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LINEAR TRANSIENT ANALYSIS OF SPATIAL CURVED BERNOULLI – EULER BEAM USING ISOGEOMETRIC APPROACH

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Abstract:

In this paper, linear transient analysis of spatial curved Bernoulli – Euler beam with circular cross – section formulated using isogeometric approach has been presented. The isogeometric approach is based on the concept that the geometry of the beam, as well as the displacement field are defined using the Non – Uniform Rational B – Spline (NURBS) functions. Governing equations of motions are derived using the basic relations of the differential geometry and continuum mechanics. In order to conduct transient analysis, time discretization has been employed using explicit integration scheme. The validation of the proposed method has been carried out for the spatial curved cantilever beam subjected to the point load with constant magnitude. The results of the free end displacements obtained using proposed method have shown excellent convergence properties and agreement with the results obtained using finite element-based software. In comparison to the conventional finite element method, less number of degrees of freedom are required in order to obtain accurate results.

Key words: Bernoulli – Euler beam, spatial curved beam, linear transient analysis

1. Introduction

Solutions of the engineering problems formulated using beam theory are usually obtained using the numerical procedures. Finite element method (FEM) represents the mostly used numerical method implemented in most software packages related to the structural analysis. The FEM is based on the physical discretization, as the domain is divided into smaller domains (finite elements), forming the mesh with the corresponding degrees of freedom. Refinement of the analyzed domain has to be conducted until the results converges.

Nowadays, geometry of curved beam is modelled using Computer – Aided Design (CAD) software packages, which are based on the Non – Uniform Rational B – Spline (NURBS) functions. NURBS basis functions enable accurate representation of the free – form curves, as well as the curves of conical section like circle, parabola, ellipse and hyperbola.

Direct relation between the FEM and CAD has not been established, which represents the main disadvantage in the design process. Therefore, the isogeometric approach (IGA) has been formulated in which direct relation between the geometry and numerical model has been established. This approach was first introduced by Hughes and his co – workers [1]. Additional advantage of this approach can be found in its refinement procedure, denoted as H –, P – and K – refinement procedure [2]. Formulation of the Bernoulli – Euler beam using IGA was first published by Hughes [1], where only straight beam has been analyzed. Greco and Cuomo [3] formulated spatial curved Bernoulli – Euler beam for the static analysis, while the dynamic formulation, applied for the free vibration analysis, was published in [4].

In this paper the linear transient analysis of an arbitrarily curved spatial beam with circular cross – section is presented. A short review of the NURBS basis function is given in Section 2, following generally explained derivation procedure of the governing equation of motion of the Bernoulli – Euler isogeometric beam element. Detailed derivation procedure can be found in [5]. In addition, explicit integration scheme used for the time discretization is given is Section 5. The procedures presented in this paper have been implemented in the original Matlab code [6]. The numerical example of an arbitrarily spatial curved beam with circular cross section has been carried out in Section 6. At the end, the main conclusions have been drawn.

2. NURBS basis functions

The spatial curved beam in Euclidean space can be represented as:

$$\mathbf{C}(\xi) = \sum_{i=0}^{n} R_{i,p}(\xi)\mathbf{P}_i \tag{1}$$

where $R_{i,p}(\xi)$ is the i-th NURBS basis function, p is the function degree, $\mathbf{P}_i$ is the control point, which discretize the beam geometry in the Euclidean space, while n is the number of control points (basis functions). In Fig. 1. the spatial beam element with corresponding control points is presented.

NURBS basis functions are constructed using the B – Spline basis functions as:

$$R_{i,p}(\xi) = \sum_{i=0}^{n} \frac{N_{i,p}(\xi) w_i}{\sum_{j=0}^{n} N_{j,p}(\xi) w_j} \tag{2}$$

where $N_{i,p}$ represents the B – Spline basis function, while w_i is the weight of the corresponding control point. The B – Spline basis functions are constructed in the parametric domain ξ using the knot vector $\boldsymbol{\xi}$. The knot vector represents the set of non-decreasing real numbers, denoted as knots (ξ_i), which represents the coordinate in the parametric domain. The B-Spline functions are derived using the Cox de Boor algorithm, where for $p = 0$:

$$N_{i,p}(\xi) = \begin{cases} 1 & \text{if } \xi \in [\xi_i, \xi_{i+1}[\\ 0 & \text{otherwise} \end{cases} \tag{3}$$

while for the p>0:

$$N_{i,p}(\xi) = \begin{cases} \dfrac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \dfrac{\xi_{i+p-1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) & \text{if } \xi \in [\xi_i, \xi_{i+p+1}[\\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Besides its capability to represents different types of curved beam geometry, application of NURBS basis functions has an advantage related to the refinement procedures, denoted as H -, P – and K – refinement procedures. Using this refinement procedures, the number of terms which represent the beam geometry can be increased without the change of the initial beam geometry. In this paper only H – refinement procedure is conducted, where the increase of the terms is conducted by knot insertion. More about NURBS and B – Spline functions can be found in [7].

3. Beam geometry

Due to the beam assumption of the rigid cross – section, all beam quantities are defined on the beam's centerline, which in general represents the spatial curve in Euclidean space. The spatial curve is defined using position vector **r**, which can be parametrized using NURBS basis functions:

$$\mathbf{r}(\xi)=\sum_{i=0}^{n} R_{i,p}(\xi)\mathbf{r}_i \tag{5}$$

where $\mathbf{r}_i$ is the *i*-th control point. Formulation of the Bernoulli – Euler beam element is conducted using a curvilinear coordinate system attached to the beam's centerline. Using basic principles of differential geometry, three vectors are defined, representing the basis of the curvilinear coordinate system. The first vector is the tangent vector **t**, derived directly from the position vector [8]. The other two vectors, normal **n** and binormal **b** are derived from the tangent vector. Due to the orthogonality of the basis vectors, normal and binormal basis vectors are located in the beam's cross – section plane.

Using orthonormal basis, an arbitrary point of the beam can be defined as:

$$\hat{\mathbf{r}}=\mathbf{r}+\eta\mathbf{n}+\varsigma\mathbf{b} \tag{6}$$

where η and ς are coordinates of the principle axes. Tangent, normal and binormal vectors of an arbitrary points can be found from its position vector, thus the relation between these vectors and the basis vectors of the beam's centerline can be established. More about the beam geometry can be found in [4].

4. Isogeometric Bernoulli – Euler beam formulation

Due to external impact on the spatial beam, an initial configuration of the beam's centerline can change its position, which is defined with the position vector of the deformed configuration:

$$\mathbf{r}^{*}=\mathbf{r}+\mathbf{u} \tag{7}$$

where **u** is the displacement vector of the beam's centerline. If the deformed configuration is parametrized as undeformed configuration, the parametrization of the displacement vector is given as:

$$\mathbf{u}(\xi)=\sum_{i=0}^{n} R_{i,p}(\xi)\mathbf{u}_i \tag{8}$$

where $\mathbf{u}_i$ is the displacement vector of the *i*-th control point. As can be observed, parameterization of the beam geometry and displacement field is defined using the same basis functions, which represents the main idea of the isogeometric approach. The Cartesian components of the displacement vectors of the control points are the degrees of freedom of the Bernoulli – Euler isogeometric beam element. In Fig. 1. undeformed and deformed configuration of the spatial curved beam element is presented with the corresponding displacement vectors.

Using convective coordinate system, the position vector of an arbitrary point in the deformed configuration is defined as:

$$\hat{\mathbf{r}}^{*}=\mathbf{r}+\eta\mathbf{n}^{*}+\varsigma\mathbf{b}^{*} \tag{9}$$

Therefore, displacement of an arbitrary point can be defined as:

$$\hat{\mathbf{u}}=\mathbf{u}+\eta\mathbf{u}_2+\varsigma\mathbf{u}_3 \tag{10}$$

where $\mathbf{u}_2$ and $\mathbf{u}_3$ represents displacement vectors of normal and binormal vectors. Applying the Bernoulli – Euler assumption, the torsional rotation of the beam's cross – section is introduced, which represents the additional degree of freedom for the Bernoulli – Euler beam in addition to the beam's centerline displacements.

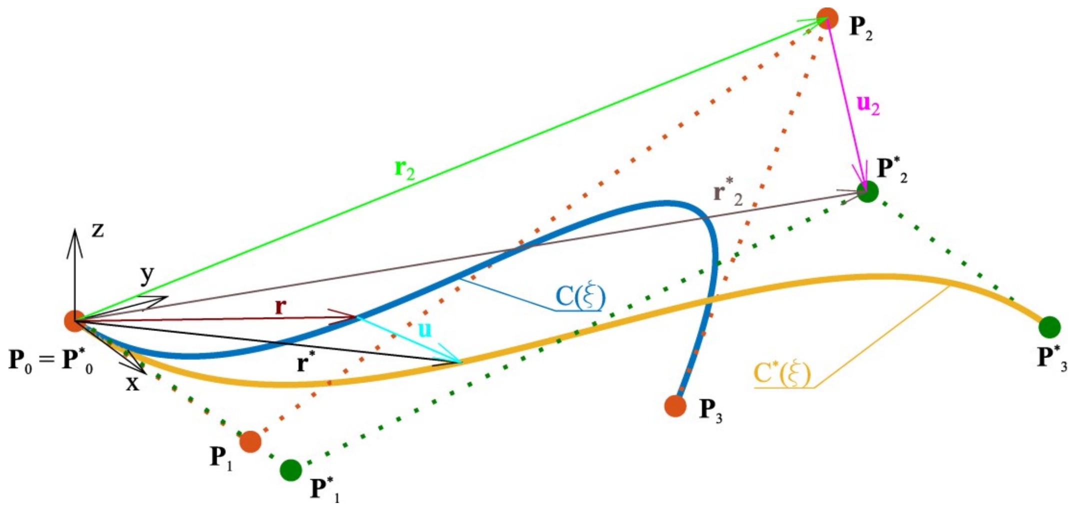

Fig. 1. Undeformed and deformed configuration of the spatial curved beam element with an arbitrary shape obtained using NURBS parameterization

Displacement of an arbitrary point represents the displacement field of the Bernoulli – Euler beam element. Acceleration field of the Bernoulli – Euler beam element is derived from the displacement field as a double time derivative. In addition, variation of the displacement field is obtained from the displacement field.

Kinematic relations of the Bernoulli – Euler beam element are derived using properties of the convective coordinate system and metric tensor of the undeformed and deformed beam configuration, while the constitutive relations are obtained using generalized Hook's law.

By substituting the above mentioned relations into the weak formulation (the principle of the virtual work) the governing equation of motion of the Bernoulli – Euler isogeometric beam element is obtained [4], which is used for the transient analysis:

$$\mathbf{M}\ddot{\ } \qquad = \mathbf{Q} \tag{11}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{Q}$ is the external load vector, $\mathbf{q}$ is the displacement vector of the control points (DOFs), while $\ddot{\ }$ is its double time derivative. The external load vector is derived from the contribution of the external forces of the virtual work.

5. Explicit integration scheme

In order to solve the governing equation of motion, which represents an ordinary differential equation, the time discretization is required. Aforementioned is applied by dividing the continuous time t into equal time intervals Δt. In this paper, the explicit integration scheme is used, which represents the direct time integration procedure, were the unknown equilibrium of the system at time t_{i+n} is calculated from the equilibrium of the known system at time t_n [9]. Using this integration scheme, the unknown quantity is the acceleration field from which the displacement increment can be found. Adding the displacement increment to the known configuration at time t_i, the configuration of the system at time t_{i+n} is obtained. It is important to emphasize that the explicit integration scheme is conditionally stable, as the time step is bounded by:

$$\Delta t_{max} = \frac{2}{\omega_{max}} \tag{12}$$

where ω_{max} is the maximum natural frequency of the system. Applying central difference method, the displacement vector of the control points at time t_{i+n} is obtained as:

$$\mathbf{q}_{n+1} = 2\cdot\mathbf{q}_n - \mathbf{q}_{n-1} + \Delta t^2 \mathbf{M}^{-1}\left(\mathbf{Q}_n + \mathbf{K}\mathbf{q}_n\right) \tag{13}$$

where $\mathbf{Q}_n$ represents the external load vector at time t_n.

6. Numerical example

In this section, the validation study of the proposed method for linear transient analysis of the spatial curved beam will be presented. The cantilever spatial beam is clamped at the first control point $\mathbf{P}_o$, defined with the coordinates (0,0,0), Fig. 2. The gravitational point load, applied at the free end of the beam, has constant magnitude during analysis. Free end displacements of the cantilever spatial curved beam subjected to the gravitational point load are analyzed. The beam material is homogeneous defined using the Young's modulus E = 31.5 GPa, the Poisson's ratio ν = 0.2 and the mass density ρ = 2500 kg/m^3. The cross-section is circular with a diameter d = 1 m. The geometry of the beam is defined using the following control points:

$$\mathbf{CP}^T = \begin{bmatrix} 0 & 2 & 4 & 3 \\ 0 & 0 & 3 & 2.5 \\ 0 & 0 & 3 & 0 \end{bmatrix} \tag{14}$$

and the 3*rd* – order basis functions defined over the knot vector and weight vector:

$$\boldsymbol{\xi}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{15}$$

$$\mathbf{w}^T = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} \tag{16}$$

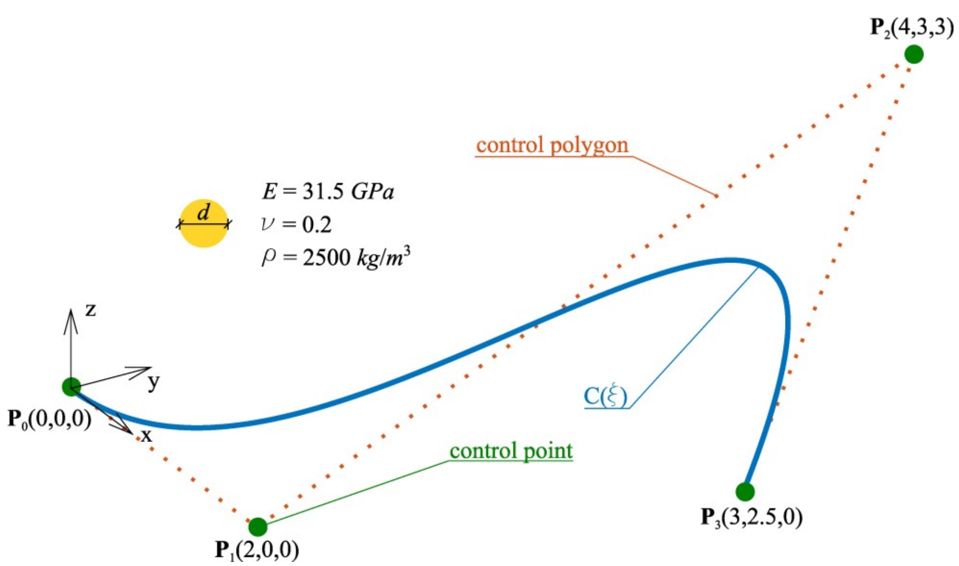

Fig. 2. An arbitrary spatial curve with corresponding control points and cross – section properties

The H – refinement procedure is applied to demonstrate the convergence property of the proposed isogeometric approach. Using this procedure, the number of DOFs is increased by knot insertion without the change of the beam geometry. In Figs. 3 – 5, the normalized displacement components in direction to the Cartesian coordinate system are presented. The displacements of the beam's free end have been divided by the load magnitude, forming normalized displacements. As can be noticed the converged results of the normalized displacements are obtained using the beam model with 72 DOFs.

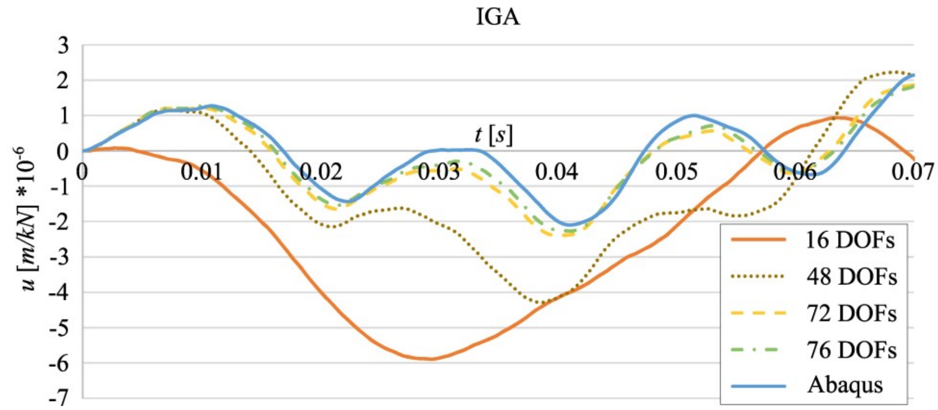

Fig. 3. Component of free end normalized displacement in direction to X coordinate obtained using IGA

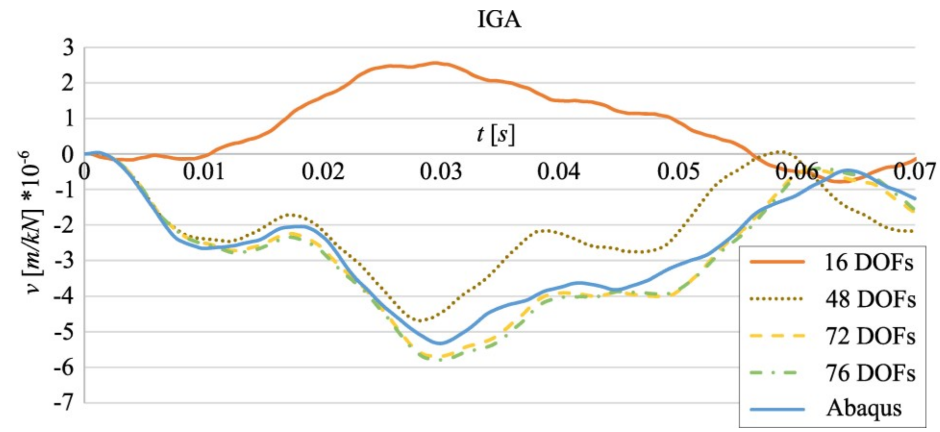

Fig. 4. Component of free end normalized displacement in direction to Y coordinate obtained using IGA

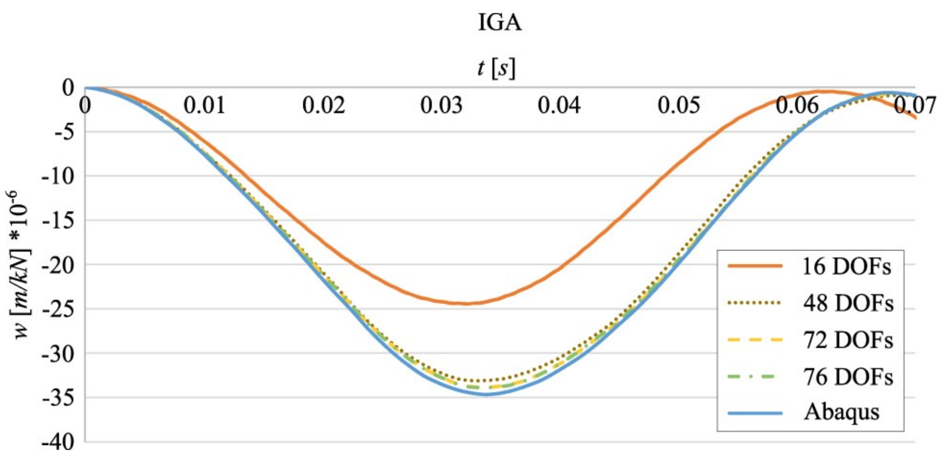

Fig. 5. Component of free end normalized displacement in direction to Z coordinate obtained using IGA

In order to demonstrate the advantage of the proposed method in the linear transient analysis of spatial curved beams, the same beam has been modelled in Abaqus/Explicit [10] using B31 elements, based on the Timoshenko beam theory. Beam element based on the Bernoulli – Euler theory is not available in the Abaqus/Explicit beam element library, therefore application of the B31 element is necessary. The results of the free end normalized displacement components are obtained for the different beam models, Figs. 6 – 8. As can be observed, the converged results of the normalized displacement components in the Cartesian coordinate system are obtained using beam model with 90 DOFs, while 72 DOFs were used to obtain converged results in the isogeometric method.

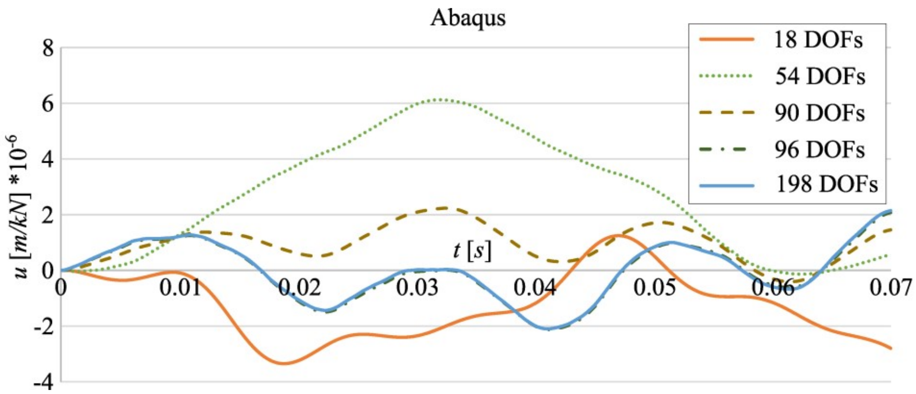

Fig. 6. Component of free end normalized displacement in direction to X coordinate obtained using FEM

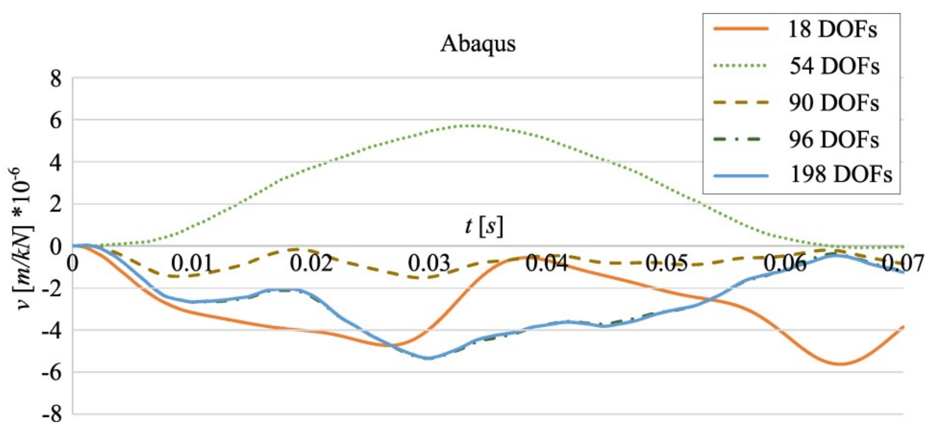

Fig. 7. Component of free end normalized displacement in direction to Y coordinate obtained using FEM

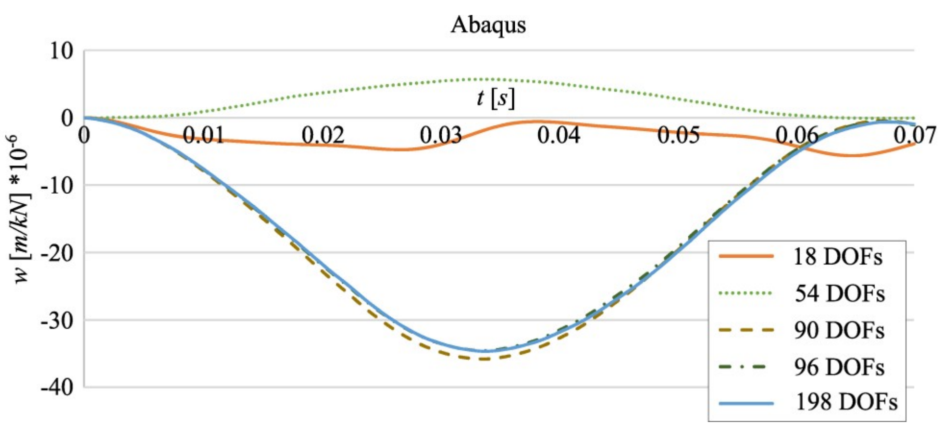

Fig. 8. Component of free end normalized displacement in direction to Z coordinate obtained using FEM

7. Conclusions

In this paper, the linear formulation of spatial curved Bernoulli – Euler beam with circular cross - section is presented using isogeometric approach. In order to conduct transient analysis of an arbitrary curved beam, explicit time integration scheme is employed, which enabled direct time integration. In order to validate the proposed method, the numerical study of the curved spatial beam subjected to point load has been carried out. Satisfactory agreement has been noticed between the results obtained using the proposed method and the results obtained using the commercial FEM software Abaqus. In addition, convergence study has been carried out using the presented approach and the FEM software Abaqus. As can be noticed, less number of DOFs is required in order to obtain converged result using isogeometric approach in comparison to the FEM. This advantage of the proposed method reduces required computational and time resources during the calculation process.

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