] and [6] - the original notes from Milanković's lectures).

- After defining the concept of a cross product of two vectors and examining it's basic properties, examples of specific application are given. In rational mechanics, circular movement of a rigid body rotating around one axis can be represented by a rotation vector. The velocity vector at point M is equal to the vector product of the rotation vector and the radius vector. Milanković marked the vectors with Gothic letters, while he used the symbol [v w] for the vector product of vectors v and w.

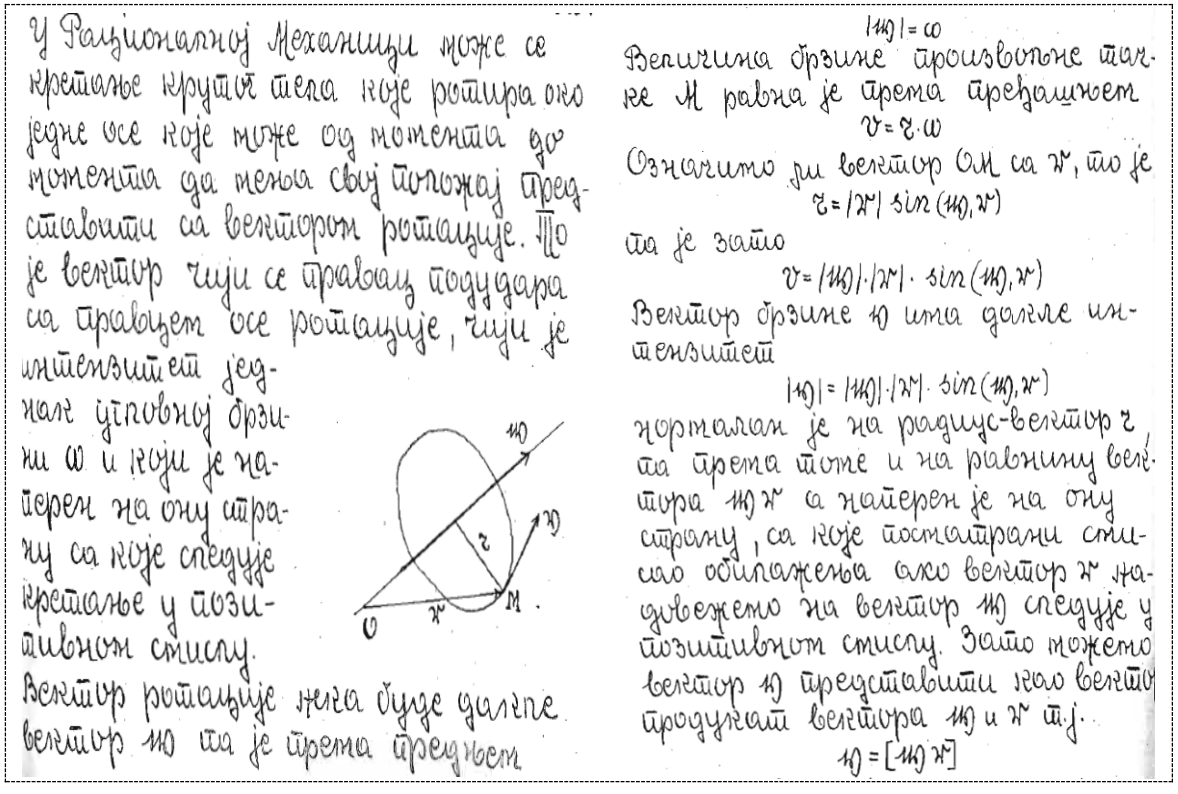


Fig. 3 *Original notes from Milanković's lectures - One application of a vector product*

- As a motivation for the introduction of vector differentiation, Milanković gives an example of a girder loaded with mechanical force and defines the limit curve (see Fig. 4). If it is a horizontal girder, then the force depends on one scalar quantitie, which represents the distance from a fixed point. Any change in this scalar implies a change in the vector in the girder arm, which in this way describes the limit curve. This practical example prepares students for the limit process that follow. The mathematical quantities that are used to describe the motion of objects can be divided into two categories – vectors and scalars. This example also makes a clear difference between vector and scalar variables. Later he explains in detail the application of vector differentiation to tangential and normal acceleration.

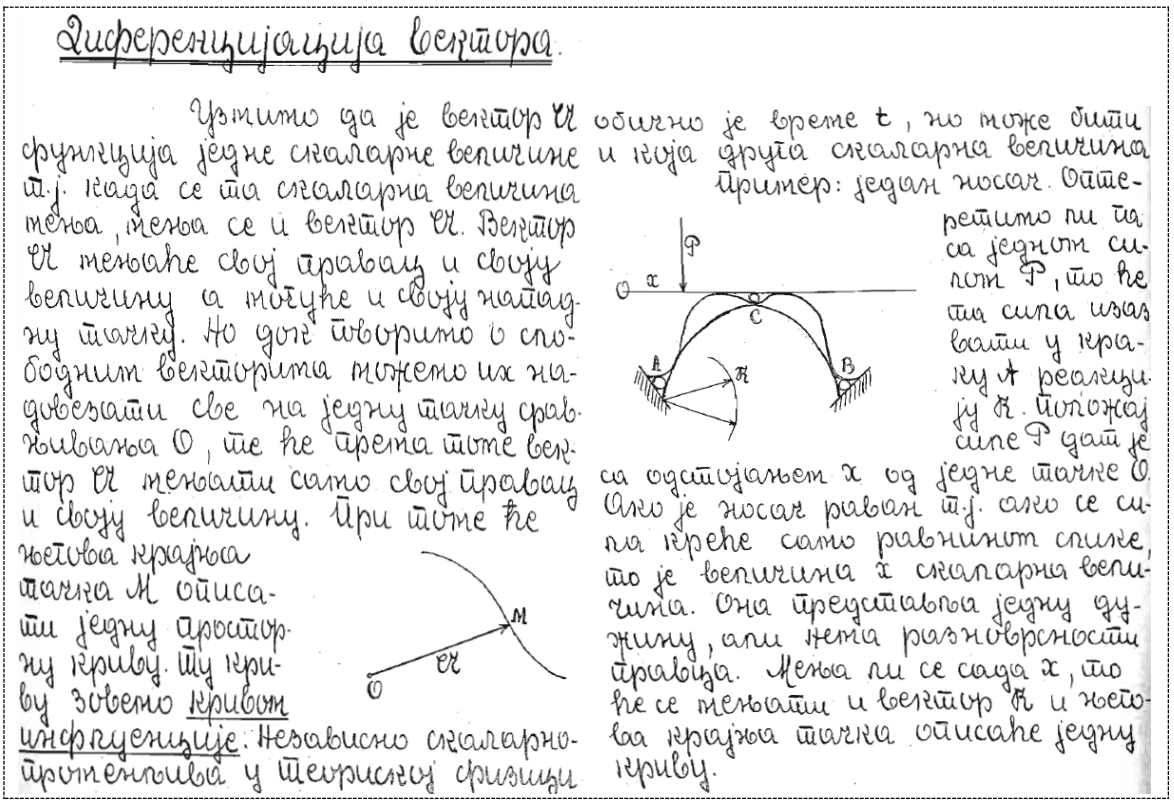
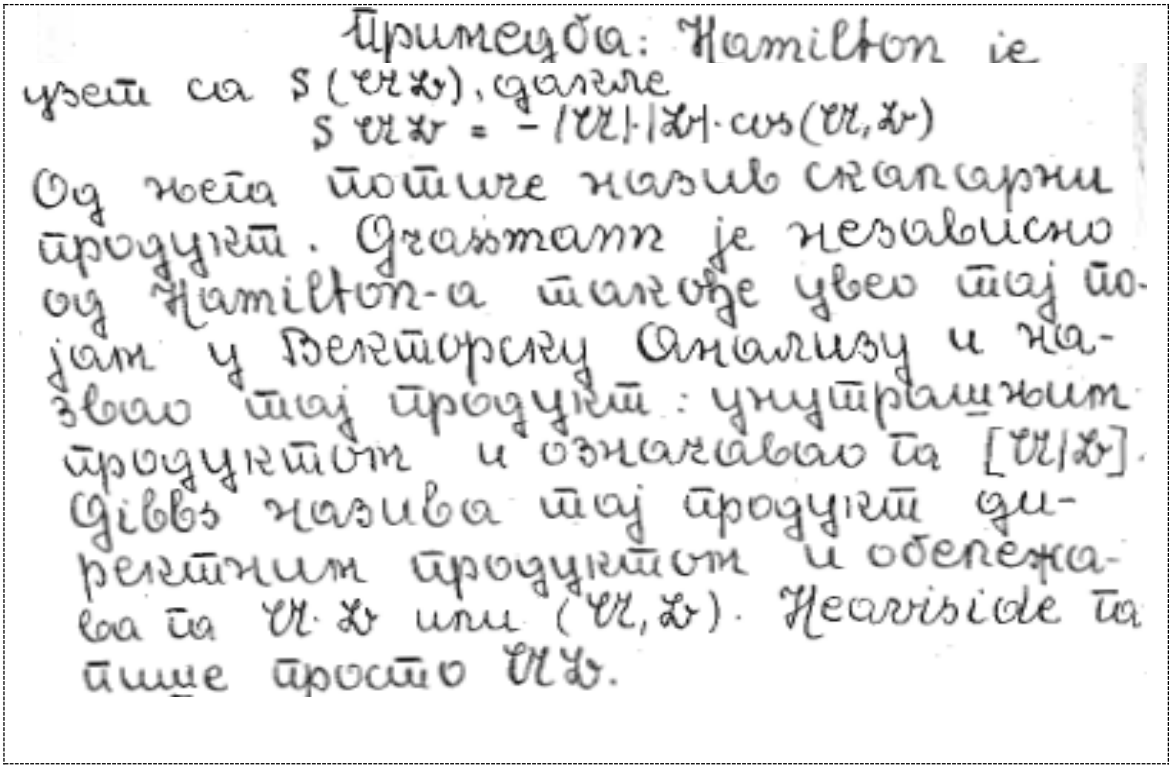


Fig. 4 *Original notes from Milanković's lectures - Differentiation of vectors. Introduction.*

The rest of the lectures is devoted to scalar and vector fields theory, the concepts of gradient and divergence, the Hamiltonian and Laplace operator and other common elements in vector calculus. Special attention is paid to the classical Stokes' and Gauss' divergence theorem and its physical applications. We refer [5] for details to the interested reader.

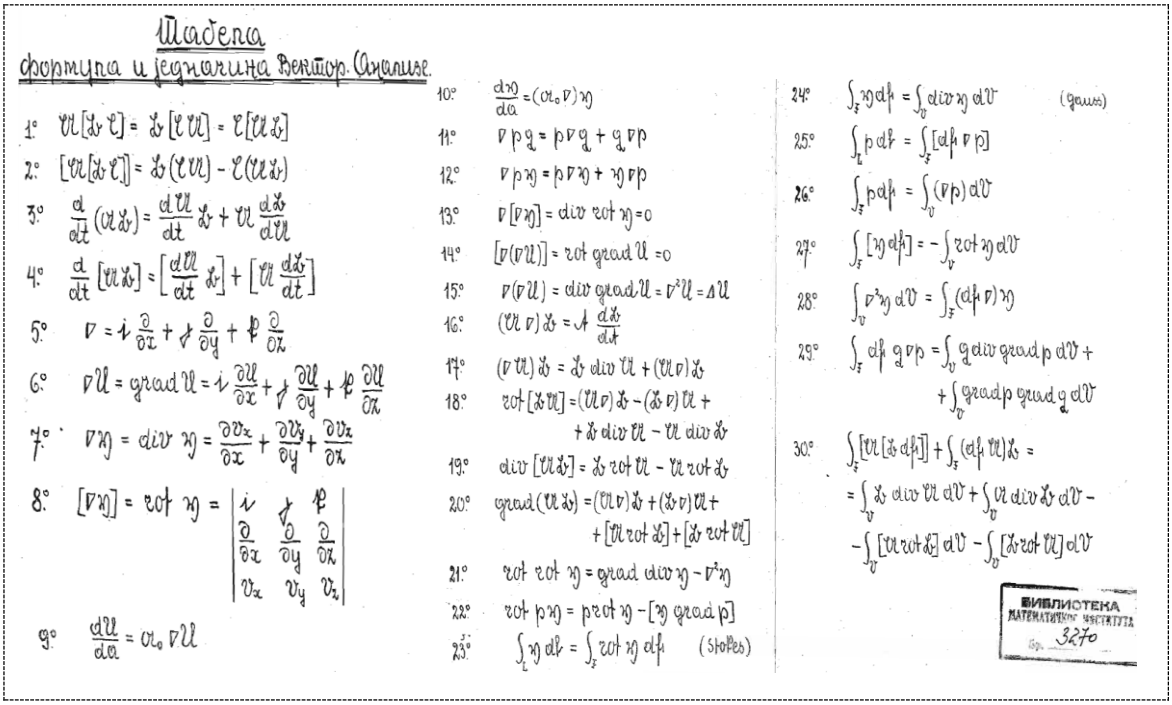
- It should be noted that Milanković also took into account and instructed students to use existing literature. He always gave a brief historical overview of the development of some mathematical concept. At that time the labels in mathematics were not quite uniform. For example, when defining the inner product, Milanković points out that Hamilton introduced this term in mathematics and introduced students to other existing labels of the same object. Grassmann used the symbol $[v|w]$, Hamilton $S(v,w)$, Heaviside vw and Gibbs (v,w) or $v \cdot w$, and the students were familiar with it.



Примедба: Hamilton је
узеш са $S(\mathfrak{U}\mathfrak{V})$, дакле
$S\,\mathfrak{U}\mathfrak{V} = -|\mathfrak{U}|\cdot|\mathfrak{V}|\cdot\cos(\mathfrak{U},\mathfrak{V})$
Од њега потиче назив скаларни
продукт. Grassmann је независно
од Hamilton-а такође увео тај по-
јам у Векторску Анализу и на-
звао тај продукт: унутрашњим
продуктом и означавао га $[\mathfrak{U}|\mathfrak{V}]$.
Gibbs назива тај продукт ди-
ректним продуктом и обележа-
ва га $\mathfrak{U}\cdot\mathfrak{V}$ или $(\mathfrak{U},\mathfrak{V})$. Heaviside га
још проста $\mathfrak{U}\mathfrak{V}$.

Fig. 5 *Original notes from Milanković's lectures - Historical overview*

- At the last lecture he worked on the recapitulation of the entire material and gave the basic formulas of the vector analysis (Fig 6.).



Шасета
формула и једначина Вектор. Анализе

1° $\mathfrak{U}[\mathfrak{V}\mathfrak{W}] = \mathfrak{V}[\mathfrak{W}\mathfrak{U}] = \mathfrak{W}[\mathfrak{U}\mathfrak{V}]$

2° $[\mathfrak{U}[\mathfrak{V}\mathfrak{W}]] = \mathfrak{V}(\mathfrak{W}\mathfrak{U}) - \mathfrak{W}(\mathfrak{U}\mathfrak{V})$

3° $\frac{d}{dt}(\mathfrak{U}\mathfrak{V}) = \frac{d\mathfrak{U}}{dt}\mathfrak{V} + \mathfrak{U}\frac{d\mathfrak{V}}{dt}$

4° $\frac{d}{dt}[\mathfrak{U}\mathfrak{V}] = \left[\frac{d\mathfrak{U}}{dt}\mathfrak{V}\right] + \left[\mathfrak{U}\frac{d\mathfrak{V}}{dt}\right]$

5° $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

6° $\nabla\mathfrak{U} = \operatorname{grad}\mathfrak{U} = i\frac{\partial\mathfrak{U}}{\partial x} + j\frac{\partial\mathfrak{U}}{\partial y} + k\frac{\partial\mathfrak{U}}{\partial z}$

7° $\nabla\mathfrak{v} = \operatorname{div}\mathfrak{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

8° $[\nabla\mathfrak{v}] = \operatorname{rot}\mathfrak{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$

9° $\frac{d\mathfrak{U}}{da} = \alpha_0\nabla\mathfrak{U}$

10° $\frac{d\mathfrak{v}}{da} = (\alpha_0\nabla)\mathfrak{v}$

11° $\nabla pq = p\nabla q + q\nabla p$

12° $\nabla p\mathfrak{v} = p\nabla\mathfrak{v} + \mathfrak{v}\nabla p$

13° $\nabla[\nabla\mathfrak{v}] = \operatorname{div}\operatorname{rot}\mathfrak{v} = 0$

14° $[\nabla(\nabla\mathfrak{U})] = \operatorname{rot}\operatorname{grad}\mathfrak{U} = 0$

15° $\nabla(\nabla\mathfrak{U}) = \operatorname{div}\operatorname{grad}\mathfrak{U} = \nabla^2\mathfrak{U} = \Delta\mathfrak{U}$

16° $(\mathfrak{U}\nabla)\mathfrak{V} = \mathfrak{U}\frac{d\mathfrak{V}}{dt}$

17° $(\nabla\mathfrak{U})\mathfrak{V} = \mathfrak{V}\operatorname{div}\mathfrak{U} + (\mathfrak{U}\nabla)\mathfrak{V}$

18° $\operatorname{rot}[\mathfrak{V}\mathfrak{W}] = (\mathfrak{W}\nabla)\mathfrak{V} - (\mathfrak{V}\nabla)\mathfrak{W} + \mathfrak{V}\operatorname{div}\mathfrak{W} - \mathfrak{W}\operatorname{div}\mathfrak{V}$

19° $\operatorname{div}[\mathfrak{U}\mathfrak{V}] = \mathfrak{V}\operatorname{rot}\mathfrak{U} - \mathfrak{U}\operatorname{rot}\mathfrak{V}$

20° $\operatorname{grad}(\mathfrak{U}\mathfrak{V}) = (\mathfrak{U}\nabla)\mathfrak{V} + (\mathfrak{V}\nabla)\mathfrak{U} + [\mathfrak{U}\operatorname{rot}\mathfrak{V}] + [\mathfrak{V}\operatorname{rot}\mathfrak{U}]$

21° $\operatorname{rot}\operatorname{rot}\mathfrak{v} = \operatorname{grad}\operatorname{div}\mathfrak{v} - \nabla^2\mathfrak{v}$

22° $\operatorname{rot}p\mathfrak{v} = p\operatorname{rot}\mathfrak{v} - [\mathfrak{v}\operatorname{grad}p]$

23° $\int_L \mathfrak{v}\,d\mathfrak{s} = \int_F \operatorname{rot}\mathfrak{v}\,df$ (Stokes)

24° $\int_F \mathfrak{v}\,df = \int_V \operatorname{div}\mathfrak{v}\,dV$ (Gauss)

25° $\int_L p\,d\mathfrak{s} = \int_F [df\nabla p]$

26° $\int_F p\,df = \int_V (\nabla p)\,dV$

27° $\int_F [\mathfrak{v}\,df] = -\int_V \operatorname{rot}\mathfrak{v}\,dV$

28° $\int_V \nabla^2\mathfrak{v}\,dV = \int_F (df\nabla)\mathfrak{v}$

29° $\int_F df\,q\nabla p = \int_V q\operatorname{div}\operatorname{grad}p\,dV + \int_V \operatorname{grad}p\operatorname{grad}q\,dV$

30° $\int_F [\mathfrak{U}[\mathfrak{V}\,df]] + \int_F (df\mathfrak{U})\mathfrak{V} = \int_V \mathfrak{V}\operatorname{div}\mathfrak{U}\,dV + \int_V \mathfrak{U}\operatorname{div}\mathfrak{V}\,dV - \int_V [\mathfrak{U}\operatorname{rot}\mathfrak{V}]\,dV - \int_V [\mathfrak{V}\operatorname{rot}\mathfrak{U}]\,dV$

Fig. 6 *Original notes from Milanković's lectures - Vector Calculus Formulas*

When writing this article, we also used [7], [8] and [9], where one can find a lot of information about the life and work of Milutin Milanković.

REFERENCES

[1] Tomić M. (1999) Uspomene na Milankovića, In: *Simpozijum Milanković - juče, danas, sutra*, Rudarsko – geološki fakultet, Beograd, p. 13-16.

[2] Milanković M. (1997) *Uspomene, doživljaji i saznanja,* Zavod za udžbenike i nastavna sredstva Srbije, Beograd, p. 935.

[3] Inđijić M. (1997) Life, scientific and professional activity of Milutin Milanković, *Bulletin Astronomique de Belgrade*, No 155, pp. 169-197.

[4] Milićević V. (2006) *Milanković u delima i slici*, Fond dr Milićević i Udruženje Milutin Milanković, Beograd, p. 177.

[5] Milanković M. (Pujić B.) (1910-1914) *Vektorska analiza sa primenama iz Teorijske fizike*, Digitalni legat Milutina Milankovića, Beograd, p. 127.

[6] Vitual library: http://elibrary.matf.bg.ac.rs

[7] Dimitrić R. (2016) *Mathematics: Abstraction and Reality - A Sketch Toward Deeper Analysis*, arXiv:1612.04254, p. 11.

[8] Petrović A. (2009) Revolution and Insolation. How Milutin Milanković has assembled the puzzle of the climate?, *Scientific Technical Review*, Vol.LIX, No.1, pp. 3-10.

[9] Spasova D., Radinović Đ., Milićević V., Maksimović S. (2009) *Milutin Milanković - putnik kroz vasionu i vekove*, Udruženje "Milutin Milanković", Beograd, p. 75.