

AN APPLICATION OF GROEBNER BASES TO PLANARITY OF INTERSECTION OF SURFACES

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Abstract

In this paper we use Groebner bases theory in order to determine planarity of intersections of two algebraic surfaces in \mathbf{R}^3 . We specially considered plane sections of certain type of conoid which has a cubic egg curve as one of the directrices. The paper investigates a possibility of conic plane sections of this type of conoid.

1. THE BASIC CONCEPT OF GROEBNER BASES

Many problems in mathematics can be treated as a problem of solvability of a system of polynomial equations:

$$(1) \quad \begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0 \\ f_2(x_1, x_2, \dots, x_n) &= 0 \\ &\vdots \\ f_m(x_1, x_2, \dots, x_n) &= 0 \end{aligned}$$

where f_1, f_2, \dots, f_m are polynomials in n variables over set of complex numbers \mathbf{C} . The main matter in the theory of systems of polynomial equations are Groebner bases. Let us define this concept through following considerations. The polynomial of several variables:

$$(2) \quad P(x_1, \dots, x_n) = \sum_{\alpha \in A_0} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

is considered as a sum over n -tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ from a finite set $A_0 \subset \mathbf{N}_0^n$ ($c_\alpha \neq 0$). The term $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called *the monomial*, while the summand $c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is called *the term of the polynomial*.

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A *monomial order* \succ is a total order on the set of all monomials, satisfying two additional properties: the ordering respects multiplication (if $u \succ v$ and w is any other monomial, then $uw \succ vw$) and the ordering is a well ordering. The *lexicographic order* over a set of monomials \succ_{lex} is defined by *order of variables* $x_1 \succ_{lex} \dots \succ_{lex} x_n$ so that: $x_1^{\alpha_1} \dots x_n^{\alpha_n} \succ_{lex} x_1^{\beta_1} \dots x_n^{\beta_n}$ iff $\alpha_1 = \beta_1, \dots, \alpha_{k-1} = \beta_{k-1}, \alpha_k > \beta_k$ for some k . Other types of monomial ordering are presented in [1]. Let us notice that polynomial (2) can be *arranged* in the given monomial order, so that the first term in the sum f is *the leading term* and denoted by $LT(f)$. Then leading term is shown as the product of *the leading coefficient* $LC(f)$ and *the leading monomial* $LM(f)$, i.e. $LT(f) = LC(f) \cdot LM(f)$.

In the polynomial ring $\mathbf{C}[x_1, \dots, x_n]$ we determine a *division algorithm*, which follows. Let a m -tuple of polynomials $F = (f_1, \dots, f_m)$ be given, arranged into monomial order \succ over the set of monomials. Then we quote an division algorithm with the following pseudo-code [2]:

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Input:     $f, f_1, \dots, f_m$ 
Output:    $r, a_1, \dots, a_m$ 
 $r := 0, a_1 := 0, \dots, a_m := 0$ 
 $p := f$ 
WHILE  $p \neq 0$  DO
   $i := 1$ 
  divisionoccured := false
  WHILE  $i \leq m$  AND divisionoccured = false DO
    IF  $LT(f_i)$  divides  $LT(p)$  THEN
       $a_i := a_i + LT(p)/LT(f_i)$ 
       $p := p - LT(p)/LT(f_i) \cdot f_i$ 
      divisionoccured := true
    ELSE
       $i := i + 1$ 
  IF divisionoccured = false THEN
     $r := r + LT(p)$ 
     $p := p - LT(p)$ 
(STOP)

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Using the previous algorithm, every polynomial $f \in \mathbf{C}[x_1, \dots, x_n]$ can be presented as follows:

$$(4) \quad f = a_1 f_1 + \dots + a_m f_m + r$$

for some $a_i, r \in \mathbf{C}[x_1, \dots, x_n]$ ($i = 1, \dots, m$), while either $r = 0$ or r is \mathbf{C} -linear combination of monomials none of which is divisible by any $LT(f_1), \dots, LT(f_m)$. The polynomial $r = REM(f, F)$ is called *the remainder* of division of f by m -tuple of polynomials F arranged into monomial order \succ . The main characteristic of the remainder r is that it is not uniquely determined in comparison to order of division by polynomials from m -tuple F . Let us remark:

$$(5) \quad LT(f) \succ LT(r).$$

For a m -tuple of polynomials $F = (f_1, \dots, f_m)$ the ideal is determined by:

$$(6) \quad I = \left\{ \sum_{i=1}^m h_i f_i \mid h_1, \dots, h_m \in \mathbf{C}[x_1, \dots, x_n] \right\}.$$

It is denoted by $\langle f_1, \dots, f_m \rangle$. Then polynomials f_1, \dots, f_m are called a *generators* for the ideal $I = \langle f_1, \dots, f_m \rangle$ in the polynomial ring $\mathbf{C}[x_1, \dots, x_n]$. Let us emphasize that Hilbert Basis Theorem claims that every ideal has a finite generating set [2]. Next, for the ideal $I = \langle f_1, \dots, f_m \rangle$ we define *the ideal of leading terms*:

$$(7) \quad \langle LT(I) \rangle = \langle \{LT(f) \mid f \in I\} \rangle.$$

In any ideal $I = \langle f_1, \dots, f_m \rangle$ it is true:

$$(8) \quad \langle LT(f_1), \dots, LT(f_m) \rangle \subseteq \langle LT(I) \rangle$$

Generating set $G = \{g_1, \dots, g_s\}$ of the ideal $I = \langle f_1, \dots, f_m \rangle$ is called a *Groebner basis* if

$$(9) \quad \langle LT(g_1), \dots, LT(g_s) \rangle = \langle LT(I) \rangle$$

Let us remark that for each permutation of s -tuple $G = \langle g_1, \dots, g_s \rangle$ the remainder $r = REM(f, G)$ is uniquely determined [2].

B. BUCHBERGER in his dissertation has shown that for each ideal there exists a Groebner basis by following algorithm [2]:

$$(10) \quad \begin{array}{l} \text{Input:} \quad F = \{f_1, \dots, f_m\} \text{--Set of generators of } I \\ \text{Output:} \quad G = \{g_1, \dots, g_s\} \text{--Groebner basis for } I \ (G \supseteq F) \\ \quad \quad G := F \\ \quad \quad \text{REPEAT} \\ \quad \quad \quad G' := G \\ \quad \quad \quad \text{FOR each pair } \{p, q\}, p \neq q, \text{ in } G' \text{ DO} \\ \quad \quad \quad \quad S := S(p, q) \\ \quad \quad \quad \quad h := REM(S, G') \\ \quad \quad \quad \quad \text{IF } h \neq 0 \text{ THEN } G := G \cup \{h\} \\ \quad \quad \quad \text{UNTIL } G = G' \\ \quad \quad \quad \text{(STOP)} \end{array}$$

In the previous algorithm we used *the S-polynomial* of f and g (in relation to the fixed monomial order \succ) which is defined by:

$$(11) \quad S = S(f, g) = \frac{LCM(LM(f), LM(g))}{LT(f)} \cdot f - \frac{LCM(LM(f), LM(g))}{LT(g)} \cdot g,$$

where LCM is the least common multiple. For different sets of generators of the ideal I Groebner basis, produced by *the Buchberger's algorithm*, is not unique. *Groebner basis is minimal* iff $LC(g) = 1$ for all $g \in G$, whereat $\{LT(g) \mid g \in G\}$ is a minimal basis of the monomial ideal $\langle \{LT(f) \mid f \in I\} \rangle$ (see Exercise 6/§7/Ch. 2 [2]). Minimal Groebner basis is not unique as well. *Reduced Groebner basis* is minimal Groebner basis for the ideal I such that for all $p \in G$ there is no monomial of p which lies in $\langle \{LT(g) \mid g \in G \setminus \{p\}\} \rangle$ [1], [2]. The following statement is true [2]:

Theorem 1.1. *Every polynomial ideal has a unique reduced Groebner basis, related to a fixed monomial order.*

Groebner bases in applications are usually considered in reduced form, as it is emphasized in [2] and [15]. In the next part of this paper we consider an application of the Groebner bases in the real three-dimensional geometry.

2. AN APPLICATION OF GROEBNER BASES TO PLANARITY OF INTERSECTION OF SURFACES

Let us consider the consistent system of two real non-linear polynomial equations with three variables:

$$(12) \quad f_1(x, y, z) = 0 \quad \wedge \quad f_2(x, y, z) = 0.$$

For the system (12) we define that *the system has planar solution* if there exists a linear-polynomial:

$$(13) \quad g = g(x, y, z) = Ax + By + Cz + D,$$

for some real constants A, B, C, D , such that every solution of the system (12) is also a solution of the linear equation:

$$(14) \quad g(x, y, z) = 0.$$

Thus we indicate that *the system has planar intersection*. Let a monomial order be determined, then the following statement is true:

Theorem 2.1. *If a Groebner basis of the system (12) contains the linear polynomial (13), then the solution of the system (12) is planar.*

The following statement is formulated for lexicographic order $x \succ_{lex} y \succ_{lex} z$:

Theorem 2.2. *Let the system (12) have the planar intersection by:*

$$(15) \quad Ax + By + Cz + D = 0,$$

for some $A, B, C, D \in \mathbf{R}$ and $A \neq 0$. If for the ideal $I = \langle f_1, f_2 \rangle$ the following is true:

$$(16) \quad y \notin \langle LT(I) \rangle \quad \wedge \quad z \notin \langle LT(I) \rangle,$$

then the linear polynomial:

$$(17) \quad \hat{g} = \hat{g}(x, y, z) = x + (B/A)y + (C/A)z + (D/A)$$

is an element of the reduced Groebner basis related to lexicographic order \succ_{lex} .

Proof. The system (12) is equivalent with the following system:

$$(12') \quad f_1(x, y, z) = 0 \quad \wedge \quad f_2(x, y, z) = 0 \quad \wedge \quad \hat{g}(x, y, z) = 0.$$

The reduced Groebner bases for systems (12) and (12') are equal. Let us prove that $\hat{g} \in I$ is element of reduced Groebner basis for system (12'). Using Buchberger algorithm, we can assume that polynomial \hat{g} is element of the Groebner basis G_1 . Let us consider minimal Groebner basis G_{min} formed from G_1 such that $\hat{g} \in G_{min}$. Next, let us prove that polynomial \hat{g} is reduced related to G_{min} (see Proposition 6./§7/Ch. 2 in [2] or Lect. 14 in [16]). By condition (16) it follows that for a linear polynomial $\hat{g} = x + (B/A)y + (C/A)z + (D/A)$ it is true that $x, y, z \notin \langle LT(G_{min} \setminus \{\hat{g}\}) \rangle$. If polynomial \hat{g} is reduced in one minimal Groebner basis, then \hat{g} is reduced related to every minimal Groebner basis (see proof of the Proposition 6./§7/Ch. 2 in [2]) and therefore polynomial \hat{g} is an element of the unique reduced Groebner basis. ■

Let us emphasize that computer algebra systems compute reduced Groebner basis. In this paper we use computer algebra system Maple and Maple Package "Groebner" which is initiated by:

with(Groebner);

Let $F := [f_1, f_2]$ be the list of two polynomials of three variables x, y, z . In Maple lexicographic order $x \succ_{lex} y \succ_{lex} z$ we denote by $\text{plex}(x, y, z)$. Then by command

Basis(F, plex(x, y, z));

Maple computes the reduced Groebner basis. If some variables are omitted from the list of variables, then the algorithm considers these variables as non-zero constants (see Appendix C/§2 (Maple) in [2] and [13], [14]).

Example 2.3. Let the system of two polynomials be given:

$$(18) \quad F := \left[z - \frac{x^2}{a^2} - \frac{y^2}{b^2}, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{x}{a} - \frac{y}{b} \right];$$

where x, y, z are variables and a, b are non-zero constants. We compute reduced Groebner basis using Maple by command $\text{Basis}(F, \text{plex}(x, y, z))$; and the result is the following list of the (arranged) polynomials:

$$(19) \quad [bx + ay - abz, 2y^2 - 2byz - b^2z^2 - b^2z]$$

Let us emphasize that the answer is the reduced Groebner basis, except for clearing denominators (so leading coefficients of the Groebner basis are polynomials in a and b) [2]. This proves that curve of intersection lies in the following plane:

$$(20) \quad bx + ay - abz = 0.$$

Let us remark that $\langle LT(I) \rangle = \langle bx, 2y^2 \rangle$ and the condition (16) is fulfilled.

Example 2.4. Let the system of two polynomials be given:

$$(21) \quad F := [x + yz + y - z^4 - 4, y - z^3 - 1];$$

where x, y, z are variables. We compute reduced Groebner basis using Maple by command `Basis(F, plex(x, y, z))`; and the result is the following list of the (arranged) polynomials:

$$(22) \quad [x + z^3 + z - 3, y - z^3 - 1]$$

Note that:

$$(23) \quad 1 \cdot (x + yz + y - z^4 - 4) - z \cdot (y - z^3 - 1) = x + y + z - 4$$

and therefore the intersection is planar. Let us emphasize that $\langle LT(I) \rangle = \langle x, y \rangle$ and condition (16) is not fulfilled.

Remark 2.5. It is possible to formulate and prove the previous theorems in more extended sense than in the given three-dimensional formulations and lexicographic order.

3. AN APPLICATION OF THE GROEBNER BASES TO PLANE SECTIONS OF ONE TYPE OF CONOID WITH BASIC CUBIC EGG CURVE

In this part, there will be considered an application of the Groebner bases theory on a surface which is obtained by motion of the system of generatrices along three directrices.

Let us accept cubic egg curve g , the right cubic hyperbolic parabola of type A [3] which lies in the plane $x - y$, for the initial curve taken as a plane section of a conoid, and also for its directrix. Starting from the definition of conoid [6] (p. 277) as a surface which has two directrices in finiteness: plane curve d_1 , and straight line d_2 , while the third directrix, straight line d_3 , lies in infinity, we consider a conoid with [11]:

(i) Cubic curve (g) in the form $g = d_1: b^2x^2 + a^2y^2 + 2dxy^2 + d^2y^2 - a^2b^2 = 0$ as directrix d_1 ($a > b > 0$, $a - b \geq d > 0$).

(ii) Straight line parallel to the axis y , in the plane $y - z$ on elevation $z = h$ as directrix d_2 ($h > 0$).

(iii) Plane $x - z$ as directrix plane of the conoid, and its infinitely distant straight line as directrix d_3 , in order to obtain a right conoid.

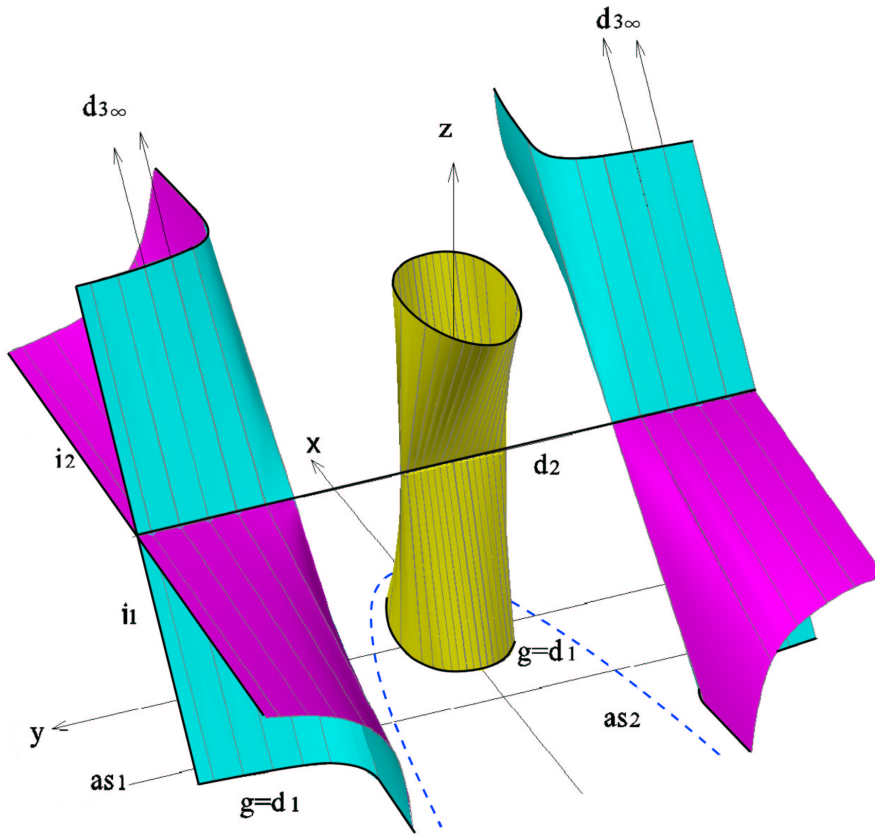


Figure 1.

Each generatrix of the surface will intersect all three directrices [7]. The surface obtained in this manner will be quintic surface [11], degenerated into a conoid of the fourth order, as shown at Figure 1, with the equation [11]:

$$(24) \quad (a^2y^2 + d^2y^2 - a^2b^2)(z - h)^2 - 2dhxy^2(z - h) + b^2h^2x^2 = 0$$

and a plane:

$$(25) \quad z = h.$$

Thus, we give an explanation of origination and the order of the surface, from the aspect of Projective Geometry.

1. As every plane F_1 parallel to the plane $x - z$, which will imply generatrices of the surface, intersects the plane $x - y$ of the curve g by a straight line parallel to the x axes [4], and each straight line must intersect the cubic curve by three points

X_1 , X_2 and X_3 , we conclude that in each point of the directrix d_2 there intersect three generatrices: i_1 , i_2 , and i_3 , of the surface. Therefore, the directrix d_2 will be triple line of the surface [5].

2. Because of the symmetry of the curve g in relation to the x axes, pair of generatrices of the conoid will intersect in the same point of the infinite directrix d_3 , so it will be double line of the surface.

3. Cubic curve $g = d_1$ has two asymptotes as_1 and as_2 , whereat the asymptote as_1 is parallel to the y axes, and as_2 is parabola [9] with x axes. Therefore, the cubic curve g has two infinite points, X_∞ and Y_∞ , of the axes x and y .

4. X_∞ represents the infinite vertex of the parabola as_2 . Every generatrix i_3 of the surface passes through this point forming the plane H_1 , jointly with the directrix d_2 . Y_∞ is a point of tangency of curve $g = d_1$ and asymptote as_1 . At the same time, it is a triple point of intersection of the directrix d_2 and the plane $x - y$.

5. Points X_∞ and Y_∞ belong to infinite straight line q_∞ of the plane $x - y$, which passes two times through those points. Straight line q_∞ represents two overlapped generatrices i_2 and i_3 which pass through the infinite point Y_∞ and are tangents to the vertex X_∞ of the parabolic asymptote as_2 (and also the curve g) at two immeasurable close points, X_2 and X_3 , which are collinear to the point Y_∞ . Therefore, generatrices $i_2(X_2Y_\infty)$ and $i_3(X_3Y_\infty)$ will be identical, and so the straight line q_∞ will be also the double line. Generatrix i_1 will be separated, so in the infinite plane ω_∞ , there will lie five lines: double line $d_{3\infty}$, double line q_∞ , and the line $i_{1\infty}$.

6. Plane $x - y$ intersects the conoidal surface by the initial cubic curve g and by the infinite double straight line q_∞ , which indicates that it is the case of the surface of fifth degree [11].

7. Considering that the generatrix i_3 is a straight line that belongs to the pencil of straight lines through the point X_∞ , which constitutes, along with directrix d_2 , the plane H_1 , the surface will be degenerated to the conoid and the plane H_1 . By eliminating the plane, the doubleness of the line q_∞ subdues, as well as the tripleness of the directrix d_2 , so the conoidal surface remains of the fourth order.

8. The horizontal plane H_1 , at the elevation $z = h$ of the directrix d_2 is asymptotic plane of the surface. It is tangential plane onto the infinite double straight line q_∞ . Besides, there will exist yet another asymptotic plane, a skew plane R , which is determined by the directrix d_2 and the asymptote as_1 of the curve $g = d_1$, as two parallel lines, and is tangential the conoid by the infinite straight line $r_\infty = i_1$.

9. The infinite plane of the space ω_∞ , intersects the conoid by one double straight line d_3 and also by the remaining straight lines q_∞ , and r_∞ , which proves that it is the surface of fourth order.

Mathematical explanation of the originating surface, initially of the fifth order is easily obtained by simple multiplication of the equation of the fourth order (24) by factor $z - h$. The equation derived in this manner: $(a^2y^2 + d^2y^2 - a^2b^2)(z - h)^3 - 2dhxy^2(z - h)^2 + b^2h^2x^2(z - h) = 0$, decomposes to an equation of the fourth

order (24) and an equation of a plane (25), as it is given in the commentary of the Projective Geometry.

Let us consider if it is possible for the plane section of the surface (24) to be a non-degenerated conic [11]. Let us emphasize that this problem is equivalent to the following system:

$$(26) \quad \left\{ \begin{array}{l} (26/i) \quad (a^2y^2 + d^2y^2 - a^2b^2)(z - h)^2 - 2dhxy^2(z - h) + b^2h^2x^2 = 0, \\ (26/ii) \quad Ax + By + Cz + D = 0; \end{array} \right.$$

which has solution by non-degenerated conic, for some real constants A, B, C, D . Using technique of Groebner bases we will prove that is not possible. The following statements are true:

Lemma 3.1. (i) *Plane $x = \alpha$, for $\alpha \in \mathbf{R} \setminus \{0\}$, has the intersection with the surface (26/i) by the curve of the fourth order. Plane $x = 0$ has the intersection with the surface (26/i) by $x = 0, z = h (|y| \leq b \vee |y| \geq ab/d)$ or $x = 0, y = \pm ab/\sqrt{a^2 + d^2}$.*

(ii) *Plane $y = \beta$, for $\beta \in \mathbf{R}$, has intersection with the surface (26/i) iff:*

$$(27) \quad |\beta| \leq b \quad \vee \quad |\beta| \geq \frac{ab}{d} \quad \left(\frac{ab}{d} > b \right);$$

then the intersection is presented as two straight lines (which determine degenerated conic).

(iii) *Plane $z = \gamma$, for $\gamma \in \mathbf{R} \setminus \{h\}$, has the intersection with the surface (26/i) by the curve of the third order. Plane $z = h$ has the intersection with the surface (26/i) by $x = 0, z = h (|y| \leq b \vee |y| \geq ab/d)$.*

Proof. (i) By substitution $x = \alpha \in \mathbf{R}$ in (26/i) we obtained:

$$\begin{aligned} & (y^2(a^2 + d^2) - a^2b^2)(z - h)^2 - 2hd\alpha y^2(z - h) + h^2b^2\alpha^2 = 0 \\ \iff & (z - h)(a^2b^2 - (a^2 + d^2)y^2) = \left(-dy^2 \pm \sqrt{(b^2 - y^2)(a^2b^2 - d^2y^2)} \right) h\alpha. \end{aligned}$$

Based on $a, d > 0$ we can conclude that the intersection is one algebraic curve of the fourth order if $\alpha \neq 0$. If $\alpha = 0$ ($\iff x = 0$), then from the previous equation follows $x = 0, z = h (|y| \leq b \vee |y| \geq ab/d)$ or $x = 0, y = \pm ab/\sqrt{a^2 + d^2}$.

(ii) By substitution $y = \beta \in \mathbf{R}$ in (26/i) we obtained:

$$\begin{aligned} & b^2h^2x^2 + (a^2\beta^2 + d^2\beta^2 - a^2b^2)z^2 - 2dh\beta^2xz + \\ & 2dh^2\beta^2x + (2a^2b^2h - 2d^2h\beta^2 - 2a^2h\beta^2)z + \\ & (a^2h^2\beta^2 + d^2h^2\beta^2 - a^2b^2h^2) = 0. \end{aligned}$$

If we solve the previous second order equation by x , from discriminant, we obtain the condition (27) for existence of the intersection via two straight lines:

$$x = \pm \frac{\beta^2d + \sqrt{(b^2 - \beta^2)(a^2b^2 - d^2\beta^2)}}{b^2h} (-z + h).$$

(iii) By substitution $z = \gamma \in \mathbf{R}$ in (26/i) we obtained:

$$b^2 h^2 x^2 + 2 d h (h - \gamma)^2 x y^2 + (a^2 + d^2)(h - \gamma) y^2 - a^2 b^2 (h - \gamma)^2 = 0$$

$$\iff x = \pm \frac{\sqrt{(b^2 - y^2)(a^2 b^2 - d^2 y^2)} - y^2 d}{b^2 h} (h - \gamma).$$

Based on $d, h > 0$ we can conclude that the intersection is one algebraic curve of the third order iff $\gamma \neq h$. If $\gamma = h$ ($\iff z = h$), then from the previous equation follows $z = h, x = 0$ ($|y| \leq b \vee |y| \geq ab/d$). ■

Remark 3.2. Starting from geometrical definition of this type of conoid, the cases (i) and (iii) will always have real plane sections.

Theorem 3.3. If the solution of the system (26) exists then it is not a non-degenerated conic.

Proof. First we assume that $C \neq 0$. Then by substitution $z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}$ in (26/i) we obtain $x - y$ projection of a possible intersection:

$$(28) \quad \begin{aligned} & \left(\frac{A^2 a^2}{C^2} + \frac{A^2 d^2}{C^2} + \frac{2 d h A}{C} \right) x^2 y^2 + \left(\frac{2 A B d^2}{C^2} + \frac{2 d h B}{C} + \frac{2 A B a^2}{C^2} \right) x y^3 + \\ & \left(\frac{B^2 d^2}{C^2} + \frac{B^2 a^2}{C^2} \right) y^4 + \left(\frac{2 A h a^2}{C} + \frac{2 A D d^2}{C^2} + \frac{2 A h d^2}{C} + \frac{2 d h D}{C} + 2 d h^2 + \right. \\ & \left. \frac{2 A D a^2}{C^2} \right) x y^2 + \left(\frac{2 B D d^2}{C^2} + \frac{2 B D a^2}{C^2} + \frac{2 B h a^2}{C} + \frac{2 B h d^2}{C} \right) y^3 + \\ & \left(h^2 b^2 - \frac{A^2 a^2 b^2}{C^2} \right) x^2 + \frac{2 A B a^2 b^2}{C^2} x y + \left(\frac{2 D h a^2}{C} + h^2 a^2 - \frac{B^2 a^2 b^2}{C^2} + \right. \\ & \left. \frac{D^2 a^2}{C^2} + h^2 d^2 + \frac{2 D h d^2}{C} + \frac{D^2 d^2}{C^2} \right) y^2 - \left(\frac{2 A D a^2 b^2}{C^2} + \frac{2 A h a^2 b^2}{C} \right) x - \\ & \left(\frac{2 B h a^2 b^2}{C} + \frac{2 B D a^2 b^2}{C^2} \right) y - h^2 a^2 b^2 - \frac{2 D h a^2 b^2}{C} - \frac{D^2 a^2 b^2}{C^2} = 0. \end{aligned}$$

The previous equation is algebraic equation of the second order iff:

$$\begin{aligned} & \frac{A^2 a^2}{C^2} + \frac{A^2 d^2}{C^2} + \frac{2 d h A}{C} = 0 \quad \wedge \\ & \frac{2 A B d^2}{C^2} + \frac{2 d h B}{C} + \frac{2 A B a^2}{C^2} = 0 \quad \wedge \\ & \frac{B^2 d^2}{C^2} + \frac{B^2 a^2}{C^2} = 0 \quad \wedge \\ & \frac{2 A h a^2}{C} + \frac{2 A D d^2}{C^2} + \frac{2 A h d^2}{C} + \frac{2 d h D}{C} + 2 d h^2 + \frac{2 A D a^2}{C^2} = 0 \quad \wedge \\ & \frac{2 B D d^2}{C^2} + \frac{2 B D a^2}{C^2} + \frac{2 B h a^2}{C} + \frac{2 B h d^2}{C} = 0. \end{aligned}$$

Solving previous system, via Maple, we obtain two subcases:

$$\mathbf{1^0.} \quad A=0, B=0, C=p, D=-ph \quad \vee \quad \mathbf{2^0.} \quad A=q, B=0, C=-\frac{a^2+d^2}{2dh}q, D=\frac{a^2+d^2}{2d}q$$

for $p, q \in \mathbf{R} \setminus \{0\}$. Therefore:

1^o. The system (26):

$$\left\{ \begin{array}{l} (a^2y^2 + d^2y^2 - a^2b^2)(z - h)^2 - 2dhxy^2(z - h) + b^2h^2x^2 = 0, \\ z - h = 0; \end{array} \right\}$$

has the reduced Groebner basis (related to lexicographic order) in the following form:

$$g_{11} := x^2 \quad \wedge \quad g_{12} := z - h.$$

Hence, the system (26) has the intersection by directrix d_2 and this intersection is not non-degenerated conic.

2^o. The system (26):

$$\left\{ \begin{array}{l} (a^2y^2 + d^2y^2 - a^2b^2)(z - h)^2 - 2dhxy^2(z - h) + b^2h^2x^2 = 0, \\ 2dhx - (a^2 + d^2)z + h(a^2 + d^2) = 0; \end{array} \right\}$$

has the reduced Groebner basis (related to lexicographic order) in the following form:

$$g_{21} := z^2 - 2zh + h^2 \quad \wedge \quad g_{22} := 2dhx - (a^2 + d^2)z + h(a^2 + d^2).$$

Hence, the system (26) has intersection by directrix d_2 and this intersection is not non-degenerated conic.

Let $C = 0$. We can consider two cases: $B = 0$ or $B \neq 0$. If $B = 0$ subcase $A = 0$ does not lead to a planar intersection. If $B = 0$ subcase $A \neq 0$ is considered in the Lemma 3.1. (case (i)). Finally, let us observe case $B \neq 0$. By substitution of $y = -\frac{A}{B}x - \frac{D}{B}$ in (26/i) we obtain $x - z$ projection of a possible intersection:

$$(29) \quad \begin{aligned} & -\frac{2dhA^2}{B^2}x^3z + \left(\frac{a^2A^2}{B^2} + \frac{d^2A^2}{B^2}\right)x^2z^2 + \frac{2dh^2A^2}{B^2}x^3 - \left(\frac{4dhAD}{B^2} + \right. \\ & \left. \frac{2ha^2A^2}{B^2} + \frac{2hd^2A^2}{B^2}\right)x^2z + \left(\frac{2a^2AD}{B^2} + \frac{2d^2AD}{B^2}\right)xz^2 + \left(\frac{4dh^2AD}{B^2} + \right. \\ & \left. \frac{h^2a^2A^2}{B^2} + h^2b^2 + \frac{h^2d^2A^2}{B^2}\right)x^2 - \left(\frac{2dhD}{B^2} + \frac{4hd^2AD}{B^2} + \frac{4ha^2AD}{B^2}\right)xz + \\ & \left(\frac{a^2D}{B^2} - a^2b^2 + \frac{d^2D}{B^2}\right)z^2 + \left(\frac{2h^2d^2AD}{B^2} + \frac{2dh^2D}{B^2} + \frac{2h^2a^2AD}{B^2}\right)x + \\ & \left(2ha^2b^2 - \frac{2ha^2D}{B^2} - \frac{2hd^2D}{B^2}\right)z + \left(\frac{h^2d^2D}{B^2} + \frac{h^2a^2D}{B^2} - h^2a^2b^2\right) = 0. \end{aligned}$$

Therefore $A = 0$ and we obtain the case (ii) from the Lemma 3.1. ■

Remark 3.4. *Let us emphasize that we may consider possible factorizations of polynomials (28) and (29) in purpose of determining a type of section by non-degenerated conic as a part of plane intersection of this egg curve based conoid.*

4. CONCLUSIONS

In the paper an application of Groebner bases is considered, in order to define planarity of two surfaces intersection. An egg curve based conoid is analyzed in particular, from the aspect of Projective Geometry. Its plane sections are analyzed from the aspect of Analytic Geometry. Using the example of the egg curve based conoid, a method is developed, based on the technique of Groebner bases, which enables investigation of existence of the assigned type of planar section. Based on the presented method, in some cases, it is possible to determine even for the other algebraic surfaces if they would imply a planar section that consists of non-degenerated conics.

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