Weighted P−partitions enumerator

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Abstract

To an extended permutohedron we associate the weighted integer points enumerator, whose principal specialization is the f-polynomial. In the case of poset cones it refines Gessel's P-partitions enumerator. We show that this enumerator is a quasisymmetric function obtained by universal morphism from the Hopf algebra of posets.

Keywords: generalized permutohedron, quasisymmetric function, poset, combinatorial Hopf algebra

1 Introduction

In the seminal paper of Aguiar, Bergeron and Sottile [2] the notion of combinatorial Hopf algebra was introduced. They explained the ubiquity of quasisymmetric functions as generating functions in enumerative combinatorics. More recently a geometric meaning of quasisymmetric enumerators is attributed. It is based on a class of convex polytopes called generalized permutohedra. The integer points enumerator associated to a generalized permutohedron is a quasisymmetric function. It was defined, and studied in the case of matroid base polytopes, by Billera, Jia and Reiner in [3], and in the case of nestohedra by Grujić in [6]. More subtle generalization which takes into account the face structure of a generalized permutohedron is introduced and studied in [8]. In this paper we consider the extended generalized permutohedra and the special case of poset cones. We prove that the integer points enumerator associated to a poset cone coincides with the universal morphism from the Hopf algebra of posets to quasisymmetric functions. The specialization for $q = 0$ is the Gessel enumerator of P-partitions which attributes the geometric meaning to this classical function.

In sections 2 we review necessary facts about quasisymmetric functions and P-partitions enumerators. In section 3 we introduce the integer points enumerator $F_q(P)$ for an extended generalized permutohedron P, which is a weighted quasisymmetric function, in the same manner as in the case of generalized permutohedra provided in [8]. The parameter q reflects the rank function of the face lattice $L(P)$. To a poset P is associated the poset cone $C(\mathsf{P})$, which is an extended generalized permutohedron and our construction produces a weighted quasisymmetric function $F_q(C(\mathsf{P}))$. In section 4 we prove Theorem 4.2, the first main result of the paper, which states that the weighted quasisymmetric function $F_q(C(\mathsf{P}))$, constructed geometrically, has an algebraic meaning as the universal morphism from a certain combinatorial Hopf algebra of posets $\mathcal{P}$ to the Hopf algebra $QSym$ of quasisymmetric functions. This result is analogous to the previous results for simple graphs [7], matroids and building sets [8], and spreads their validity to the case of extended generalized permutohedra. The main theorem is followed by various examples, and statements about behavior of the enumerator of the opposite poset and under the action of the antipode. In Theorem 5.8 in section 5, it is shown that for a well labelled poset P and $q = 0$ our enumerator specializes to the classical Gessel's P-partitions enumerator. We also provide an example of posets with the same P-partitions enumerators but which are distinguished by corresponding weighted quasisymmetric enumerators.

2 Quasisymmetric functions

A *composition* α of a positive integer n, $\alpha \models n$, is an ordered list $(\alpha_1, \ldots, \alpha_k)$ of positive integers such that $\alpha_1 + \cdots + \alpha_k = n$. The *monomial quasisymmetric function* M_α indexed by the composition α is an element of the commutative algebra of formal power series in the countable ordered set of variables $\mathbf{x} = (x_1 < x_2 < x_3 < \cdots)$ defined by

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}.$$

The *algebra of quasisymmetric functions* $\mathcal{Q}Sym$, spanned by M_α when α runs over all compositions, is a subalgebra of the algebra of formal power series. The algebra $\mathcal{Q}Sym$ is a graded, connected Hopf algebra (see [5], Proposition 5.8). The homogeneous component $QSym_n$ is spanned by $\{M_\alpha\}_{\alpha \models n}$. Let $\zeta_{\mathcal{Q}} : \mathcal{Q}Sym \to \mathbf{k}$ be a linear multiplicative functional defined on the monomial basis by

$$\zeta_{\mathcal{Q}}(M_\alpha) = \begin{cases} 1, & \text{if } \alpha = (n) \text{ for } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The Hopf algebra $\mathcal{Q}Sym$ equipped with the character ζ_Q is the terminal object in the category of combinatorial Hopf algebras.

Theorem 2.1 ([2], Theorem 4.1)**.** *For a combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ there is a unique morphism of graded Hopf algebras $\Psi : H \to \mathcal{Q}Sym$ such that*

$$\Psi \circ \zeta_{\mathcal{Q}} = \zeta.$$

For a homogeneous element h of degree n the coefficients $\zeta_\alpha(h)$, $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ of $\Psi(h)$ in the monomial basis are given by

$$\zeta_\alpha(h) = \zeta^{\otimes k} \circ (p_{\alpha_1} \otimes \cdots \otimes p_{\alpha_k}) \circ \Delta^{k-1}(h),$$

where p_i is the projection of $\mathcal{H}$ on the i-th homogeneous component $\mathcal{H}_i$ and Δ^{k-1} is the $(k-1)-$fold coproduct map of $\mathcal{H}$.

For $F \in \mathcal{Q}Sym$ and $m \in \mathbb{N}$, let $\mathbf{ps}^1$ denotes the *principal specialization*

$$\mathbf{ps}^1(F)(m) = F(\underbrace{1, \ldots, 1}_{m}, 0, 0 \ldots).$$

We have

$$\mathbf{ps}^1(M_\alpha)(m) = \binom{m}{k(\alpha)},$$

where $k(\alpha)$ is the number of parts of $\alpha = (\alpha_1, \ldots, \alpha_k)$. Specially, for $m = -1$ we have

$$\mathbf{ps}^1(M_\alpha)(-1) = \binom{-1}{k(\alpha)} = (-1)^{k(\alpha)}.$$

For a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$ let $D(\alpha) \subseteq [n-1]$ be a subset defined by $D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. We say that $\alpha \models n$ *refines* $\beta \models n$, and write $\beta \preceq \alpha$, if $D(\beta) \subseteq D(\alpha)$.
Another important basis of $\mathcal{Q}Sym$ is the basis of *fundamental quasisymmetric functions* defined by

$$L_\alpha = \sum_{\alpha \preceq \beta} M_\beta.$$

2.1 Quasisymmetric enumerator of $\mathsf{P}-$partitions

A *labelled poset* P is a poset on some finite subset of positive integers. A $\mathsf{P}-$*partition* is a function $f : \mathsf{P} \to \mathbb{N}$ such that

- $i <_{\mathsf{P}} j$ and $i <_{\mathbb{Z}} j$ implies $f(i) \leq f(j)$,
- $i <_{\mathsf{P}} j$ and $i >_{\mathbb{Z}} j$ implies $f(i) < f(j)$.

Definition 2.2. A poset P is a *well labelled poset* if $i <_{\mathsf{P}} j$ implies $i >_{\mathbb{Z}} j$. In that case $\mathsf{P}-$partition is a function $f : \mathsf{P} \to \mathbb{N}$ such that

$$i <_{\mathsf{P}} j \quad \text{implies} \quad f(i) < f(j).$$

Denote by $\mathcal{A}(\mathsf{P})$ the set of all $\mathsf{P}-$partitions. Define *the enumerator of* $\mathsf{P}-$*partitions* by

$$F_{\mathsf{P}}(\mathbf{x}) = \sum_{f \in \mathcal{A}(\mathsf{P})} x_{f(1)} x_{f(2)} \cdots x_{f(n)}.$$

Proposition 2.3 ([5], Proposition 5.18)**.** *For a totally ordered labelled poset* $\mathsf{P} = \{i_1 <_{\mathsf{P}} i_2 <_{\mathsf{P}} \cdots <_{\mathsf{P}} i_n\}$ *the enumerator of* P*-partitions is equal to the fundamental quasisymmetric function*

$$F_{\mathsf{P}}(\mathbf{x}) = L_{\alpha(\mathsf{P})},$$

where $\alpha(\mathsf{P}) \models n$ *is a composition such that* $D(\alpha(\mathsf{P})) = \{j : i_j >_{\mathbb{Z}} i_{j+1}\}$.

Theorem 2.4 ([5], Theorem 5.19)**.** *For a labelled poset* P,

$$F_{\mathsf{P}}(\mathbf{x}) = \sum_{\mathsf{l} \in \mathcal{L}(\mathsf{P})} F_{\mathsf{l}}(\mathbf{x}),$$

where the sum is over the set $\mathcal{L}(\mathsf{P})$ *of all linear extensions* l *of* P.

Example 2.5. Figure 1 presents two posets with their enumerators of P-partitions. Note that the poset P_1 is not well labelled, while the poset P_2 is.

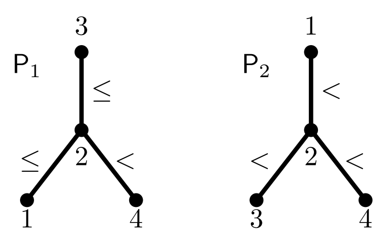

Figure 1: $F_{\mathsf{P}_1}(\mathbf{x}) = L_{(1,3)} + L_{(2,2)}$ and $F_{\mathsf{P}_2}(\mathbf{x}) = L_{(2,1,1)} + L_{(1,1,1,1)}$

3 Extended generalized permutohedra

A *standard* $(n-1)-$*dimensional permutohedron* Pe^{n-1} is the convex hull of the orbit of a point $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ with increasing coordinates $a_1 < a_2 < \cdots < a_n$

$$Pe^{n-1} = \text{conv}\{(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) \ : \ \sigma \in \mathfrak{S}_n\},$$

where $\mathfrak{S}_n$ is the permutation group of $[n]$.

Proposition 3.1 ([10], Proposition 2.6)**.** *The* $d-$*dimensional faces of* Pe^{n-1} *are in one-to-one correspondence with set compositions* $\mathcal{C} = C_1|C_2|\cdots|C_{n-d}$ *of* $[n]$. *The face corresponding to the set composition* $\mathcal{C}$ *is given by the* $n-d$ *linear equations*

$$\sum_{i \in C_1 \cup C_2 \cup \cdots \cup C_k} x_i \quad = \quad a_1 + a_2 + \cdots + a_{|C_1 \cup C_2 \cup \cdots \cup C_k|},$$

for $1 \leq k \leq n-d$.

A set composition $\mathcal{C} = C_1|C_2|\cdots|C_{n-d}$ defines the *flag* $\mathcal{F}$ of subsets

$$\mathcal{F} \,:\, \emptyset = F_0 \subset F_1 \subset \cdots \subset F_{n-d-1} \subset F_{n-d} = [n],$$

of the length $|\mathcal{F}| = n - d$, where $F_i = C_1 \cup C_2 \cup \cdots \cup C_i$, for $1 \leq i \leq n - d$. There is an obvious order reversing one−to−one correspondence between the face lattice of the permutohedron $L(Pe^{n-1})$ and the lattice of flags of subsets of the set $[n]$. Using this correspondence we will label faces of the standard permutohedron Pe^{n-1} by flags of subsets of $[n]$. We have

$$\dim(\mathcal{F}) = n - |\mathcal{F}|.$$

The normal fan $\mathcal{N}(Pe^{n-1})$ of the standard permutohedron Pe^{n-1} is the fan of the *braid arrangement*, given by hyperplanes $\{x_i = x_j\}_{1\leq i<j\leq n}$, in $\mathbb{R}^n$. The cones of the braid arrangement fan are called *braid cones.* The braid cone $C_{\mathcal{F}}$ at the face $\mathcal{F}$ is determined by

- $x_p = x_q$ if $p, q \in F_{i+1} \setminus F_i$, for some $0 \leq i \leq |\mathcal{F}| - 1$,
- $x_p \leq x_q$ if $p \in F_i \setminus F_{i-1}$ and $q \in F_{i+1} \setminus F_i$, for some $1 \leq i \leq |\mathcal{F}| - 1$.

We have $\dim(C_{\mathcal{F}}) = |\mathcal{F}|$. The fan $\mathcal{N}_1$ is a *refinement* of $\mathcal{N}_2$ (or $\mathcal{N}_2$ is a *coarsement* of $\mathcal{N}_1$) if every cone of $\mathcal{N}_1$ is contained in a cone in $\mathcal{N}_2$ (or if every cone in $\mathcal{N}_2$ is a union of cones of $\mathcal{N}_1$).

Definition 3.2. A convex polytope Q is an $(n-1)$-dimensional *generalized permutohedron* if the braid arrangement fan $\mathcal{N}(Pe^{n-1})$ refines the normal fan $\mathcal{N}(Q)$.

For a generalized permutohedron Q there is a map $\pi_Q : L(Pe^{n-1}) \rightarrow L(Q)$ between face latices, determined by $\pi_Q(\mathcal{F}) = G$ if and only if the relative interior of the braid cone $C^\circ_{\mathcal{F}}$ is contained in the relative interior C°_G of the normal cone C_G at the face $G \in L(Q)$. We say that the flag $\mathcal{F}$ is *normal* to the face G. Denote by $\mathsf{F}(G) = \{\mathcal{F} : C^\circ_{\mathcal{F}} \subseteq C^\circ_G\}$ the set of normal flags to a face G. By Proposition 2.3 in [8], we have

$$\sum_{\mathcal{F}\in\mathsf{F}(G)} (-1)^{|\mathcal{F}|} \;=\; (-1)^{n-1-\dim(G)}. \tag{1}$$

Definition 3.3. An *extended generalized permutohedron* P is a polyhedron whose normal fan $\mathcal{N}(P)$ is a coarsening of a subfan of the braid arrangement fan.

3.1 Quasisymmetric enumerator $F_q(P)$

A vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{Z}^n_+$ defines the *weight function* $\omega^* : \mathbb{R}^n \rightarrow \mathbb{R}$ by $\omega^*(x) := \langle\omega, x\rangle$ where $\langle\,,\rangle$ is the standard scalar product in $\mathbb{R}^n$. The weight function ω^* on Pe^{n-1} is maximized along a unique face $\mathcal{F}_\omega$ of Pe^{n-1} determined by the condition that the vector ω lies in the relative interior of its normal cone $\omega \in C^\circ_{\mathcal{F}_\omega}$. The flag $\mathcal{F}_\omega$ satisfies the following conditions

- ω is constant on $F_i \setminus F_{i-1}$, for $1 \leq i \leq k$,
- $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$, for $1 \leq i \leq k-1$,

where k is the length of $\mathcal{F}_\omega$. Let $M_{\mathcal{F}}$ be the *enumerator* of positive integer vectors $\omega \in \mathbb{Z}_+^n$ in relative interior of the corresponding braid cone $C_{\mathcal{F}}$

$$M_{\mathcal{F}} \;=\; \sum_{\omega \,\in\, \mathbb{Z}_+^n \,\cap\, C_{\mathcal{F}}^\circ} x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n}. \tag{2}$$

Note that $\omega \in C_{\mathcal{F}}^\circ$ if and only if $\mathcal{F}_\omega = \mathcal{F}$. The enumerator $M_{\mathcal{F}}$ is a monomial quasisymmetric function indexed by composition

$$\mathrm{type}(\mathcal{F}) = (|F_1 \setminus F_0|, |F_2 \setminus F_1|, \ldots, |F_k \setminus F_{k-1}|).$$

A *weight function* ω^* on an extended generalized permutohedron P is maximized along a unique face G_ω of P which is determined by the condition $\omega \in C_{G_\omega}^\circ$. For a face G of P we define a *quasisymmetric enumerator* $F_q(G)$ by

$$F_q(G) = q^{\dim(G)} \sum_{\omega \in \mathbb{Z}_+^n \,\cap\, C_G^\circ} x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n}.$$

By (2) it follows immediately that

$$F_q(G) = q^{\dim(G)} \sum_{\mathcal{F} \in \mathsf{F}(G)} M_{\mathcal{F}}.$$

Definition 3.4. For an extended generalized permutohedron P with the face lattice $L(P)$ let

$$F_q(P) = \sum_{G \in L(P)} F_q(G).$$

Let $f(P, q) := f_0 + f_1 q + f_2 q^2 + \cdots + f_{n-1} q^{n-1}$ be the *$f-$polynomial* of $(n-1)-$dimensional polyhedron P. The coefficient f_i is the number of $i-$dimensional faces of polyhedron P. The following proposition describes the $f-$polynomial of extended generalized permutohedron P in terms of the weighted quasisymmetric enumerator $F_q(P)$.

Proposition 3.5. *The $f-$polynomial of an $(n-1)-$dimensional extended generalized permutohedron P is determined by the principal specialization*

$$f(P, q) \;=\; (-1)^{n-1} \mathbf{ps}^1(F_{-q}(P))(-1).$$

Proof. It follows from (1) that

$$\mathbf{ps}^1(F_{-q}(G))(-1) = (-1)^{n-1} q^{\dim(G)}.$$

Therefore

$$\mathbf{ps}^1(F_{-q}(P))(-1) = (-1)^{n-1} \sum_{G \in L(P)} q^{\dim(G)}.$$

3.2 Poset cone

For a poset P on $[n]$ denote by $\mathsf{P}|_S$ the *restriction* of P to $S \subseteq [n]$. A subset $S \subseteq [n]$ is called *ideal* of P, denoted by $S \lhd \mathsf{P}$, if no element of $[n] \setminus S$ is less than an element of S.

Definition 3.6. The *poset cone* of a poset P on $[n]$ is an extended generalized permutohedron given by

$$C(\mathsf{P}) = \mathrm{cone}\{e_i - e_j : i <_{\mathsf{P}} j\},$$

where $e_1, e_2, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.

Proposition 3.7 ([1] Proposition 15.1)**.** *The generating rays of $C(\mathsf{P})$ are determined by the vectors $e_i - e_j$ corresponding to the cover relations $i \lessdot_{\mathsf{P}} j$ in P.*

It is shown in [1] that the poset cone is described by

$$C(\mathsf{P}) = \left\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i = 0 \ \text{ and } \ \sum_{s \in S} x_s \geq 0 \ \text{ for all } \ S \lhd \mathsf{P}\right\}.$$

Therefore

$$\dim(C(\mathsf{P})) = n - c(\mathsf{P}), \tag{3}$$

where $c(\mathsf{P})$ is the number of connected components of the poset P.

Let $\mathsf{C} : i_1, i_2, \dots, i_n$ be a cyclic sequence of elements of P where every consecutive pair is comparable in P. We say that C is a *circuit* of P. Circuits consist of *up−edges* $i_j <_{\mathsf{P}} i_{j+1}$ and *down−edges* $i_j >_{\mathsf{P}} i_{j+1}$.

Definition 3.8. A subposet Q of a poset P on $[n]$ is *positive* if for every circuit C all down−edges of C are in Q if and only if all up−edges of C are in Q. Let $\mathrm{Pos}(\mathsf{P})$ be the set of all positive subposets of P.

The faces of the poset cone $C(\mathsf{P})$ are characterized by the following lemma.

Lemma 3.9 ([1], Lemma 15.3)**.** *Let P be a poset on $[n]$. The faces of the poset cone $C(\mathsf{P}) \subseteq \mathbb{R}^n$ are precisely the poset cones $C(\mathsf{Q})$ as Q ranges over positive subposets of P.*

By Definition 3.4 and Lemma 3.9, it follows that the weighted quasisymmetric enumerator $F_q(C(\mathsf{P}))$ for a poset P can be expressed as

$$F_q(C(\mathsf{P})) = \sum_{\mathsf{Q} \in \mathrm{Pos}(\mathsf{P})} \sum_{\mathcal{F} \in \mathsf{F}(C(\mathsf{Q}))} q^{\dim(C(\mathsf{Q}))} M_{\mathcal{F}}. \tag{4}$$

Proposition 3.10. *A vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^n$ lies in $\mathcal{N}(C(\mathsf{P}))$ if and only if $\omega_i \leq \omega_j$ for any $i \leq_{\mathsf{P}} j$.*

Proof. A vector $\omega \in \mathbb{Z}_+^n$ lies in $\mathcal{N}(C(\mathsf{P}))$ if and only if ω^* is maximized over $C(\mathsf{P})$ at some its face. The weight function ω^* on the ray generated by $e_i - e_j$ is given by

$$\omega^*(t(e_i - e_j)) = \langle \omega, t(e_i - e_j) \rangle = t(\omega_i - \omega_j), t \geq 0.$$

Consequently, we have:

1. If $\omega_i > \omega_j$, then $t(\omega_i - \omega_j) \geq 0$ for $t \geq 0$ and maximum of ω^* is not achieved along the ray generated by $e_i - e_j$.
2. If $\omega_i < \omega_j$, then $t(\omega_i - \omega_j) \leq 0$ for $t \geq 0$ and maximum along the ray generated by $e_i - e_j$ is achieved at the vertex of $C(\mathsf{P})$.
3. If $\omega_i = \omega_j$, then $t(\omega_i - \omega_j) = 0$ for $t \geq 0$ and ω^* is maximized along the ray generated by $e_i - e_j$. □

Corollary 3.11. *A flag $\mathcal{F}$ is normal to the face $C(\mathsf{Q})$ for some positive subposet $\mathsf{Q} \in \mathrm{Pos}(\mathsf{P})$, i.e. $\mathcal{F} \in \mathsf{F}(C(\mathsf{Q}))$ if and only if for each $\omega \in C^\circ_{\mathcal{F}}$ holds that*

1. $\omega_i = \omega_j$ *for all* $i \leq_{\mathsf{Q}} j$,
2. $\omega_i < \omega_j$ *for all* i, j *which are incomparable in* Q *and* $i \leq_{\mathsf{P}} j$.

The following examples illustrates the concepts introduced in connection with poset cones.

Example 3.12. Let P be a poset on $[4]$ defined by covering relations $1, 2 < 3, 4$. The generated rays of the poset cone $C(\mathsf{P})$ are determined by vectors $e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4$. The normal fan $\mathcal{N}(C(\mathsf{P}))$ is described by inequalities $\omega_1, \omega_2 \leq \omega_3, \omega_4$, see Figure 2. The list of circuits of P is the following

$$1 < 4 > 2 < 3, 1 < 3 > 2 < 4, 4 > 2 < 3 > 1, 3 > 2 < 4 > 1,$$

$$2 < 3 > 1 < 4, 2 < 4 > 1 < 3, 3 > 1 < 4 > 2, 4 > 1 < 3 > 2.$$

Its facets are determined by positive subposets given by

$$1 < 3, 4; 2 < 3, 4; 1, 2 < 3; 1, 2 < 4.$$

Direct calculation gives $F_q(C(\mathsf{P})) = q^3 M_{(4)} + 2q^2(M_{(1,3)} + M_{(3,1)}) + 4qM_{(1,2,1)} + M_{(2,2)} + 2M_{(1,1,2)} + 2M_{(2,1,1)} + 4M_{(1,1,1,1)}$.

4 Hopf algebra $\mathcal{P}$

The set of isomorphism classes of finite posets linearly generates the $\mathbf{k}-$vector space $\mathcal{P}$

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n,$$

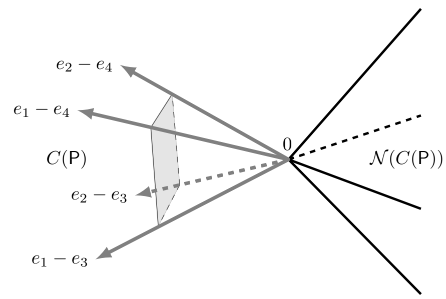

Figure 2: Poset cone and its normal fan

where $\mathcal{P}_n$ is the homogeneous component of degree n spanned by posets on $[n]$. The space $\mathcal{P}$ is a graded, commutative and non-cocommutative Hopf algebra (see [2], Example 2.3) with the *multiplication*

$$[\mathsf{P}_1] \cdot [\mathsf{P}_2] = [\mathsf{P}_1 \sqcup \mathsf{P}_2],$$

where $\mathsf{P}_1 \sqcup \mathsf{P}_2$ is disjoint union of posets, and the *comultiplication*

$$\Delta([\mathsf{P}]) = \sum_{S \lhd \mathsf{P}} [\mathsf{P}|_S] \otimes [\mathsf{P}|_{[n]\setminus S}].$$

Definition 4.1. A flag $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_k = [n]$ is a *flag of ideals* of poset P, denoted by $\mathcal{F} \lhd \mathsf{P}$, if $F_i \lhd \mathsf{P}$, for all $0 < i \leq k$. Let

$$\mathfrak{F}(\mathsf{P}) = \{\mathcal{F} : \mathcal{F} \lhd \mathsf{P}\}.$$

There is a map $\mathfrak{F}(\mathsf{P}) \to \mathcal{P}_n$ given by

$$\mathcal{F} \mapsto \mathsf{P}/\mathcal{F} = \prod_{i=1}^{k} \mathsf{P}|_{F_i \setminus F_{i-1}}.$$

A poset $\mathsf{P}/\mathcal{F}$ is a weak subposet of P, i.e. if $i \leq j$ in $\mathsf{P}/\mathcal{F}$ then $i \leq j$ in P. Let $\mathsf{P}/\mathcal{F} = \mathsf{P}_1 \sqcup \mathsf{P}_2 \sqcup \cdots \sqcup \mathsf{P}_m$ be the decomposition into connected components. We say that P_u and P_v are *incomparable* if elements from P_u and P_v are mutually incomparable in a poset P. Otherwise, P_u is *smaller* than P_v if no element of P_v is less than an element of P_u.

4.1 Quasisymmetric enumerator $F_q(C(\mathsf{P}))$

We extend the basic field $\mathbf{k}$ into the field of rational functions $\mathbf{k}(q)$ and define the character $\zeta_q : \mathcal{P} \to \mathbf{k}(q)$ with

$$\zeta_q([\mathsf{P}]) = q^{\mathrm{rk}(\mathsf{P})} \quad \text{where} \quad \mathrm{rk}(\mathsf{P}) = n - c(\mathsf{P}).$$

Let $\Psi_q : (\mathcal{P}, \zeta_q) \to (\mathcal{Q}Sym, \zeta_{\mathcal{Q}})$ be the unique morphism of combinatorial Hopf algebras over $\mathbf{k}(q)$, given by Theorem 2.1 with

$$\Psi_q([\mathsf{P}]) = \sum_{\alpha \models n} (\zeta_q)_\alpha(\mathsf{P}) M_\alpha. \tag{5}$$

The coefficient corresponding to a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n$ is determined by

$$(\zeta_q)_\alpha(\mathsf{P}) = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathrm{type}(\mathcal{F}) = \alpha}} \prod_{j=1}^{k} q^{\mathrm{rk}(\mathsf{P}|_{F_j \setminus F_{j-1}})} = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathrm{type}(\mathcal{F}) = \alpha}} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})},$$

where

$$\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = \sum_{j=1}^{k} \mathrm{rk}(\mathsf{P}|_{F_j \setminus F_{j-1}}) = n - \sum_{j=1}^{k} c(\mathsf{P}|_{F_j \setminus F_{j-1}}).$$

Thus, the equation (5) can be expressed as

$$\Psi_q([\mathsf{P}]) = \sum_{\mathcal{F} \in \mathfrak{F}(\mathsf{P})} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} M_{\mathcal{F}}.$$

To a poset P are associated two weighted quasisymmetric functions, $F_q(C(\mathsf{P}))$ with a geometric meaning based on the combinatorics of the poset cone $C(\mathsf{P})$ and $\Psi_q([\mathsf{P}])$ with an algebraic meaning based on the Hopf algebra structure on finite posets. The following theorem shows that these two functions are equal.

Theorem 4.2. *For a poset P the quasisymmetric enumerator function $F_q(C(\mathsf{P}))$ associated to a poset cone $C(\mathsf{P})$ coincides with the value of the universal morphism at $[\mathsf{P}]$ from the combinatorial Hopf algebra of posets $\mathcal{P}$ to $\mathcal{Q}Sym$*

$$F_q(C(\mathsf{P})) = \Psi_q([\mathsf{P}]).$$

Proof. The weighted enumerator $F_q(C(\mathsf{P}))$ is described by (4). We need to show that

$$\mathfrak{F}(\mathsf{P}) = \{\mathcal{F} : \mathcal{F} \in \mathsf{F}(C(\mathsf{Q})) \text{ for some } \mathsf{Q} \in \mathrm{Pos}(\mathsf{P})\},$$

and $\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = \dim(C(\mathsf{Q}))$, for $\mathcal{F} \in \mathsf{F}(C(\mathsf{Q}))$. Let P be a poset on the set $[n]$.

$\supseteq$: Suppose $\mathcal{F} \in \mathsf{F}(C(\mathsf{Q}))$ for some $\mathsf{Q} \in \mathrm{Pos}(\mathsf{P})$. If $\mathsf{Q} = \mathsf{Q}_1 \sqcup \ldots \sqcup \mathsf{Q}_m$ is the decomposition into connected components, by (3) we have $\dim(C(\mathsf{Q})) = n - m$. For $\omega \in C^{\circ}_{\mathcal{F}}$, by Corollary 3.11, we deduce the following facts:

- ω is constant on each Q_u, hence $\mathsf{Q}_u \subseteq F_i \setminus F_{i-1}$ for some $i = 1, \ldots, |\mathcal{F}|$.
- if $\mathsf{Q}_u, \mathsf{Q}_v \subset F_i \setminus F_{i-1}$ for some $i = 1, \ldots, |\mathcal{F}|$, then Q_u and Q_v are incomparable. Otherwise, if $i \leq_{\mathsf{P}} j$ for some $i \in \mathsf{Q}_u, j \in \mathsf{Q}_v$, then ω^* is maximized along the ray generated by $e_i - e_j$, contrary to $\mathcal{F} \in \mathsf{F}(C(\mathsf{Q}))$.

- if $\mathsf{Q}_u \subseteq F_i \setminus F_{i-1}$, $\mathsf{Q}_v \subseteq F_j \setminus F_{j-1}$ and Q_u is smaller than Q_v, then $i <_{\mathbb{Z}} j$. Otherwise, if $i >_{\mathbb{Z}} j$, then $\omega_i > \omega_j$ for some $i \in \mathsf{Q}_u$, $j \in \mathsf{Q}_v$ such that $i <_{\mathsf{P}} j$, hence ω^* does not reach the maximum.

We conclude that $\mathcal{F} \in \mathfrak{F}(\mathsf{P})$ and $\mathsf{P}/\mathcal{F} = \mathsf{Q}$, consequently $\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = n - m = \dim(C(\mathsf{Q}))$.

$\subseteq$: Let $\mathcal{F} \lhd \mathsf{P}$ be a flag of ideals of the poset P and $\mathsf{C} : i_1, i_2, \ldots, i_n$ be a circuit with all down-edges belonging to $\mathsf{P}/\mathcal{F}$. The level function of elements of P according to the flag $\mathcal{F}$, defined by $l(i) = \min\{a \mid i \in F_a\}, i \in \mathsf{P}$, is nondecreasing along the circuit C. It means that l is constant on C, i.e. $\mathsf{C} \subset F_a \setminus F_{a-1}$, for some $1 \leq a \leq |\mathcal{F}|$. Particulary, all up-edges of the circuit C are in $\mathsf{P}/\mathcal{F}$. The same is true for circuits with all up-edges in $\mathsf{P}/\mathcal{F}$ and we conclude that $\mathsf{P}/\mathcal{F}$ is a positive subposet of P. Corollary 3.11 gives $\mathcal{F} \in \mathsf{F}(C(\mathsf{P}/\mathcal{F}))$ and the proof is finished by the obvious identity $\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = \dim C(\mathsf{P}/\mathcal{F})$. □

By Theorem 4.2 and Proposition 3.5 we obtain the following expression for the f-polynomial.

Corollary 4.3. *Let $C(\mathsf{P})$ be the poset cone associated to a poset P on the ground set $[n]$. The $f-$polynomial of $C(\mathsf{P})$ is given by*

$$f(C(\mathsf{P}), q) = (-1)^{n-1}\mathbf{ps}^1(\Psi_{-q}([\mathsf{P}]))(-1).$$

Example 4.4. Let st_n be the poset on $[n]$ with covering relations $i \lessdot n$, for $1 \leq i \leq n-1$.

Let $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_k = [n]$ be a flag of ideals of the poset st_n. Then $n \in F_k$ and two different elements of $[n]$ are in the same connected component in $\mathsf{st}_n/\mathcal{F}$ if and only if the both are in $F_k \setminus F_{k-1}$. Therefore

$$\mathrm{rk}_{\mathsf{st}_n}(\mathcal{F}) = |F_k \setminus F_{k-1}| - 1.$$

The number of flags corresponding to the faces of $C(\mathsf{st}_n)$ of dimension i is equal to $\binom{n-1}{i}$, which implies

$$F_q(C(\mathsf{st}_n)) = \sum_{i=0}^{n-1} \binom{n-1}{i} \left(M_{(1)}^{n-1-i}\right)_{i+1} q^i,$$

where for $F \in \mathcal{Q}Sym$, $F \mapsto (F)_i$ is the linear extension of the map given on monomial basis by $M_\alpha \mapsto M_{(\alpha,i)}$. By Proposition 4.3, the corresponding $f-$polynomial is equal to

$$f(C(\mathsf{st}_n), q) = \sum_{i=0}^{n-1} \binom{n-1}{i} q^i = (1+q)^{n-1}.$$

Example 4.5. Let l_n be a linear poset on $[n]$ and $\mathcal{F} \in \mathfrak{F}(\mathsf{l}_n)$. For $1 \leq i \leq |\mathcal{F}|$ all components of $F_i \setminus F_{i-1}$ are connected, so

$$c(F_i \setminus F_{i-1}) = 1 \text{ and } \mathrm{rk}(\mathsf{l}|_{F_{i+1} \setminus F_i}) = |F_{i+1} \setminus F_i| - 1.$$

It implies that $\mathrm{rk}_{\mathsf{I}_n}(\mathcal{F}) = n - |\mathcal{F}|$. We have

$$F_q(C(\mathsf{I}_n)) = \sum_{i=0}^{n-1} \left(\sum_{\alpha\,:\,k(\alpha)=n-i} M_\alpha \right) q^i.$$

Since $|\{\alpha \models n \;:\; k(\alpha) = n - i\}| = \binom{n-1}{i}$, the corresponding f-polynomial is equal to

$$f(C(\mathsf{I}_n), q) = \sum_{i=0}^{n-1} \binom{n-1}{i} q^i = (1+q)^{n-1}.$$

We obtain that $f(C(\mathsf{st}_n), q) = f(C(\mathsf{I}_n), q)$. Actually, it is a consequence of a more general fact.

Proposition 4.6. *Let* P *be a poset on* $[n]$ *whose Hasse diagram is a tree. Then*

$$f(C(\mathsf{P}), q) = (1+q)^{n-1}.$$

Proof. The poset P has $n-1$ covering relations. Proposition 3.9 implies that the generating rays of $C(\mathsf{P})$ are $n-1$ linearly independent vectors $e_i - e_j$ where $j \lessdot i$. Hence, the $k-$face of $C(\mathsf{P})$ is generated by k generating rays, so $f_k(C(\mathsf{P})) = \binom{n-1}{k}$. □

Example 4.7. Let $\mathsf{K}_{m,n}$ be the poset on the set $[m+n]$ such that for all $i \in [m]$ and $j \in [m+n] \setminus [m]$ hold $i \lessdot j$. The Hasse diagram of $\mathsf{K}_{m,n}$ is the complete bipartite graph $K_{m,n}$.

We have

$$F_q(C(\mathsf{K}_{m,n})) = \left(M_{(1)}^m\right) \circ \left(M_{(1)}^n\right) + \\ + \sum_{k=1}^{m+n-1} q^k \sum_{t_1+t_2=k+1} \binom{m}{t_1}\binom{n}{t_2} \left(M_{(1)}^{m-t_1} \circ M_{(k+1)} \circ M_{(1)}^{n-t_2}\right),$$

where $1 \leq t_1 \leq m$, $1 \leq t_2 \leq n$ and $\circ$ is the *concatenation product* defined on monomial basis by $M_\alpha \circ M_\beta := M_{\alpha\,\cdot\,\beta}$. Note that this includes the case $\mathsf{st}_n = \mathsf{K}_{n-1,1}$. The principal specialization evaluated at -1 gives

$$f(C(\mathsf{K}_{m,n}, q) = 1 + \sum_{k=1}^{m+n-1} q^k \sum_{t_1+t_2=k+1} \binom{m}{t_1}\binom{n}{t_2}.$$

4.1.1 Opposite poset

For a flag $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_k = [n]$, the *opposite flag* $\mathcal{F}^{op}$ is defined by

$$\mathcal{F}^{op} : \emptyset \subset [n] \setminus F_{k-1} \subset [n] \setminus F_{k-2} \subset \cdots \subset [n] \setminus F_1 \subset [n].$$

The normal cone corresponding to the opposite flag is the opposite cone

$$C_{\mathcal{F}^{op}} = -C_{\mathcal{F}} \text{ and } \mathrm{type}(\mathcal{F}^{op}) = \mathrm{rev}(\mathrm{type}(\mathcal{F})),$$

where $\mathrm{rev}(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. The *opposite poset* P^{op} to a poset P is the poset on the same set such that $i <_{\mathsf{P}^{op}} j$ if and only if $j <_{\mathsf{P}} i$.

Proposition 4.8. *Let* P *be a poset on* $[n]$*, the quasisymmetric enumerator function corresponding to the opposite poset* P^{op} *is determined by*

$$F_q(C(\mathsf{P}^{op})) \;=\; \mathrm{rev}(F_q(C(\mathsf{P})),$$

where for $F \in \mathcal{Q}Sym$*,* $F \mapsto \mathrm{rev}(F)$ *is the linear extension of the map given on monomial basis by* $M_\alpha \mapsto M_{\mathrm{rev}(\alpha)}$*.*

Proof. Note that $\mathcal{F} \in \mathfrak{F}(\mathsf{P})$ if and only if $\mathcal{F}^{op} \in \mathfrak{F}(\mathsf{P}^{op})$. The statement follows from $\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = \mathrm{rk}_{\mathsf{P}^{op}}(\mathcal{F}^{op})$. □

4.1.2 Action of the antipode on $F_q(C(\mathsf{P}))$

The geometric interpretation of the action of the antipode on $\mathcal{Q}Sym$ is given by the following lemma.

Lemma 4.9 ([8], Lemma 4.5)**.** *The antipode* S *on the monomial quasisymmetric function* $M_{\mathcal{F}}$ *associated to a flag* $\mathcal{F}$ *acts by*

$$S(M_{\mathcal{F}}) \;=\; (-1)^{|\mathcal{F}|+1} \sum_{\mathcal{G} \,\preceq\, \mathcal{F}^{op}} M_{\mathcal{G}},$$

where $\mathcal{G} \preceq \mathcal{F}^{op}$ *if and only if* $\mathcal{F}^{op} \subseteq \mathcal{G}$ *as faces of* Pe^{n-1}*.*

The next theorem extends the similar statement proven for generalized permutohedra in [8] to the case of poset cones.

Theorem 4.10. *If* P *is a poset on the set* $[n]$*, the antipode* S *acts on the quasisymmetric enumerator function* $F_q(C(\mathsf{P}))$ *by*

$$S(F_q(C(\mathsf{P})) = (-1)^n \sum_{\mathcal{G} \in\, \mathfrak{F}(\mathsf{P})} f(C(\mathsf{P}/\mathcal{G}), -q) M_{\mathcal{G}^{op}}.$$

Proof. Since $\mathcal{F} \in \mathfrak{F}(\mathsf{P})$ and $\mathcal{G} \preceq \mathcal{F}$ implies $\mathcal{G} \in \mathfrak{F}(\mathsf{P})$ we have

$$\begin{aligned} S(F_q(C(\mathsf{P}))) &= \sum_{\mathcal{F} \in\, \mathfrak{F}(\mathsf{P})} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} S(M_{\mathcal{F}}) = \sum_{\mathcal{F} \in\, \mathfrak{F}(\mathsf{P})} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} (-1)^{|\mathcal{F}|+1} \sum_{\mathcal{G} \preceq \mathcal{F}^{op}} M_{\mathcal{G}} \\ &= \sum_{\mathcal{G} \in\, \mathfrak{F}(\mathsf{P}^{op})} M_{\mathcal{G}} \sum_{\substack{\mathcal{F} \in\, \mathfrak{F}(\mathsf{P}) \\ \mathcal{G} \preceq \mathcal{F}^{op}}} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} (-1)^{|\mathcal{F}|+1}. \end{aligned}$$

By equivalences $\mathcal{G} \preceq \mathcal{F}^{op}$ if and only if $\mathcal{G}^{op} \preceq \mathcal{F}$ and $\mathcal{G} \in \mathfrak{F}(\mathsf{P}^{op})$ if and only if $\mathcal{G}^{op} \in \mathfrak{F}(\mathsf{P})$, the last equality becomes

$$S(F_q(C(\mathsf{P}))) = (-1)^n \sum_{\mathcal{G}^{op} \in \mathfrak{F}(\mathsf{P})} M_{\mathcal{G}} \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathcal{G}^{op} \preceq \mathcal{F}}} (-q)^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} (-1)^{|\mathcal{F}|+1+n+\mathrm{rk}_{\mathsf{P}}(\mathcal{F})}$$
$$= (-1)^n \sum_{\mathcal{G} \in \mathfrak{F}(\mathsf{P})} M_{\mathcal{G}^{op}} \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathcal{G} \preceq \mathcal{F}}} (-q)^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} (-1)^{|\mathcal{F}|+1+n+\mathrm{rk}_{\mathsf{P}}(\mathcal{F})}.$$

By the proof of Theorem 4.2, we have that $\mathsf{P}/\mathcal{G}$ is positive subposet of P and $\mathcal{G}$ is a normal flag to the face $C(\mathsf{P}/\mathcal{G})$. By the same argument, the faces of $C(\mathsf{P}/\mathcal{G})$ are of the form $C(\mathsf{P}/\mathcal{F})$ for flags of ideals $\mathcal{F} \in \mathfrak{F}(\mathsf{P})$ which satisfy $\mathcal{G} \preceq \mathcal{F}$. Therefore, according to the identities (1) and $\mathrm{rk}_{\mathsf{P}}(\mathcal{F}) = \dim C(\mathsf{P}/\mathcal{F})$, we obtain

$$f(C(\mathsf{P}/\mathcal{G}), q) = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathcal{G} \preceq \mathcal{F}}} (-1)^{n+1+|\mathcal{F}|+\mathrm{rk}_{\mathsf{P}}(\mathcal{F})} q^{\mathrm{rk}_{\mathsf{P}}(\mathcal{F})}.$$

□

5 Function $F(\mathsf{P})$

In this section we show that for a well labelled poset the enumerator function $F_q(C(\mathsf{P}))$ specializes at $q = 0$ to the enumerator of $\mathsf{P}-$partitions. For a poset P on the set $[n]$ let

$$F(\mathsf{P}) = F_0(C(\mathsf{P})) = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathsf{P}/\mathcal{F} \text{ discrete}}} M_{\mathcal{F}}.$$

The sum is over flags of ideals of P such that $\mathsf{P}/\mathcal{F}$ is a *discrete poset* (poset with no covering relations).

Proposition 5.1. *The principal specialization* $\mathbf{ps}^1$ *of* $F(\mathsf{P})$ *at* $m = -1$ *results in*

$$\mathbf{ps}^1(F(\mathsf{P}))(-1) = (-1)^{n-1}.$$

Proof. The poset cone $C(\mathsf{P})$ has a unique vertex, so the required equation follows from Proposition 3.5. □

Let $\mathsf{P}_1 * \mathsf{P}_2$ be the series composition of posets $\mathsf{P}_1, \mathsf{P}_2$ which is a poset on the disjoint union of elements of $\mathsf{P}_1, \mathsf{P}_2$ with the order relation defined by $i \leq_{\mathsf{P}_1 * \mathsf{P}_2} j$ if and only if $i \leq_{\mathsf{P}_1} j$ or $i \leq_{\mathsf{P}_2} j$ or $i \in \mathsf{P}_1, j \in \mathsf{P}_2$.

Proposition 5.2. *If* $\mathsf{P} = \mathsf{P}_1 * \mathsf{P}_2 * \cdots * \mathsf{P}_n$ *then*

$$F(\mathsf{P}) = F(\mathsf{P}_1) \circ F(\mathsf{P}_2) \circ \cdots \circ F(\mathsf{P}_n),$$

where $\circ$ *is the concatenation product described in Example 4.7.*

Proof. The proof follows from the fact that $F(\mathsf{P})$ is the sum over all flags of ideals $\mathcal{F}$ such that $\mathsf{P}/\mathcal{F}$ is a discrete poset. Such a flag is the consecutive composition of flags of the same type corresponding to components $\mathsf{P}_1, \dots, \mathsf{P}_n$. □

Example 5.3. A poset $\mathsf{K}_{m,n}$ can be expressed as the series composition of discrete posets $\mathsf{d}(m) * \mathsf{d}(n)$ on $[m], [n]$. By Proposition 5.2, we obtain $F(\mathsf{K}_{m,n}) = F(\mathsf{d}(m)) \circ F(\mathsf{d}(n)) = M_{(1)}^m \circ M_{(1)}^n$.

Proposition 5.4. *If* $\mathrm{Max}(\mathsf{P})$ *is the set of maximal elements of a connected poset* P *on the set* $[n]$ *then*

$$F(\mathsf{P}) = \sum_{\emptyset \neq A \subseteq \mathrm{Max}(\mathsf{P})} (F(\mathsf{P}|_{[n]\setminus A}))_{|A|}.$$

Particulary, if $\mathrm{Max}(\mathsf{P}) = \{v\}$ *then* $F(\mathsf{P}) = \big(F(\mathsf{P}|_{[n]\setminus\{v\}})\big)_1$.

Proof. For a flag of ideals $\mathcal{F} : \emptyset = F_0 \subset F_1 \subset \cdots \subset F_m = [n]$ let $F_{\mathrm{end}} = F_m \setminus F_{m-1}$. If $\mathsf{P}/\mathcal{F}$ is a discrete poset, we have that $\mathsf{P}|_{F_{\mathrm{end}}}$ is discrete too, so $F_{\mathrm{end}} \subseteq \mathrm{Max}(\mathsf{P})$. Therefore

$$F(\mathsf{P}) = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathsf{P}/\mathcal{F} \text{ discrete}}} M_{\mathcal{F}} = \sum_{\emptyset \neq A \subseteq \mathrm{Max}(\mathsf{P})} \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathsf{P}/\mathcal{F} \text{discrete} \\ F_{\mathrm{end}} = A}} M_{\mathcal{F}} = \sum_{\emptyset \neq A \subseteq \mathrm{Max}(\mathsf{P})} (F(\mathsf{P}|_{[n]\setminus A}))_{|A|}.$$

□

Remark 5.5. The direct calculation shows that $F(\mathsf{P})$ distinguishes all non-isomorphic posets on $[n]$, for $n \in \{1, 2, 3, 4, 5, 6\}$.

The weighted enumerator function invariant F_q contains more information about posets than its specialization at $q = 0$. The following example of posets with the same quasisymmetric invariant $F(\mathsf{P})$ is borrowed from [9, Example 4.9].

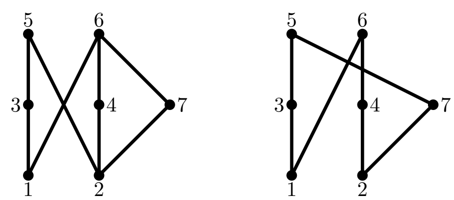

Figure 3: Posets with the same quasisymmetric invariant $F(\mathsf{P})$

Example 5.6. Let P_1 and P_2 be posets on Figure 3. The direct calculation gives

$$\begin{aligned}
F(\mathsf{P}_1) = F(\mathsf{P}_2) = {} & M_{(2,3,2)} + 2M_{(1,1,3,2)} + 2M_{(2,3,1,1)} \\
& + M_{(1,3,2,1)} + M_{(1,2,3,1)} + M_{(2,1,3,1)} + M_{(1,3,1,2)} \\
& + 3M_{(1,2,2,2)} + 3M_{(2,1,2,2)} + 3M_{(2,2,1,2)} + 3M_{(2,2,2,1)} \\
& + 2M_{(1,2,2,1,1)} + 6M_{(2,1,1,1,2)} + 6M_{(1,2,2,1,1)} \\
& + 7M_{(2,1,1,2,1)} + 7M_{(1,2,1,1,2)} + 8M_{(1,1,2,1,2)} + 8M_{(2,1,2,1,1)} \\
& + 8M_{(1,2,1,2,1)} + 8M_{(1,1,2,2,1)} + 9M_{(1,1,1,2,2)} + 9M_{(2,2,1,1,1)} \\
& + 20M_{(1,1,1,1,1,2)} + 23M_{(1,1,1,1,2,1)} + 24M_{(1,1,1,2,1,1)} \\
& + 24M_{(1,1,2,1,1,1,)} + 23M_{(1,2,1,1,1,1)} + 20M_{(2,1,1,1,1,1)} \\
& + 4M_{(1,1,3,1,1)} + 3M_{(1,3,1,1,1)} + 3M_{(1,1,1,3,1)} + 66M_{(1,1,1,1,1,1,1)}.
\end{aligned}$$

Consider flags $\emptyset \subset \{i\} \subset [6] \setminus \{j\} \subset [6]$, for $i \in \{1,2\}$ and $j \in \{5,6\}$ which are the only flags of ideals of the type $(1,5,1)$ of posets P_1 and P_2. The coefficients by $M_{(1,5,1)}$ in $F_q(C(\mathsf{P}_i)), i = 1,2$ are

$$\zeta_{(1,5,1)}(\mathsf{P}_1) = 2q^4 + q^3 + q^2 \text{ and } \zeta_{(1,5,1)}(\mathsf{P}_2) = q^4 + 3q^3,$$

which shows that $F_q(C(\mathsf{P}_1)) \neq F_q(C(\mathsf{P}_2))$.

The following theorem gives a geometric interpretation of the enumerator $F_{\mathsf{P}}(\mathbf{x})$ of $\mathsf{P}-$partitions for a well labelled poset P. Recall from Definition 2.2 that

$$F_{\mathsf{P}}(\mathbf{x}) = \sum_{f \in \mathcal{A}(\mathsf{P})} x_{f(1)} x_{f(2)} \cdots x_{f(n)},$$

where $f \in \mathcal{A}(\mathsf{P})$ if and only if $i <_{\mathsf{P}} j$ implies $f(i) < f(j)$, for all $i, j \in \mathsf{P}$.

Proposition 5.7. *For a well labelled poset* P *holds*

$$\mathcal{A}(\mathsf{P}) \;=\; \{\omega \in C^{\circ}_{\mathcal{F}} \;:\; \mathcal{F} \in \mathfrak{F}(\mathsf{P}) \text{ and } \mathsf{P}/\mathcal{F} \text{ is a discrete poset}\}.$$

Proof. We have $\omega \in C^{\circ}_{\mathcal{F}}$ for some $\mathcal{F} \in \mathfrak{F}(\mathsf{P})$ such that $\mathsf{P}/\mathcal{F}$ is a discrete poset if and only if ω^* is maximized uniquely at the vertex of $C(\mathsf{P})$. By Proposition 3.10 and definition of P-partitions of well labeled posets, the both sets are characterized by the same condition. □

Theorem 5.8. *For a well labelled poset* P *holds* $F(\mathsf{P}) = F_{\mathsf{P}}(\mathbf{x})$.

Proof. By (2) and Proposition 5.7

$$F(\mathsf{P}) = \sum_{\substack{\mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathsf{P}/\mathcal{F} \text{ discrete}}} M_{\mathcal{F}} = \sum_{\substack{\omega \in \mathbb{Z}^n_+ \cap C^{\circ}_{\mathcal{F}} \\ \mathcal{F} \in \mathfrak{F}(\mathsf{P}) \\ \mathsf{P}/\mathcal{F} \text{ discrete}}} x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n} = \sum_{\omega \in \mathcal{A}(\mathsf{P})} x_{\omega_1} x_{\omega_2} \cdots x_{\omega_n},$$

what is exactly the expression for $F_{\mathsf{P}}(\mathbf{x})$. □

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