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Treći međunarodni naučni skup

moNGeometrija 2012

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Zbornik radova

Serbia, Novi Sad, June 21st – 24th 2012

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ONE APPLICATION OF THE CONE SURFACES ON THE ERDÖS-MORDELL INEQUALITY

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Abstract

We discuss the spatial interpretation of the Erdös-Mordell inequality on the area of triangle ABC, and also consider the plane extension of this inequality.

Key words: triangle, surface, cone, inequality, Erdös-Mordell inequality

1. INTRODUCTION

Let triangle *ABC* be given in Euclidean plane. Denote by R_A, R_B and R_C the distances from the point *M* to the vertices *A*, *B* and *C* respectively, and denote by $r_a,\ r_b$ and r_c the distances from the point *M* to the sides *BC*, *CA* and *AB* respectively (*Figure 1*).

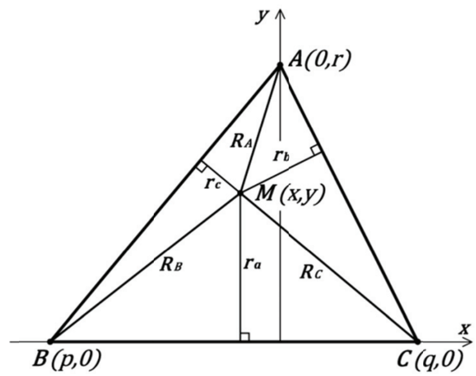

Figure 1: Erdös-Mordell inequality

In this paper we consider the extension of Erdös-Mordell Inequality:

$$R_A + R_B + R_C \geq 2(r_a + r_b + r_c). \tag{1}$$

Proofs of the Erdös-Mordell inequality are based on the following inequalities:

$$R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c \tag{2}$$

$$R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \tag{3}$$

$$R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b \tag{4}$$

Note that V. Pambuccian recently proved that the Erdös-Mordell inequality is equivalent to non-positive curvature [9].

In this paper, a spatial interpretation of inequalities (2), (3) and (4) will be given. All the symbolic calculus and all the figures in the research were done using MAPLE software package.

Let the triangle *ABC* be given, with the vertices *A*=$A(0,r)$, *B*=$B(p,0)$, *C*=$C(q,0)$, $p \neq q,\ r \neq 0$. Without diminishing generality, let $p < q$. Denote by *M*=$M(x,y)$ an arbitrary point in the plane of the triangle *ABC*. The distance from the point *M* to the point *A* and the distance from the point *M* to the lines *b* and *c* are given by functions:

$$R_A = \sqrt{x^2 + (y - r)^2} \tag{5}$$

$$r_b = \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} \tag{6}$$

$$r_c = \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}} \tag{7}$$

The inequality (2) is equal to the following inequality:

$$|q - p|\sqrt{r^2 + p^2}\sqrt{r^2 + q^2}\sqrt{x^2 + (y - r)^2} \geq (r^2 + p^2)|-qy - rx + qr| + (r^2 + q^2)|py + rx - pr|, \tag{8}$$

for which we will give a spatial interpretation in the section 2.

1.1. The division of the triangle plane area

Let us define the following terms:

$$AB(x, y) = -qy - rx + qr \tag{9}$$

$$AC(x, y) = py + rx - pr \tag{10}$$

For the inequality (8) the plane $\mathbb{R}^2$ is divided into the following areas λ_i (i=1…4).

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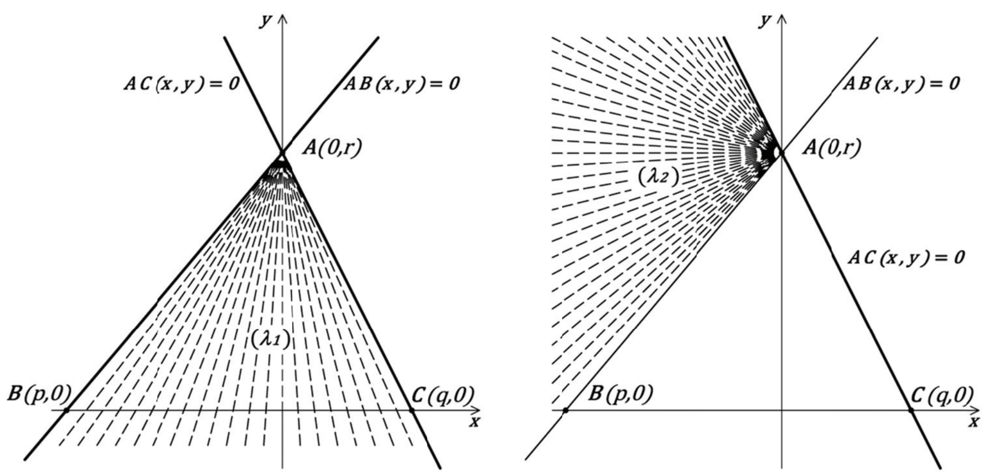
$$(\lambda_1): \begin{cases} AB(x,y) \geq 0 \\ AC(x,y) \geq 0 \end{cases} \tag{11}$$

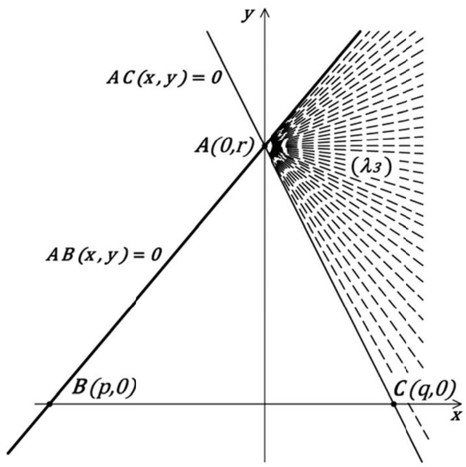
$$(\lambda_2): \begin{cases} AB(x,y) < 0 \\ AC(x,y) \geq 0 \end{cases} \tag{12}$$

$$(\lambda_3): \begin{cases} AB(x,y) \geq 0 \\ AC(x,y) < 0 \end{cases} \tag{13}$$

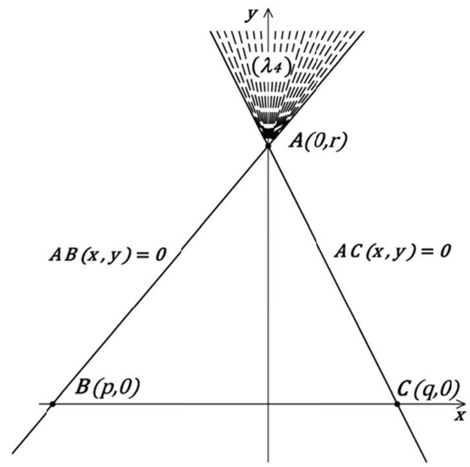
$$(\lambda_4): \begin{cases} AB(x,y) < 0 \\ AC(x,y) < 0 \end{cases} \tag{14}$$



a) Area $\boldsymbol{\lambda_1}$ *b) Area* $\boldsymbol{\lambda_2}$



c) Area λ_3



d) Area λ_4

Figure 2: The triangle plane division into areas λ_i (i=1…4) for the vertex A

2. THE SPATIAL INTERPRETATION

Let us define the following function in $\mathbb{R}^3$:

$$z = |q-p|\sqrt{r^2+p^2}\sqrt{r^2+q^2}\sqrt{x^2+(y-r)^2}$$

$$-\left((r^2+p^2)|-qy-rx+qr|+(r^2+q^2)|py+rx-pr|\right), \quad (15)$$

which determins surface $z = z(x,y)$. The observed surface fulfills the following implicit equation:

$$(q-p)^2(r^2+p^2)(r^2+q^2)(x^2+(y-r)^2) =$$

$$(z+(r^2+p^2)|-qy-rx+qr|+(r^2+q^2)|py+rx-pr|)^2 \quad (16)$$

The surface (16) is graphically displayed in *Figure 2a*. For the surface $z = z(x,y)$ it is valid:

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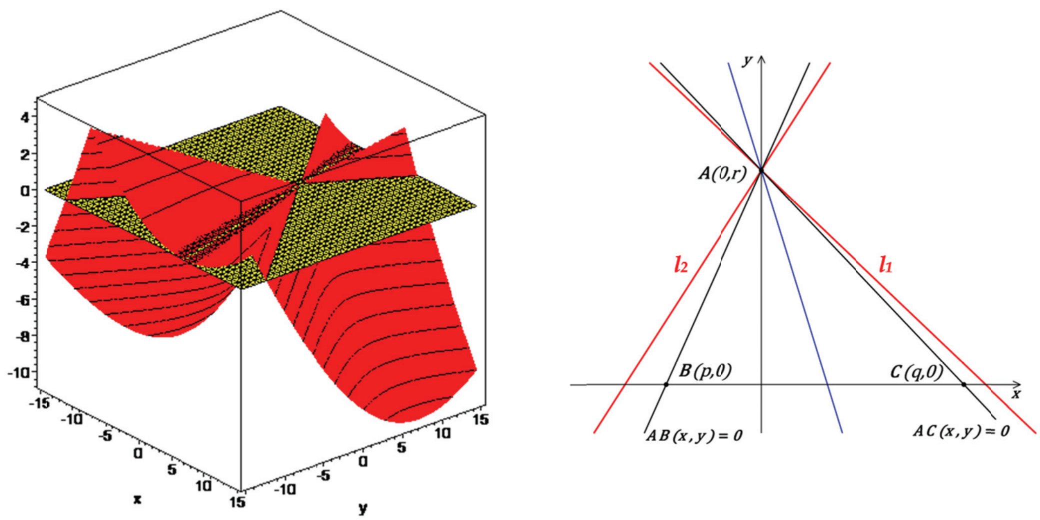
$$(2) \iff z = z(x, y) > 0 \tag{17}$$

For *z=0* i.e. in the plane *Oxy,* we obtain the intersecting straight lines, as shown in the *Figure 2b*, (whereby the straight lines of the initial triangle *ABC* are shown as well).



a) $z = z(x, y)$ *b)* $z = 0$

Figure 3

2.1 The division of the conical surfaces according to the areas λ_i (i=1...4)

The general equation of a cone is:

$$Cone(x, Y, z) : \hat{a}x^2 + \hat{b}Y^2 + \hat{c}z^2 + 2\hat{u}xY + 2\hat{v}Yz + 2\hat{w}xz = 0, \; Y = y - r \tag{18}$$

with the apex at (*0, r, 0*). This is a homogenous equation of degree 2.

The surface (16) shown in *Figure* contains parts of four conic surfaces given by the equations (19) – (22). The graphical representation of these surfaces is displayed in *Table 1* for $\angle BAC < \pi/2$, *Table 2 for* $\angle BAC = \pi/2$ and *Table 3* for $\angle BAC > \pi/2$.

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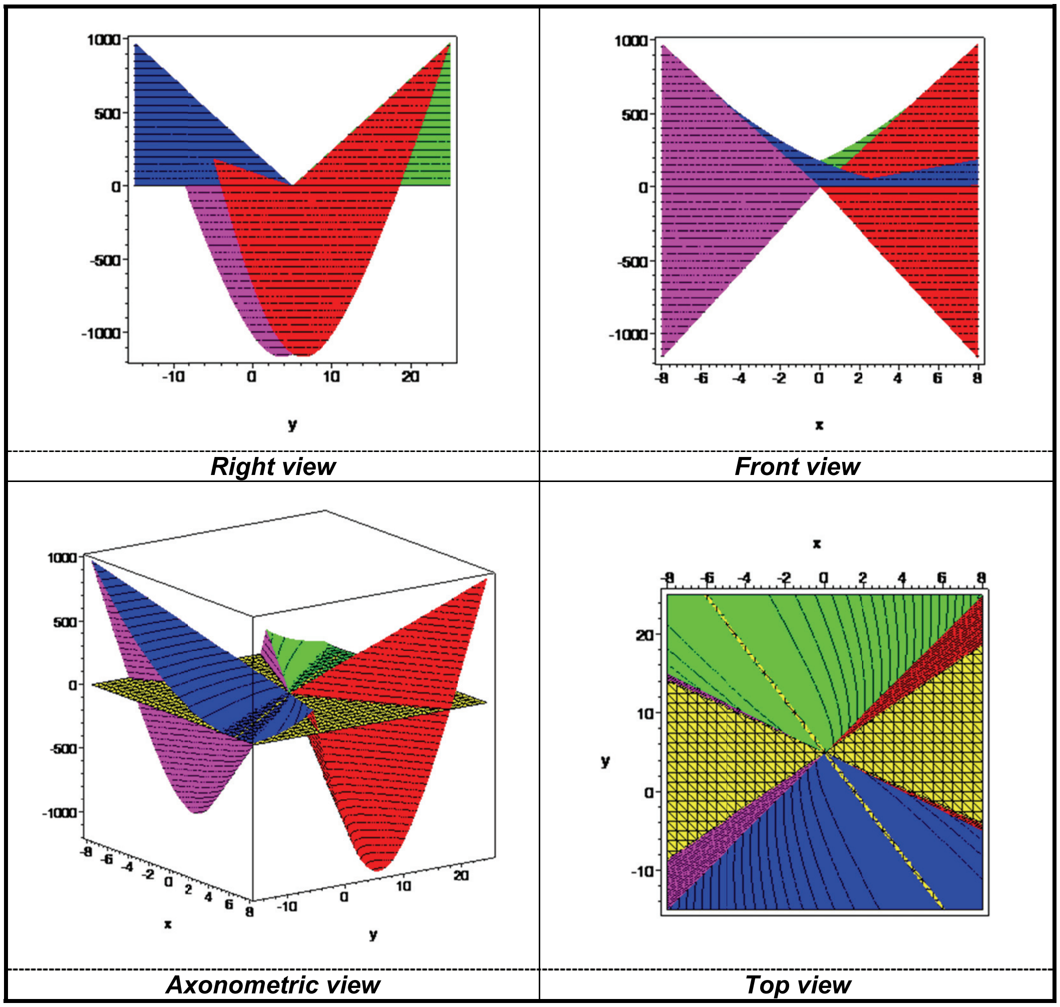

Figure 4: The surface $z = z(x, y)$ displayed in three standard projections, with an axonometric view, for $\angle BAC < \pi/2$

$$
(\lambda_1):\ \begin{cases}
\hat{a} = (p-q)^2(pq-r^2)^2 \\
\hat{b} = r^2(p-q)^2(p+q)^2 \\
\hat{c} = -1 \\
\hat{u} = -r(p+q)(p-q)^2(pq-r^2) \\
\hat{v} = (p-q)(pq-r^2) \\
\hat{w} = r(p-q)(p+q)
\end{cases} \tag{19}
$$

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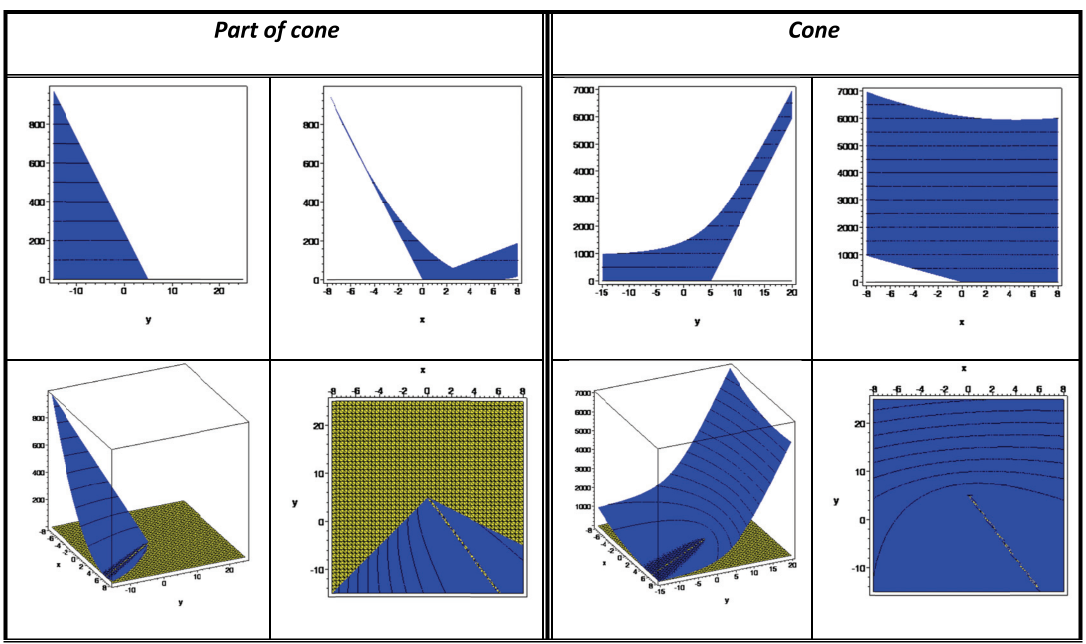
$$(\lambda_2):\begin{cases}\hat{a}=(pq+r^2)\left(pq(p-q)^2-r^2\left((p-q)^2+2(q^2+p^2+2r^2)\right)\right)\\ \hat{b}=-4pq(p^2+r^2)(q^2+r^2)+r^2(p-q)^2(p+q)^2\\ \hat{c}=-1\\ \hat{u}=-r(p+q)(pq+r^2)(p^2+q^2+2r^2)\\ \hat{v}=(p+q)(pq+r^2)\\ \hat{w}=r(p^2+q^2+2r^2)\end{cases} \tag{20}$$

$$(\lambda_3):\begin{cases}\hat{a}=(pq+r^2)\left(pq(p-q)^2-r^2\left((p-q)^2+2(q^2+p^2+2r^2)\right)\right)\\ \hat{b}=-4pq(p^2+r^2)(q^2+r^2)+r^2(p-q)^2(p+q)^2\\ \hat{c}=-1\\ \hat{u}=-r(p+q)(pq+r^2)(p^2+q^2+2r^2)\\ \hat{v}=-(p+q)(pq+r^2)\\ \hat{w}=-r(p^2+q^2+2r^2)\end{cases} \tag{21}$$

$$(\lambda_4):\begin{cases}\hat{a}=(p-q)^2(pq-r^2)^2\\ \hat{b}=r^2(p-q)^2(p+q)^2\\ \hat{c}=-1\\ \hat{u}=-r(p+q)(p-q)^2(pq-r^2)\\ \hat{v}=-(p-q)(pq-r^2)\\ \hat{w}=-r(p-q)(p+q)\end{cases} \tag{22}$$



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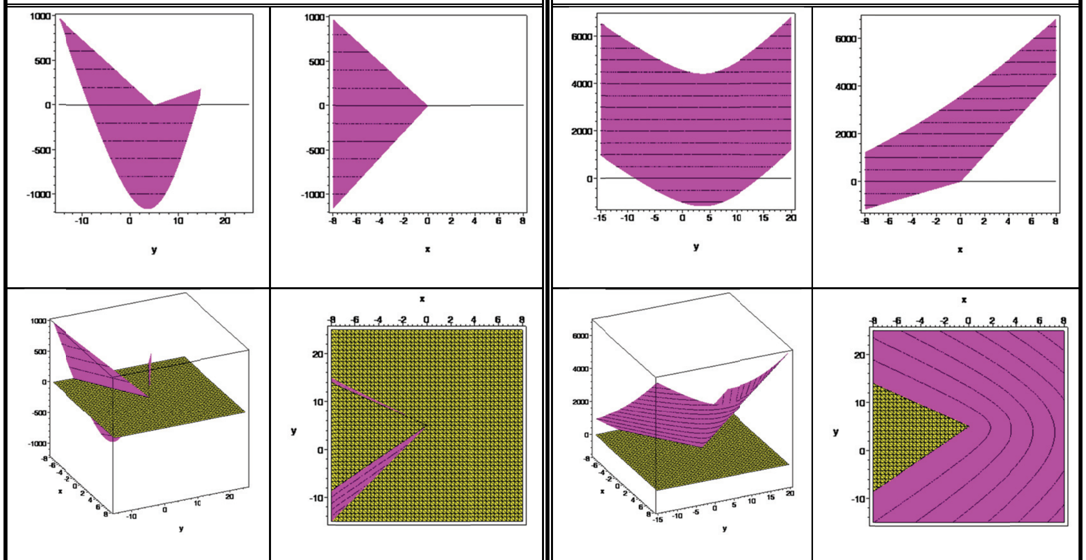
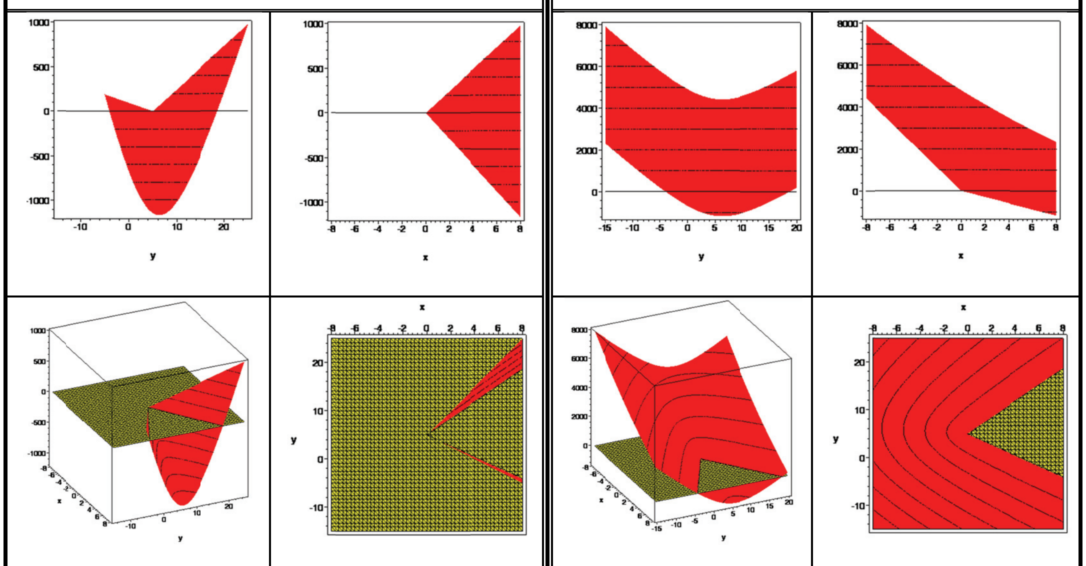
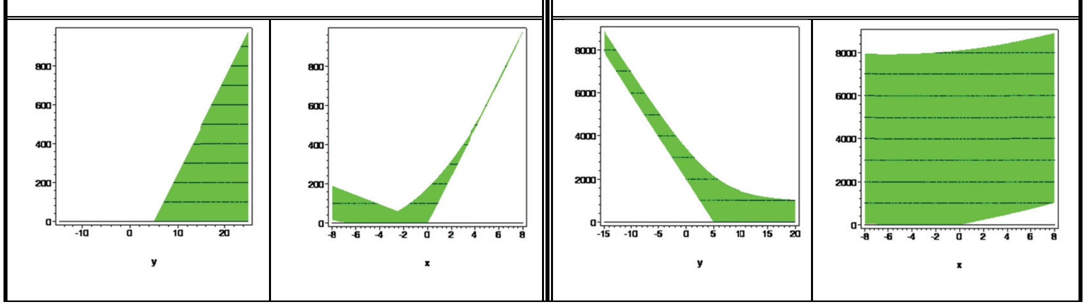
Area λ_1



Area λ_2



Area λ_3



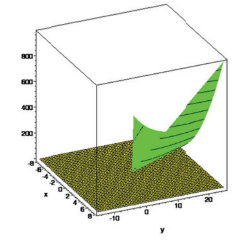
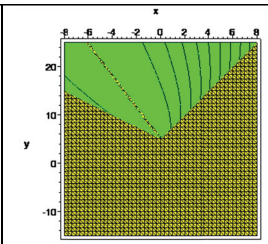
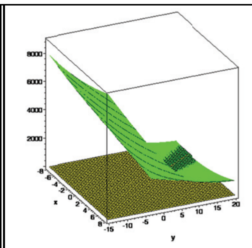
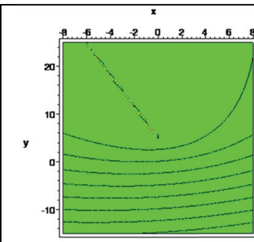
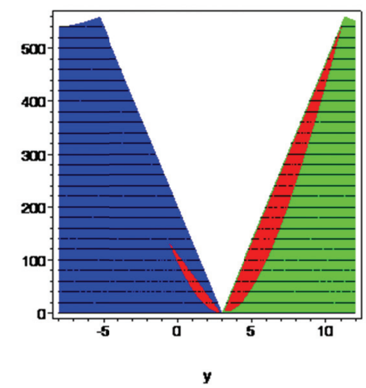
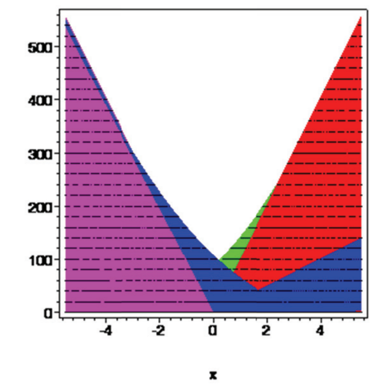
			
Area λ_4			

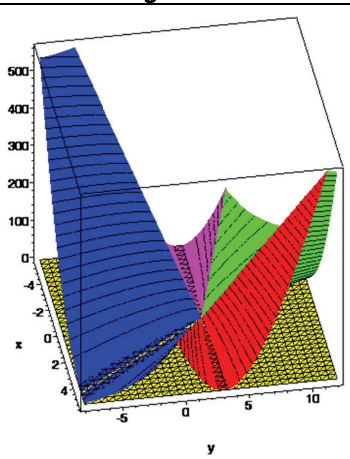
Table 2: The display of all the conic surfaces' parts in the areas λ_i (i=1…4) for $\angle BAC < \pi / 2$



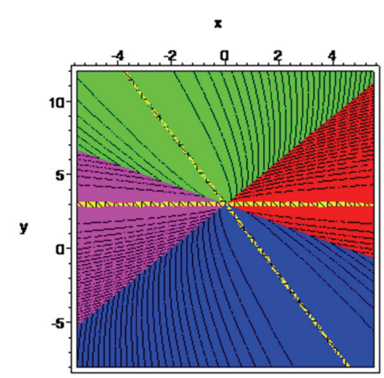
Right view



Front view



Axonometric view



Top view

Figure 5: The surface $z = z(x, y)$ displayed in three standard projections, with an axonometric view, for $\angle BAC = \pi / 2$

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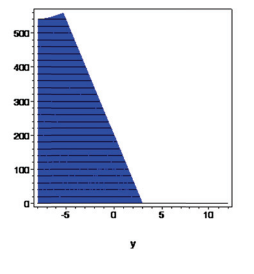
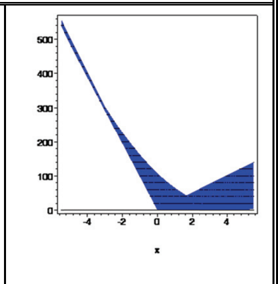
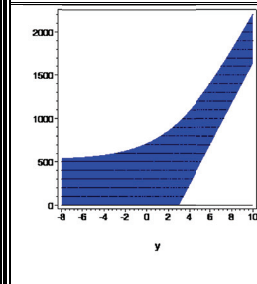
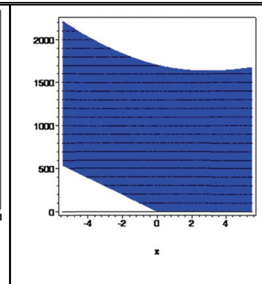
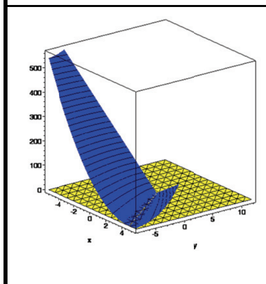
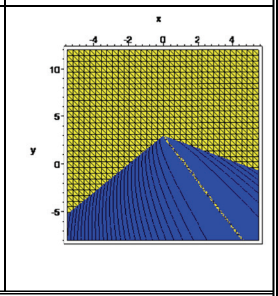
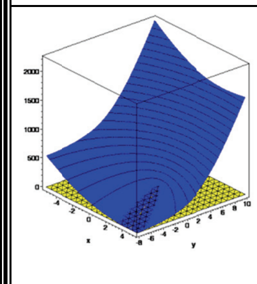
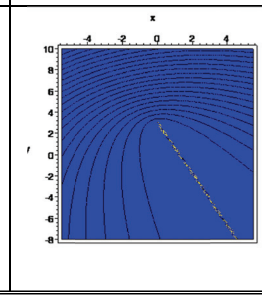
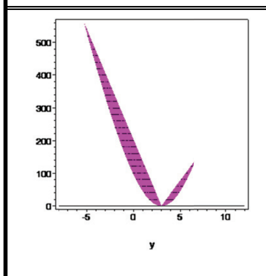
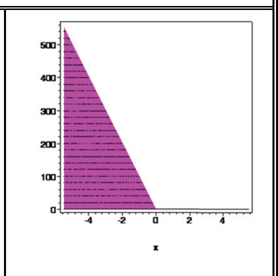
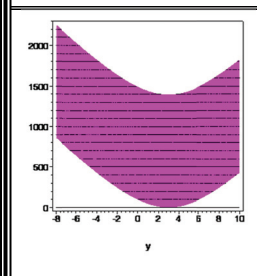
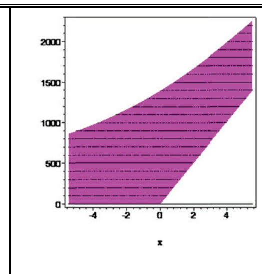
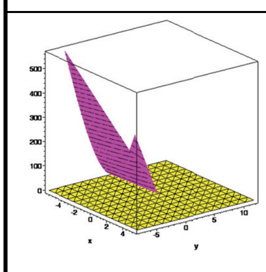
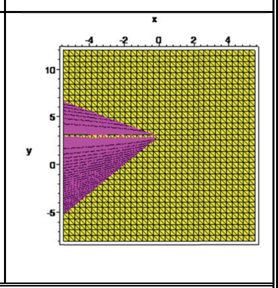
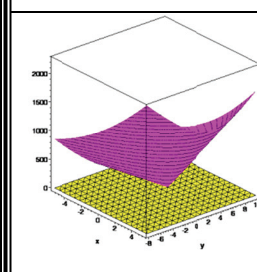
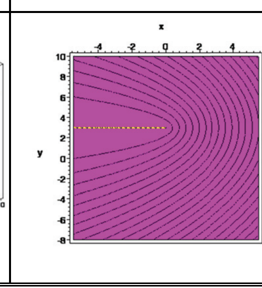
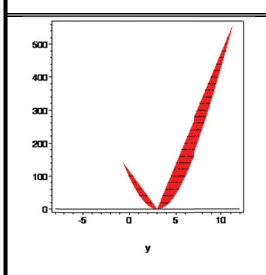
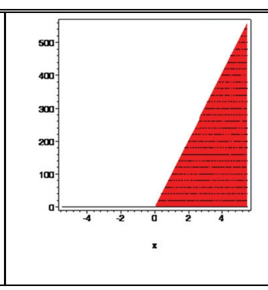
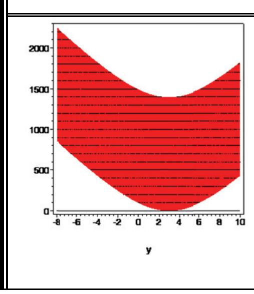
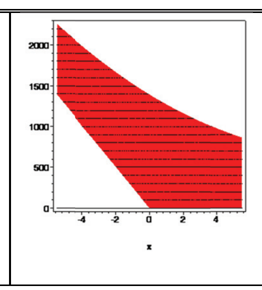
Part of cone		***Cone***	
			
			
Area λ_1			
			
			
Area λ_2			
			

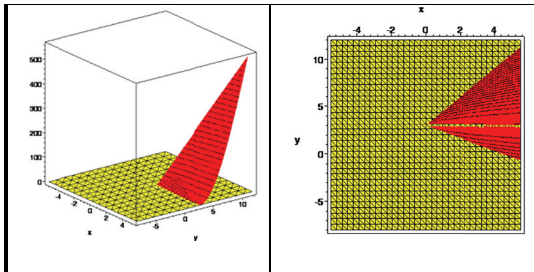
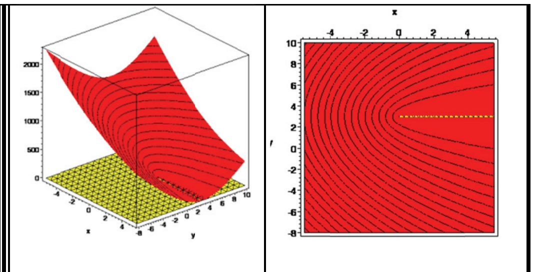
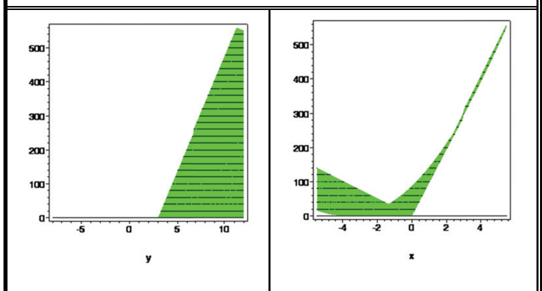
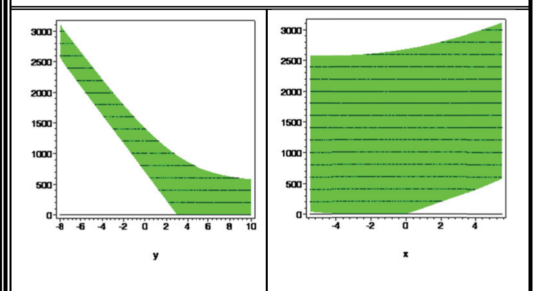
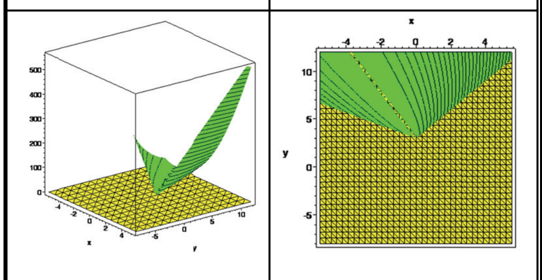
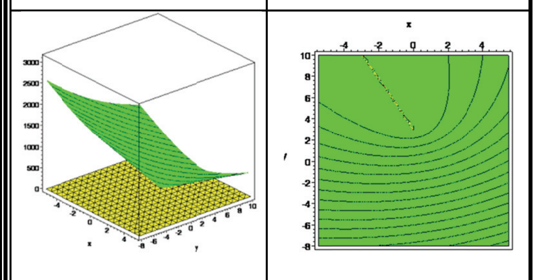
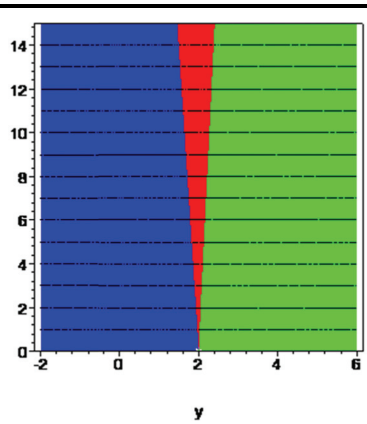
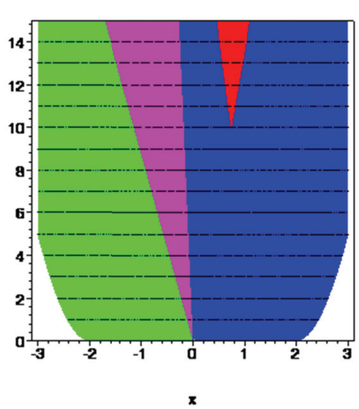
	
Area λ_3	
	
	
Area λ_4	

Table 3: The display of all the conic surfaces' parts in the areas λ_i *(i=1…4) for* $\angle BAC = \pi / 2$

	
Right view	***Front view***

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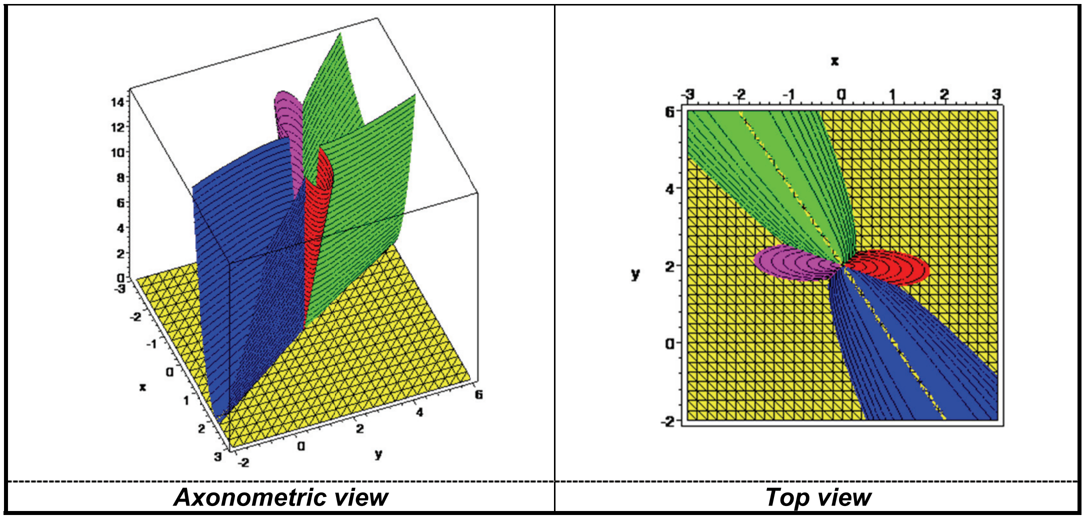
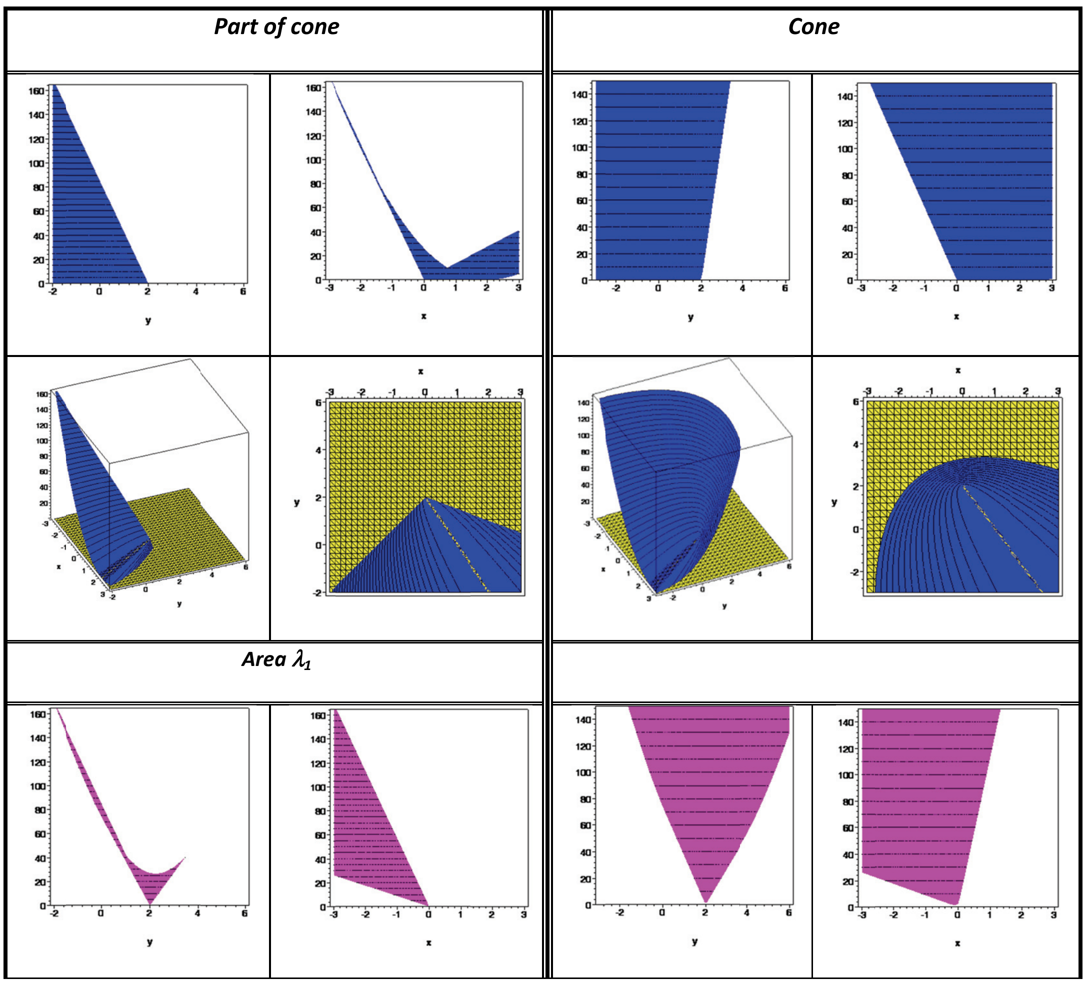


Figure 6: The surface $z = z(x, y)$ displayed in three standard projections, with an axonometric view, for $\angle BAC > \pi / 2$



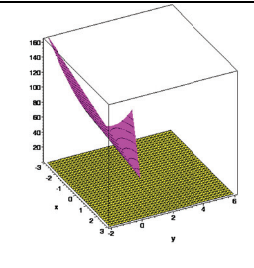
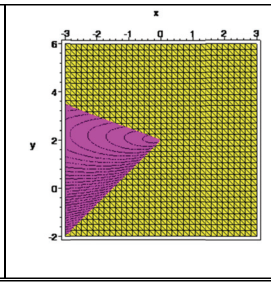
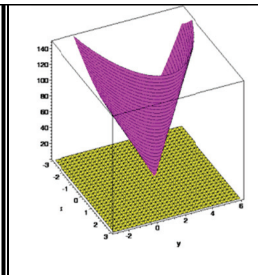
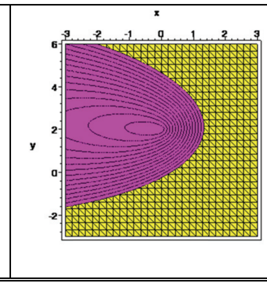
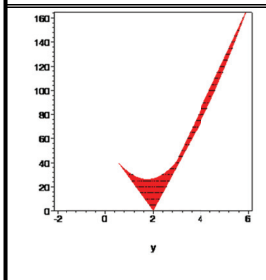
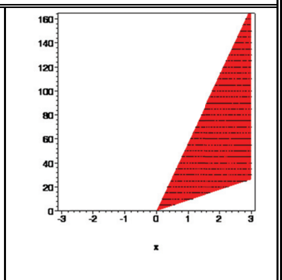
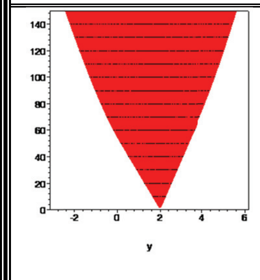
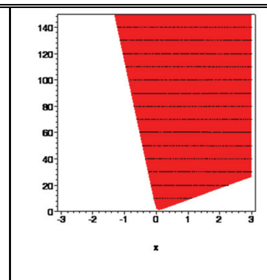
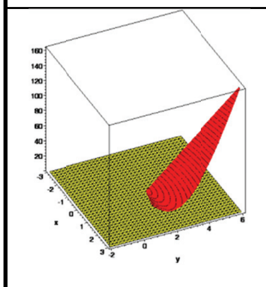
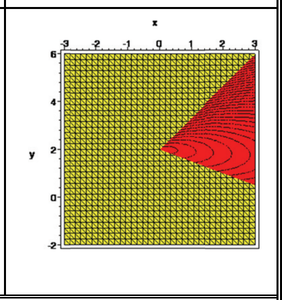
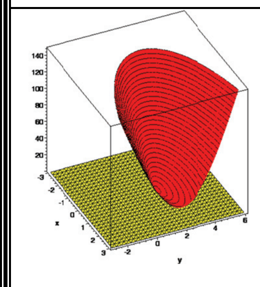
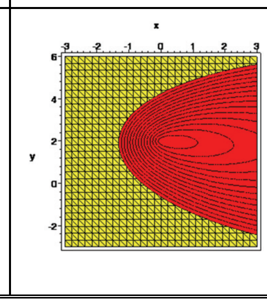
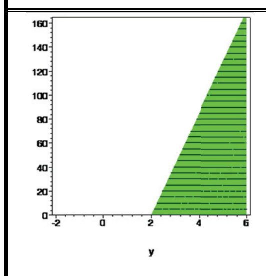
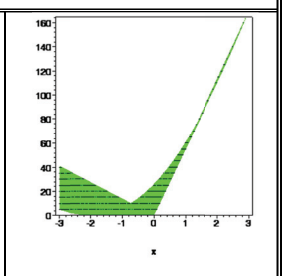
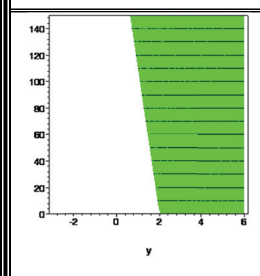
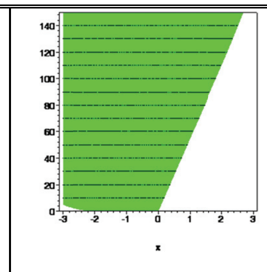
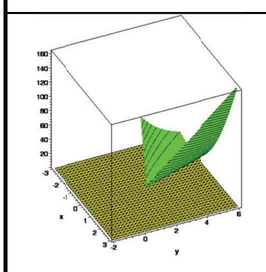
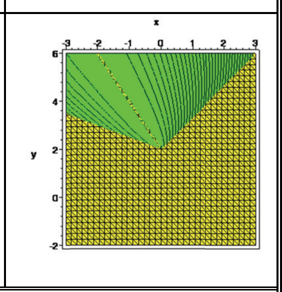
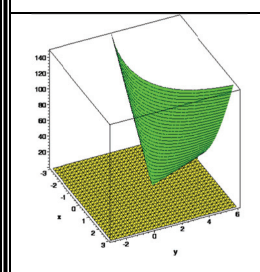
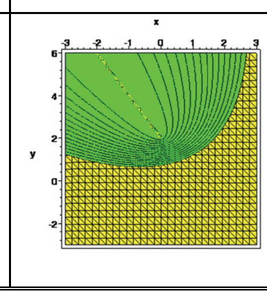
			
Area λ_2			
			
			
Area λ_3			
			
			
Area λ_4			

Table 4: The display of the conic surfaces' parts in the areas λ_i *(i=1…4) for* $\angle BAC > \pi/2$

In [3] and [7], diverse proves of the inequality (2) are given for the points in the interior of the triangle *ABC*. In the paper [6] it has been shown that the inequality (2) is valid in the area of the angle between the lines $l_1: y = k_2 x + r$ and $l_2: y = k_3 x + r$.

Consider the point $A(x_0, y_0)$ of the triangle *ABC.* Let us calculate $z_0 = z(x_0, y_0)$, then $z_0 > 0$. Set a ray from the point *A* through the point *P*=$P(x_0, y_0, z_0)$. Then all the points of the ray observed are above the plane *Oxy*. The above proves that the inequality (2) extends from the triangle *ABC* to the whole area of the angle in the vertex *A*, which contains the triangle *ABC*. This conclusion also holds for all the rays *AP* in the area of the angle between the straight lines l_1 and l_2 which are contained in the triangle *ABC* (according to [6]).

The paper [6] shows that the area $\boldsymbol{E_A}$ for the vertex *A*:

$$\boldsymbol{E_A} = \left\{(x,y) \middle| R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c\right\}, \tag{23}$$

represents the area of the angle between the lines l_1 and l_2, which contains the triangle *ABC*. For the vertices *B* and *C*, let us define

$$\boldsymbol{E_B} = \left\{(x,y) \middle| R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c\right\}, \tag{24}$$

$$\boldsymbol{E_C} = \left\{(x,y) \middle| R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b\right\}. \tag{25}$$

Respectively, according to [6]. The analogous surfaces, as for the surface $z = z(x, y)$ are to be formed for the vertices *B* and *C*, as well. Based on the previous spatial analysis of the inequality (2), as for the (3) and (4), the inequality

$$R_A + R_B + R_C \geq \left(\frac{c}{b} + \frac{b}{c}\right) r_a + \left(\frac{c}{a} + \frac{a}{c}\right) r_b + \left(\frac{a}{b} + \frac{b}{a}\right) r_c, \tag{26}$$

is valid in the intersection of the areas:

$$E = E_A \cap E_B \cap E_C. \tag{27}$$

Hence we obtain a spatial interpretation of the proof in the following statement.

Theorem. The Erdös-Mordell inequality (1) is valid in the area ***E***.

3. CONCLUSIONS

By setting a conical surface with the vertex at the point *A* of the triangle *ABC,* and by repeating the procedure for the vertices *B* and *C*, we perform a spatial interpretation of Erdös-Mordell inequality proof, which confirms the extension of the inequality given in [6].

The previous procedure can provide spatial interpretation of proves of various planimetric inequalities based on the distances $R_A, R_B, R_C, r_a,\ r_b$ and r_c, such as Child inequality [4].

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