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TILINGS WITH DIAMOND, STAR AND PINEAPPLE SHAPES BASED ON THE GEOMETRY OF THE REGULAR PENTAGON

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ABSTRACT

This paper presents a gallery of tessellations created by combining the following equilateral shapes: diamonds and stars, which can be further assembled into a pineapple shape. These three shapes can be decomposed into simpler shapes: an isosceles triangle and a rectangle. The triangle is formed by a subdivision of the regular pentagon into five equal sections, so that each have a base of length $\boldsymbol{a}$ *and legs of length* $\boldsymbol{b}$*. The rectangle is created by having the same* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *for its sides. The diamond is formed by two such triangles and one rectangle, while the star is formed by a radial arrangement of five triangles back into the pentagon onto which five more triangles are added (elevated). Using these two shapes, we can tile the Euclidean plane without overlaps and gaps in different ways, including the pentagonal matrix. They can be further assembled into a "pineapple" shape, which can also tile the plane arranged in different ways, using only one shape (tile). We present several examples that include: periodic, non-periodic, rotational, radial and free-form tessellations. These shapes, in addition to their visual attractiveness and decorativeness which can be used in design, also hide the connection with patterns that can be found in nature, similar to Turing patterns.*

Keywords: pentagon, star, tiling, isosceles triangle, rectangle.

1. INTRODUCTION

Tilling the plane is one of the oldest problems in geometry, and yet it is still relevant in modern days. Both in the theoretical and in the practical sense, solving the fitting of congruent or incongruent planar figures, captures the attention of both scientists and designers. Nowadays, with the advent of graphics software, generating geometric patterns based on the division of the Euclidean plane has been greatly facilitated compared to earlier epochs when the tool for solving these problems was restricted to classical accessories: ruler, compass and knowledge of trigonometry. However, in addition to the computer aided approach, this problem is especially addressed by scientists from the aspect of group theory.

In this paper, we treat the problem purely from the aspect of constructive geometry, 2D transformations and CAD, which facilitates the construction and generation of identical regular or irregular polygons. The emphasis is on the possibilities and diversity of pattern creation, so a gallery will be displayed as an overview of the assorted solutions. To obtain them, in addition to geometric thinking, there is also a merit of spatial understanding and some creativity, which still offers solutions that escape algorithms.

The starting point of the research is the shape of a regular pentagon. This polygon is the first in a series of single-digit-sided polygons that "detaches" in its geometry from the polygons that surround it: equilateral triangle, square and hexagon. These polygons, including the octagon, appear in many convex regular-faced polyhedra, while in the Euclidean, *k*-uniform tillings (Grünbaum et al., 1977) they partake with only one other polygon: dodecagon. However, the regular pentagon appears in a number of regular-faced polyhedra, for example in the Platonic solid, dodecahedron, and also in Archimedean and Johnson solids (Johnson 1966), although it does not participate in 2D tilings with regular polygons (Grünbaum et al., 1977). Trigonometrically, no other regular polygon complements the pentagon to the point of assembling even a local patch without gaps and overlaps, let alone the entire plane. Therefore, this shape has been a challenge for solving the tiling problem from antiquity to the present day. It is well known that progenitors of many geometric settings, such as Dürer (1525) or Kepler (1619) dealt with this problem. Each of them gave their own solutions which, in addition to the regular pentagon, included other forms (Lück, 2000). In Dürer's solution, it was a radial tilling that included rhombuses in addition to the pentagon. In Kepler’s, a more complex one, the forerunner of later solutions in the domain of symmetric groups, in addition to the pentagon we encounter a "star" polygon (pentagram, pentagon stelation), a decagon, and another form merging two decagons, which he named a "monster." The pentagon, as unsolvable for tiling with other regular polygons, remained a problem not only as a shape of the tile, but also as a shape of the base to be tiled. Many mathematicians dealt with this problem after Kepler, but it was not before the 20th century that the satisfactory solution was found. Robert Penrose (1074) gave his result that met the requirements of 5-fold symmetry. His solutions appears in three variations and only in one of them does the regular pentagon take place. Along with the pentagon, two more shapes appear in this solution: the “boat” and the rhombus. In the other two solutions, Penrose reduced the number of shapes to only two, and neither of them is a regular polygon. In one of the variations the shapes are "darts" and "kites", and in the other they are "thick" and "thin" rhombus.

Guided by these examples, the solutions in this paper are given, using the geometry of the pentagon and its angular measures, while the pentagon itself does not appear as such. Actually, in the presented solutions, the pentagon-based field can be tiled in 5-fold symmetry using the proposed tile shapes, namely: “diamond” (in fact: elongated rhombus), “star” (not a stelation of the pentagon as in Kepler’s solution, but its elevation) and “pineapple” (a shape formed as a conglomerate of two diamonds and one star). All these shapes are equilateral and made up of two primary shapes: an isosceles triangle and a rectangle. The ratios of their sides correspond to the ratio of the side $\boldsymbol{a}$ of the pentagon and the radius $\boldsymbol{R} = \boldsymbol{b}$ of its circumscribed circle. The procedure for obtaining these shapes and the tilings themselves will be described below.

2. DEFINING TILE SHAPES AND THEIR CONFIGURATIONS

This paper is a continuation of the research presented in the paper (Obradović et al., 2021) (in print). As explained in the aforementioned source, the inspiration for the presented solution(s) of a pentagon based tiling, and consequently for the shapes of the tiles, came from the spatial covering of a specific polyhedral composition. The polyhedra that constitute it are: concave cupolae of the second sort (Obradović et al., 2008) with the pentagonal base, minor type (*CC-II- 5.m*). The covering itself consists of equilateral triangles and squares that fit exactly on the triangular faces of *CC-II-5.m*, thus building a complex corrugated polyhedral surface. Orthogonally projected onto the plane of the cupolas’ bases, we obtain shapes: an isosceles triangle and a rectangle, whose sides have a ratio corresponding to the ratio of the side of the regular pentagon ($\boldsymbol{a}$) and the radius of the circle circumscribed around it ($\boldsymbol{b}$). Thereby: $a: b = 1: \frac{\varphi}{\sqrt{(1+\varphi^2)}}$. As is well known, when it comes to a regular pentagon, φ represents the golden ratio. In other hand, the triangles are created by radially subdividing pentagons into five equal sections, while rectangles can be seen as polygons inserted between them within a regular decagon (Fig. 1 a). Such a disposition of shapes can also be identified as an orthogonal projection of the Johnson solid J5 (Johnson 1966), a pentagonal cupola.

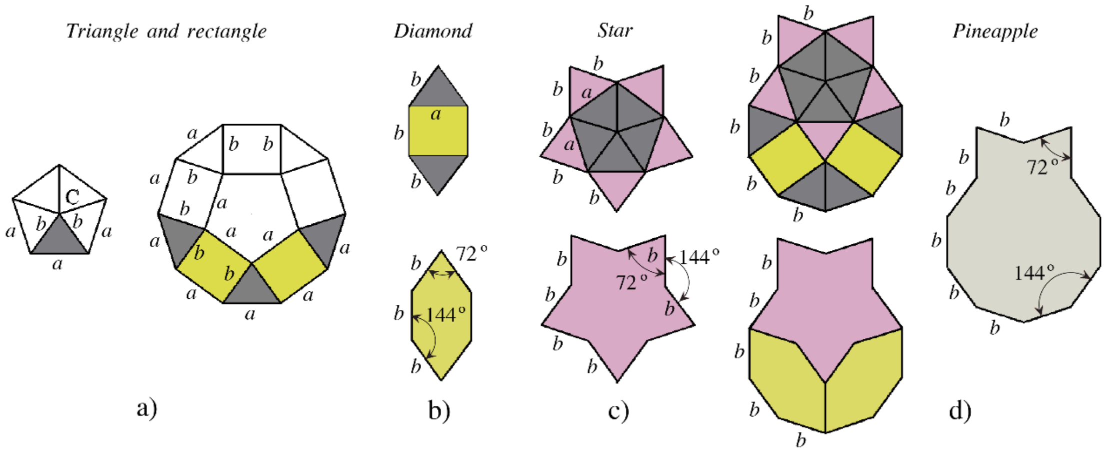


Figure 1: Shapes of the tiles used for tiling: a) triangle and rectangle, b) diamond, c) star, d) pineapple

By combining these two shapes, we get shapes of "diamond" and "star". The diamond is actually an elongated rhombus composed of two isosceles triangles reflexively symmetrical across the sides of triangle a, between which a rectangle is placed (Fig. 1 b). The star is a shape created by elevation (augmentation) of a pentagon, by adding a new isosceles triangle onto each of its sides, reflexively symmetrical across the side a itself (Fig. 1 c). These two shapes, thus, always contain starting triangles and rectangles, so they can be substituted by such elementary shapes in another form of the solution. Just as the diamond and the star are made up of triangles and rectangles, we can create new shape out of them as well. Such a shape is named a "pineapple" (Fig. 1 d), and it contains one star and two diamonds, reflexively symmetrical in relation to the axis of the star overlapped with the side of the diamond itself. (This shape is named “shield” in (Li et al., 2017), but we keep the name already used in Obradović et al., 2021.)

We can manipulate this shape easier than the composition of the star and the diamonds, and even tile a whole plane solely by this single shape. Therefore, in the illustrations given in the paper, we will encounter the shape of pineapple along with diamonds and stars, although it can be reduced to more and more elementary shapes: first to diamonds and stars, and then to triangles and rectangles. Hence, the method of creating tilings, and the rule we use to generate them is the substitution rule.

2.1 Tile grouping and forming the aggregations of tiles

In order to form a tiling, it is necessary to arrange the tiles next to each other without overlapping and gaps. Such fitting of the chosen tiles is enabled by their angular coordinates. As we can see in Fig. 1, angles of 72° and 144° (216°) appear as the angular measured of the tiles, which easily complement each other to 2π angle. This ensures that, with the given shapes, we can continue arraying the tiles in an arrangement that may be geometrically determined, but also arbitrary, even generative. To facilitate the tiling generation, instead of individual sequence of tiles, we can use their grouping into aggregates that we can use either as new tiles, or as a kind of “core” from which we proceed in arranging the tiles.

In Fig. 2, some of such shapes are given, as an illustration of the aggregation of tiles that we use to expedite pattern establishing.

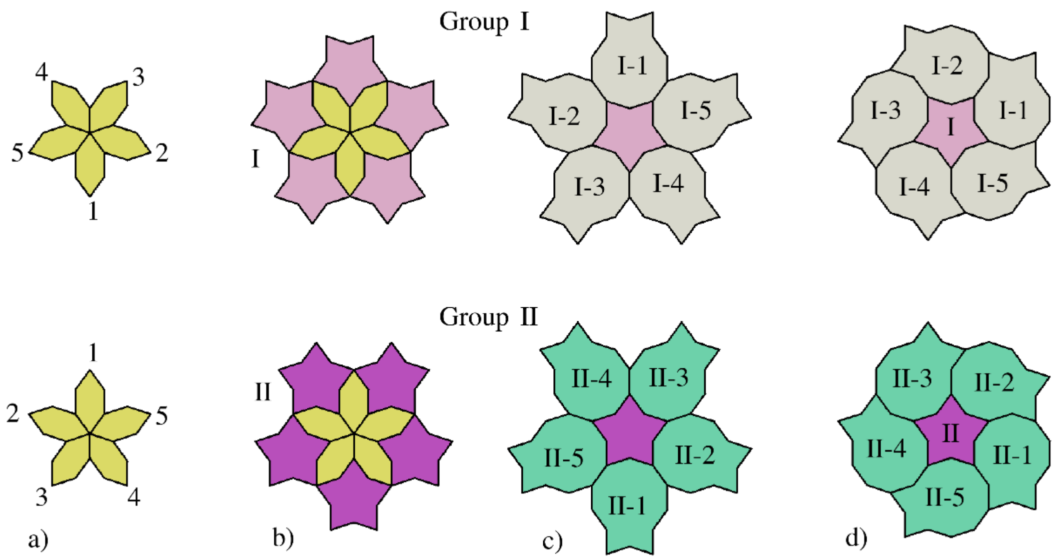


Figure 2: Grouping the tiles into aggregations: a) diamonds, b) stars of Groups I and II, c) pineapples of Groups I and II, s) tiles of Group I and II aggregated in the new “lettuce” formation

We see that diamonds can be arranged in the positions rotated by 72° in relation to the adjacent one, establishing 5-fold symmetry. Whether placed with a vertical diamond "up" or "down" in such an arrangement, their five possible positions remain the same (1, 2, 3, 4 and 5). However, this will not apply to stars and pineapples. Thus, if we rotate a star around its centroid by 180°, we get two different positions: “I” and “II”. Position "I" is shown with the star point on the vertical axis "down", and position "II" with the star point "up".

The pineapple can have positions as in Fig. 2 c, and 2d. If the "star" section of the pineapple is set in position “I”, we have a pineapple tile of Group I in which five different positions of the tile itself are possible (shown in light gray). We also have another five positions of pineapple tiles in Group II (shown in green), so there are a total of 10 positions of pineapple tiles.

Note: In order to make it easier to follow the resulting patterns, the different positions of the tiles from Groups I and II are shown in different colors.

In the following examples, we will see how these tiles can be arranged in different types of tiling: from periodic, non-periodic, rotational, radial to free form.

3. PERIODIC TILING WITH DIAMONDS, STARS AND PINEAPPLES

Periodic tiling implies such an arrangement of tiles (prototiles), or regions (fundamental region, composed of several prototiles), where they are transformed by translation. If we can establish a relation of translation to the entire tiling, then we have a periodic sequence of tiles. K-uniform tilings are actually all periodic tilings.

In Fig. 3 we see pineapple tiles of Group II-1 forming a periodic tiling (Fig. 3 a). We can break them to diamonds and stars (Fig. 3 b), and then to triangles and rectangles (Fig. 3 c). It is clear that in an identical way we can form a periodic tiling using any of the remaining 9 positions of pineapple tiles, just as we can convert a given solution into any of them by a simple rotation for $n \cdot 36^o$, where $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

By combining tiles of Group I and Group II, we can also obtain periodic tiling. Tile I-1 and II-1 adhere to each other without overlapping or gaps, so that they can be arranged in rows, as shown in Fig. 4 a. Also, using the feature that the pineapple tiles of the same numbers from Groups I and II adhere perfectly onto each other, we can form different zigzag repetitive tilings, as shown in Fig. 4b and c, where we have an alternating series of tiles I-2 and I-3 with II-2 and II-3. Depending on the number of tiles in a row $(2, 3, 4, \dots n)$ and the multiplicity of rows of tiles from the same group, we can get an infinite number of different solutions based on this same scheme.

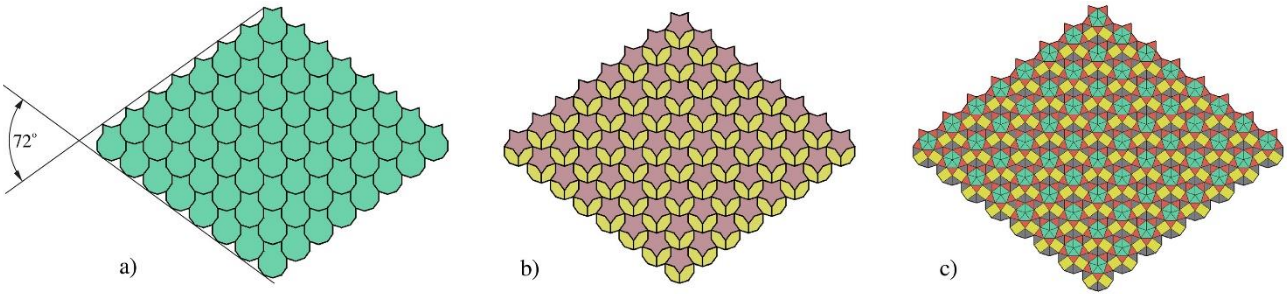


Figure 3: Periodic tilings with: a) pineapples, b) stars and diamonds, c) triangles and rectangles

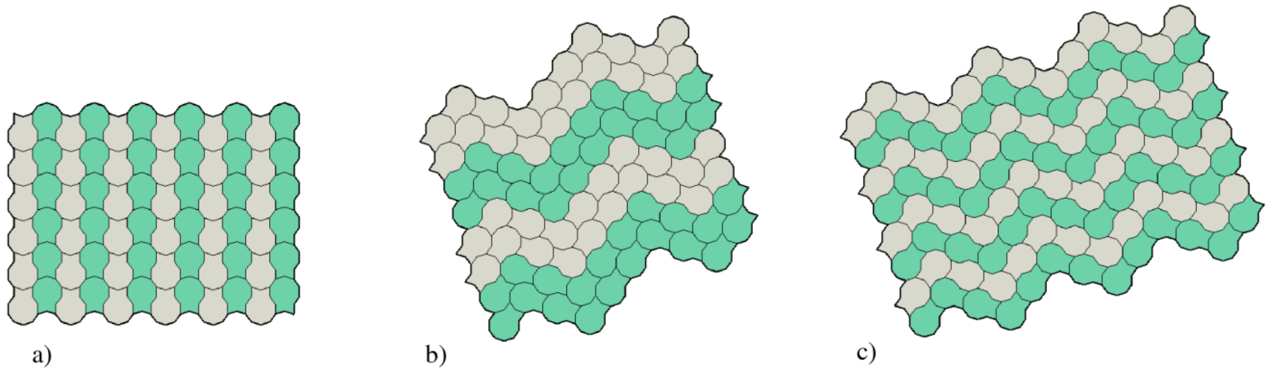


Figure 4: Periodic tilings with combinations of the tiles from Groups I and II arranged in rows

In Fig. 5 we see the tiling obtained using an aggregation of pineapple tiles of Group II, as in Fig. 2 d. In addition to this shape, pineapple tiles: II-4, I-4 and I-2 are used, together with star tiles both of I and II position and diamond tiles, in order to "clog" the gaps between these aggregations. Such tiling is named "garden" because the shapes used are reminiscent of vegetable shapes: leaf, flower, pineapple, lettuce.

This surely does not exhaust all the possibilities of creating periodic tilings with these shapes of tiles. Also, combinations of different "patches" of periodic arrangements are possible, i.e. with locally organized tiles in periodically tiled regions that complement each other. The layout of these regions and the way the tiles are arranged within them depend on the designer’s intention, creativity and the geometric needs of the tiling surface itself, so they can have a decorative and even artistic role along with the geometric one.

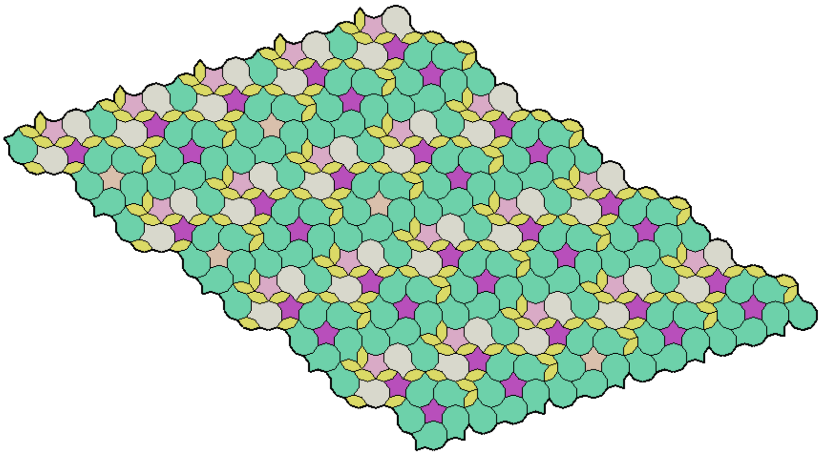


Figure 5: The “garden” tiling composed of tiles and the aggregations of tiles from both Groups I and II

Fig. 6 shows two examples of crossing two or more periodic pineapple tillings into one, supplemented by diamond and star tiles.

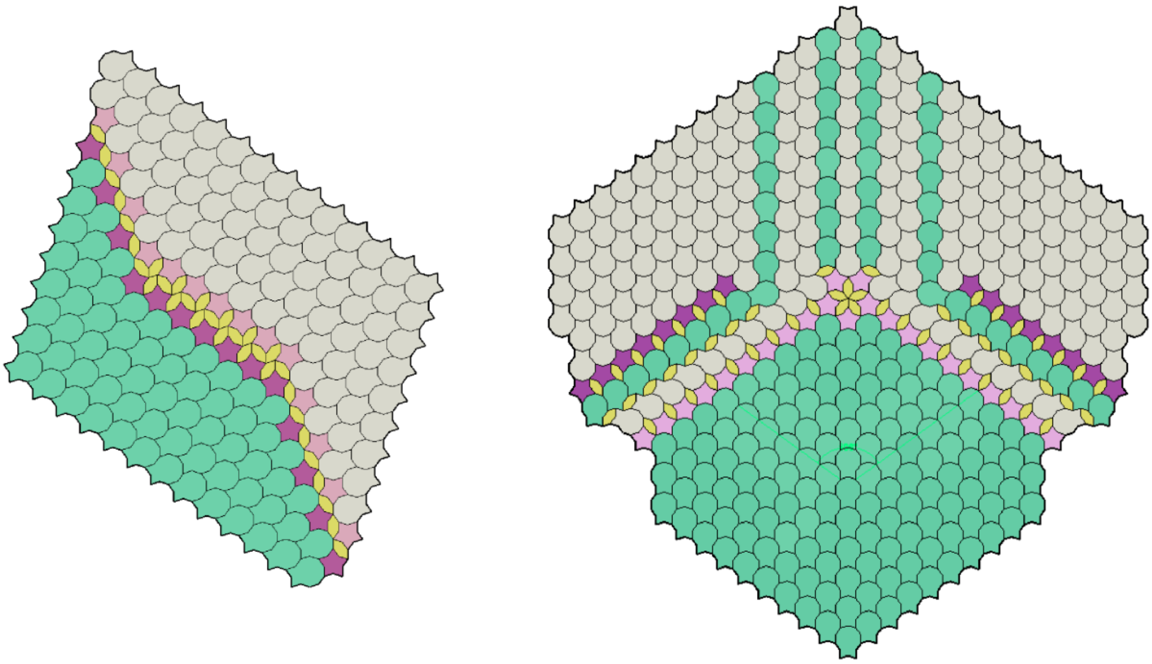


Figure 6: Tiles obtained by combining regions periodically tiled by the pineapple tiles from Groups I and II

4. NON-PERIODIC TILING WITH DIAMONDS, STARS AND PINEAPPLES

Non-periodic tilings, as their name suggests, cannot occur by applying (only) translation. The presence of local translation is possible in these arrangements of tiles, but in general, they occur by using other transformations or procedures. For example, they can be created by dividing the tiles of a periodic tiling, so that we get a non-repeating pattern. More demanding to create, especially for n-fold symmetries where seamless fitting of adjacent regions is required, are tilings obtained by rotation (perhaps most common for polygonal, and thus pentagonal bases). Such tilings can be radial, displaced radial (Shawcross, 2012) or aperiodic. Aperiodic tiling excludes translation and we do not find it even in local regions. The problem of aperiodic tiling is one of the most challenging in the whole subject of tiling the Euclidean plane and is relatively recent (solved in 20th century by R. Berger, R. M. Robinson and R. Penrose). The most famous among them is Penrose tiling, based on the geometry of a regular pentagon, as well as the cases considered in this paper. However, unlike Penrose tilling where rotational symmetry can be established locally in multitude of different centers, in the given solutions there is one center only.

Here, we present a several examples of tilings with rotational symmetry, created by rotating planar regions within a sector of 72 degrees, using a polar array of 5 such elements within a range of 2π. The differences between them are mainly reflected in the organization of tiles within a given region.

The tiles can be arranged by applying translation alone, and by using a single tile (pineapple), with the exception of one, the star, in the center of rotation, as shown on Fig. 7. This is exactly the tiling that emerges as a projection of the spatial "covering" (Fig. 7 d), presented and explained in Obradović et al. (2021). Such an arrangement of tiles within a pentagonal base that can be spread to infinity. It is actually a radial tiling, because all tiles are arranged in rows that radiate from the center of rotation onwards to infinity.

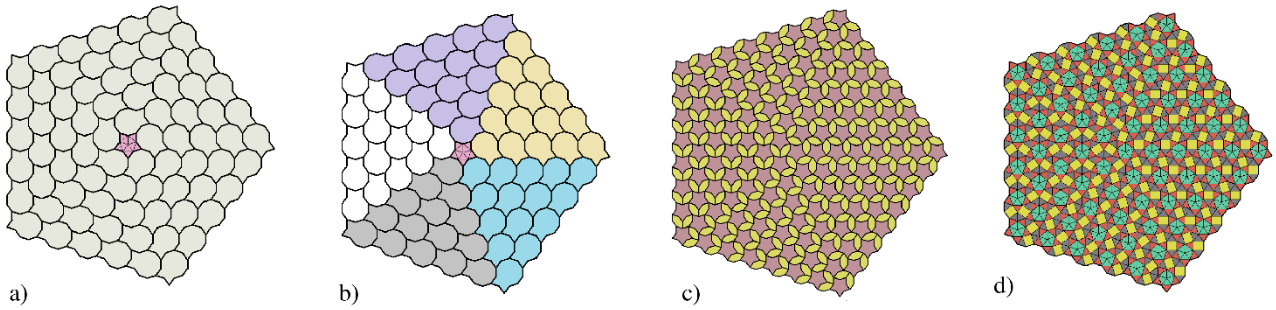


Figure 7: Radial, pentagon based tiling with a) pineapples, b) pineapples in periodic regions, c) the tiling reduced to diamonds and stars, d) the same tiling reduced to triangles and rectangles

4.1. Radial and rotational tilings

Radial tiling can be obtained in many different ways, and one of them is shown in Fig. 8 a, where clusters of diamond tiled fill the zones between rows of pineapple ones.

If we place the rows of tiles so that their main directions do not cut across the center of rotation but past it, we encounter the so-called "displaced" radial tiling (Shawcross, 2012), but it is still a rotational tilling with a periodic and predictable arrangement of tiles, which can be spread to infinity with an evident pattern of further sequence.

In some of the examples (Fig. 8 a, c) we see that diamond tiles play the role of independent factors that form entire local regions without the presence of other tiles. They can also form tiling independently, as shown in Obradović et al. (2021) (see Fig. 6 a). Also, we notice that in such solutions, we start from the "core" in the center of rotation, defined by a rotationally symmetrical arrangement of tiles. In the examples given in Fig. 8 a-c, these solutions corresponds to the group of tiles shown in Fig. 2b, Group I, while in the solutions in Fig. 8 e-f in the "core" we find the groups of tiles shown in Figure 2 d (or their variation, in Fig. 8 d).

Further sequence of tiles, starting from the given "core", can be quite diverse, almost arbitrary, but certainly with respect to the rules of tile fitting. What we have to keep in mind is the fit of two adjacent regions rotated by 72° (in the polar array of 5 items in the 2π range), which reflects on the definite layout of the tiling.

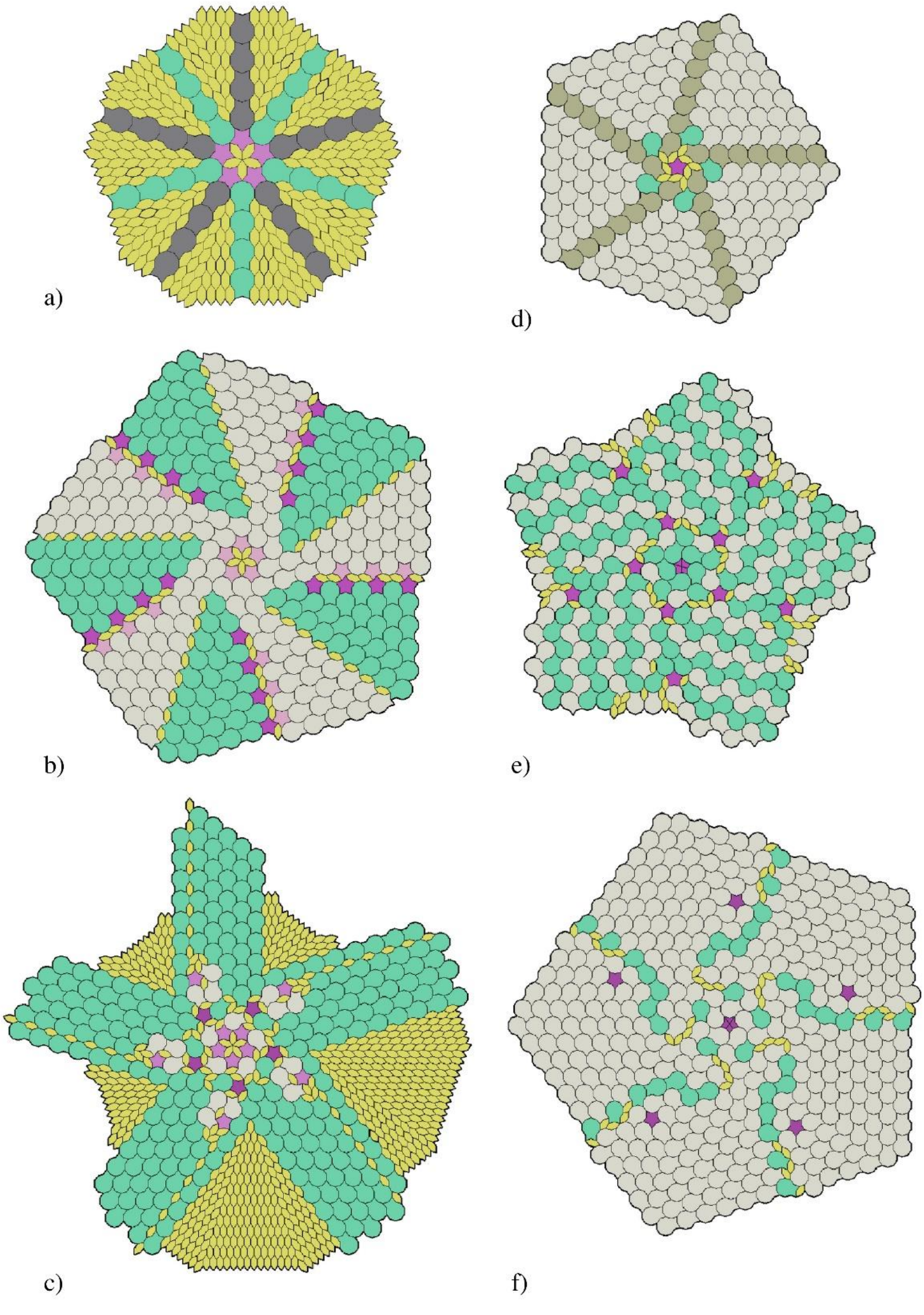
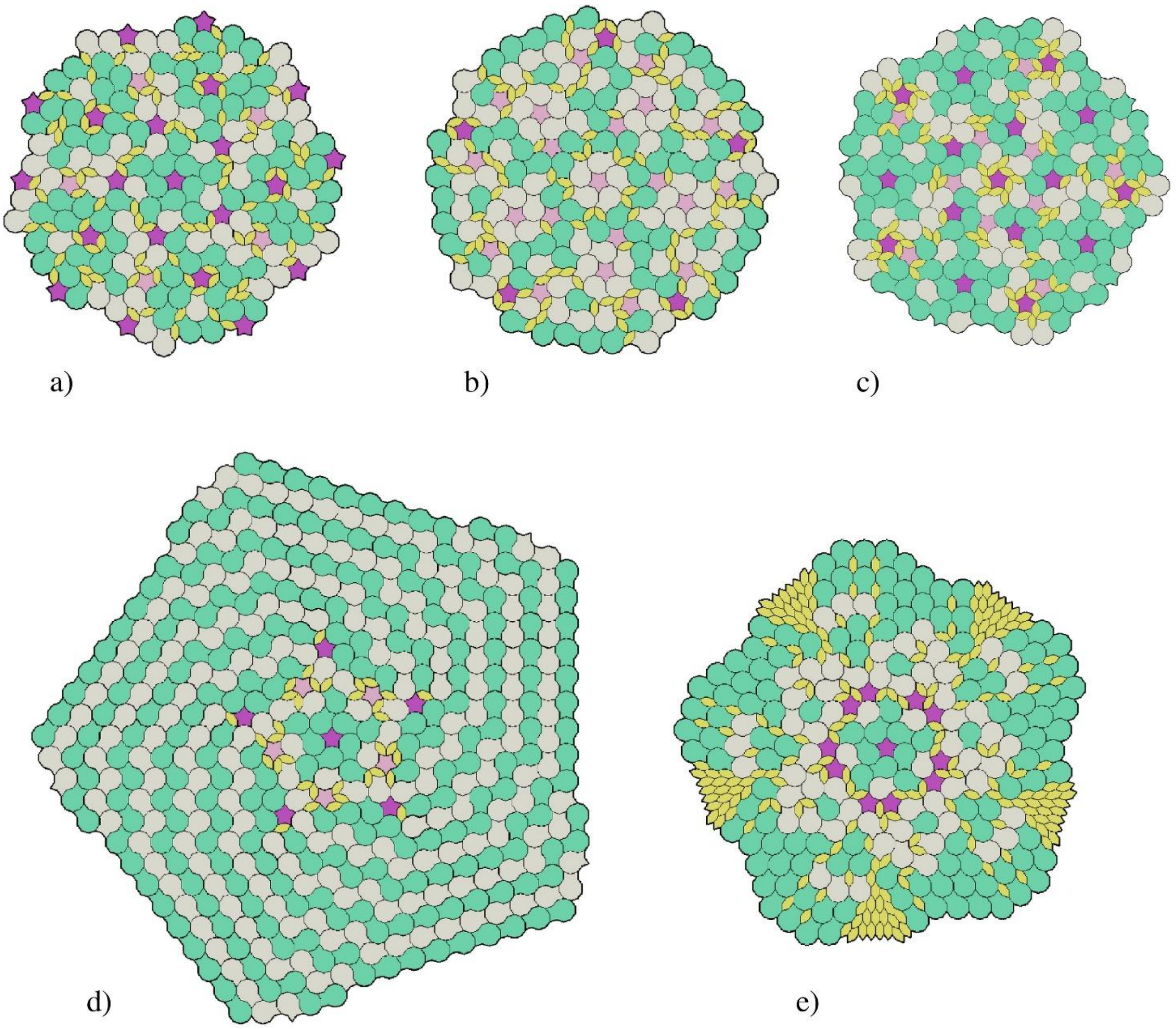


Figure 8: Some examples of radial tilings

If the tiles do not respect the stringing within the selected region, i.e. a radial arrangement in relation to the center of rotation, but form distinct patterns with possible new centers of rotation, as in Fig. 9 a, b, c, and e, it is a simple rotational tiling. Although these solutions may be seemingly similar, they actually differ not only in the selection of tiles from different Groups I and II, but also in their disposition. It is important to note that even a slight change in the position of a single tile, leads to a completely different arrangement of the subsequent tiles. Thus, we can create a multitude of different solutions.

The solution in Fig. 9 d could almost be treated as (displaced) radial or even spiral, but still, due to the irregular change of the tiles' directions from the different groups (Groups I and II) and their formation by rotation, it is placed among the rotational tilings.



a) b) c) d) e)

Figure 9: Some examples of rotational tilings

4.2 Tilings with single (central) decagon

The solutions given in Fig. 10 represent a certain anomaly among the solutions formerly shown, because another, fourth shape of tile appears here: a decagon. It is present as a single tile in the center of the tiling, but despite of disrupting the established set of tiles, an arrangement obtained by its introduction gives a fairly orderly disposition of tiles, moreover, with a lesser number of different tile shapes.

In the examples shown in Fig. 10 a, c and d, we see that, in addition to the central decagon, only diamonds and pineapples appear, and the stars appear in the case given in Fig. 10 b, only in the first ring around the decagon. Such solutions often give spiral arrangements of tiles, or concentric, as in Fig. 10 d. In this example we can see that, as in the case given in Fig. 4, the number of rows in which the tiles of Group I and Group II alternate can be taken arbitrarily.

In all these examples of rotational tilings, we can notice that the regularities are more noticeable the closer to the center of rotation the observed region is. As the distance increases, the number of tiles within the unit region becomes larger and their positions may be more diverse. Hence, their arrangement can produce conflicts with the adjacent unit region after rotation, but also within the region itself. These conflicts can be resolved by more suitable arrangement of the given three shapes of tiles. Yet, the further we move away from the center, the more reflection is needed to resolve the fitting of the tiles. The author takes the liberty of saying that solving these tillings can be just as much fun and challenging as solving enigmatics or computer games (similar to Tetris). This means that the solutions shown are valid only in the zone close to the center of rotation, but their further expansion can have several different solutions.

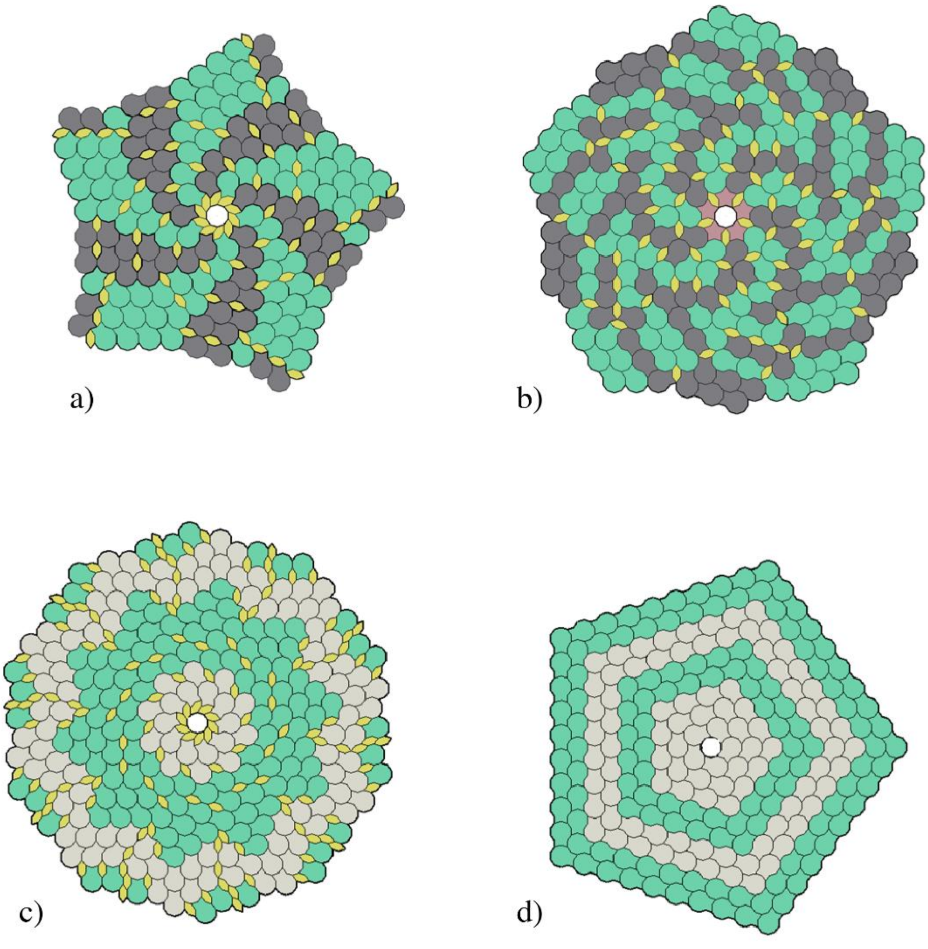


Figure 10: Examples of rotational tilings with a central decagonal tile

4.3 Combined and free-form solutions

The tiles can be combined and arranged in much complex ways. We can create tillings using previously formed aggregations of tiles, e.g. the one given in Fig. 2 d (Group II). Such a new shape, reminiscent of a flowering plant is named "lettuce", as in one of the previous solutions (see Fig. 5). We start with the given aggregations of pineapple tiles of Group II and array them in two parallel rows, starting from the position of the same shape in the center of the tilling (Fig. 11). Filling the gaps between them is performed with new pineapple and diamond tiles. After the rotation by 72°, we get zones that are not covered with tiles. To facilitate the “patching” of these empty zones, we also use parts of the "lettuce" aggregation, made of 3 pineapple tiles arranged around the central star. As we can notice, only the tiles from Group II are present in this solution.

For a clearer picture, they are shown in different colors: the pineapple tiles that make up the "lettuce" formation are colored green, and the tiles that fill the gaps are colored red. The diamonds that can only have their five positions, regardless of group, are given in yellow. In this way we get a solution that can be reduced to only two shapes: diamonds and stars, but also to another two: triangles and rectangles, as elementary shapes. The solution itself also belongs to rotational tilings.

As mentioned above, with these tiles we can play and create all kinds of different patterns, including free-form, and even generative. One of such free-form solutions is given in Fig. 12. Here we can see that, for such solutions, also some ready-made aggregations of tiles can be used. For easier tracking of solutions and eventually for achieving decorative patterns, we can display the tiles and aggregations by using different colors. Laying the tiles further on, in order to tile the entire plane, now is much simpler, since it is not conditioned by the fitting with the adjacent region after rotation.

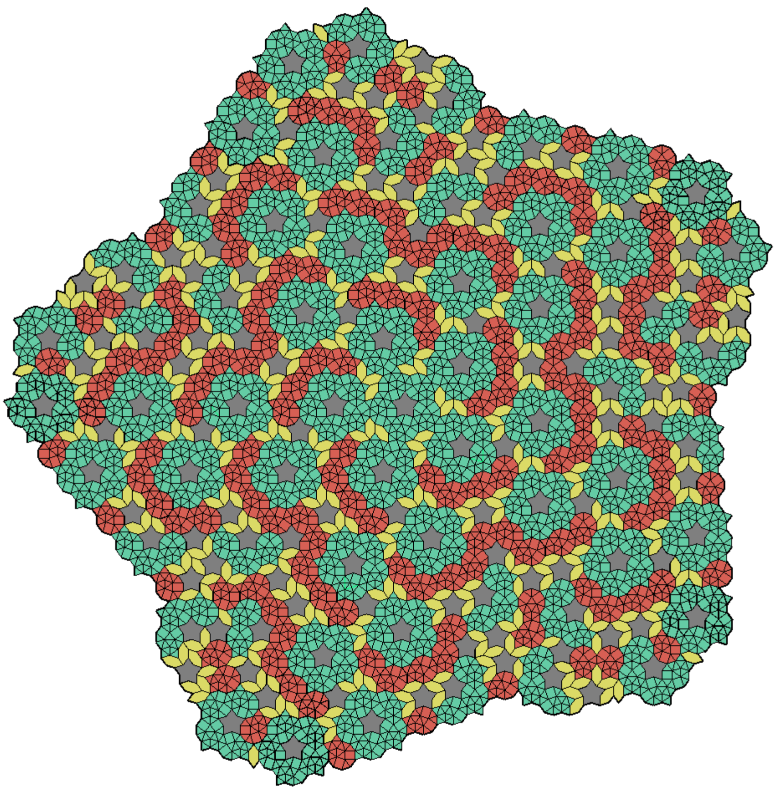


Figure 11: Tiling obtained by using “lettuce” aggregations of Group II tiles

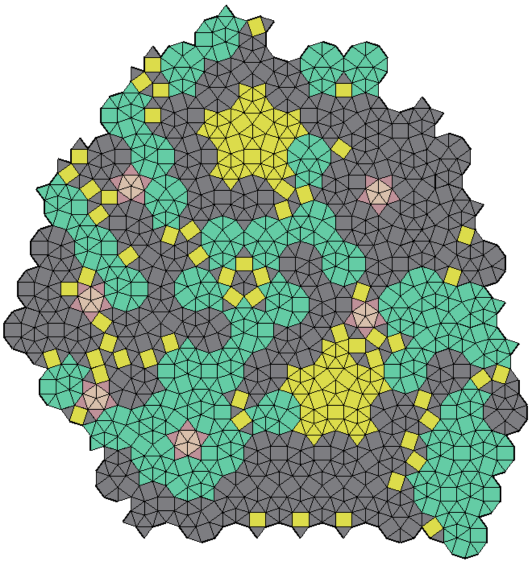


Figure 12: Free-form tiling using pineapples, stars and diamonds, broken into triangles and rectangles

Finally, we give some of the presented solutions, reduced to the shapes of triangles and rectangles. As we can see in Fig. 13, the arrangement of these elementary shapes results in patterns that are quite reminiscent of the Turing patterns. We can encounter them in marine organisms such as corals, sea urchins (which also have 5-fold symmetry), or even some species of fish (Ellison, 2019). In this way, we can try to decipher and link the formation of these bionic patterns with geometric ones.

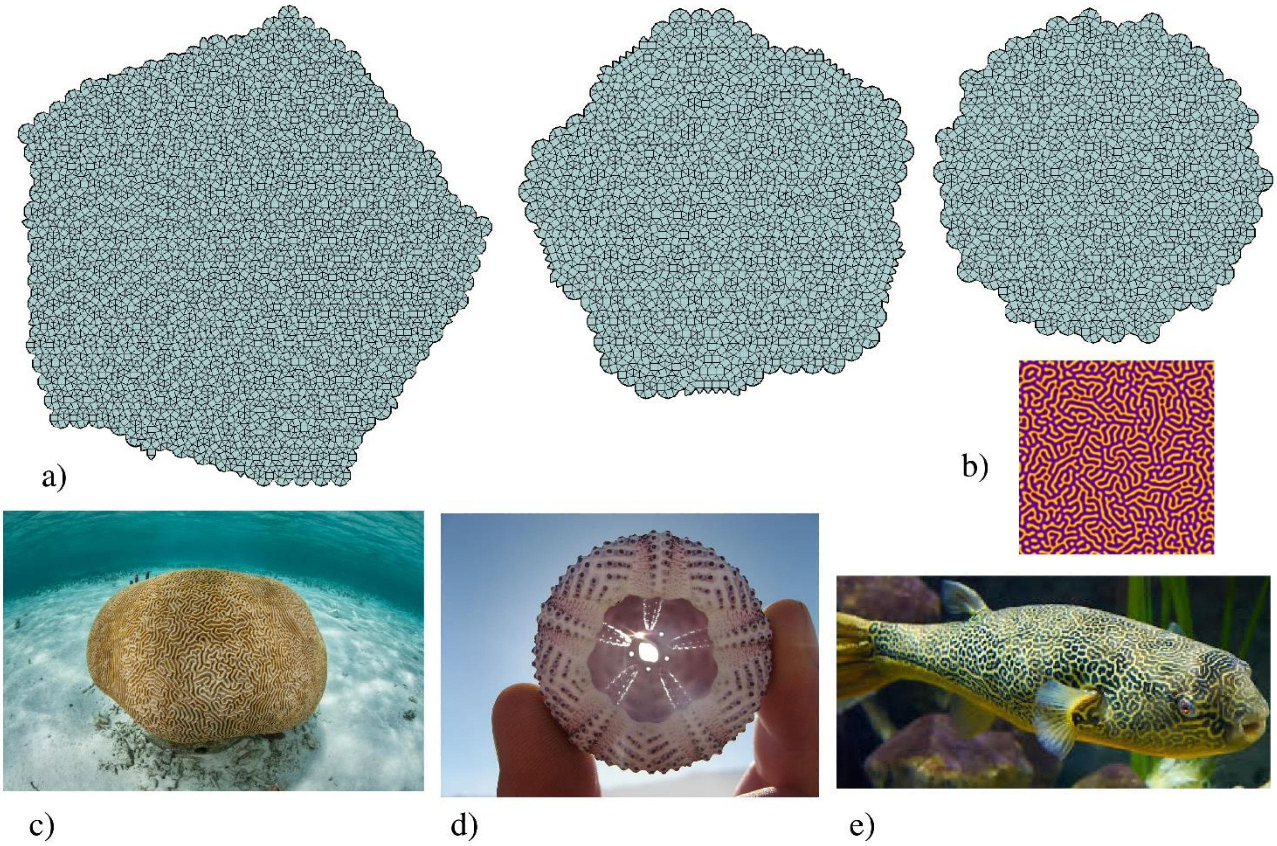


Figure 13: a) Patterns that emerge by reducing some of the above solutions into triangles and rectangles and their resemblance to b) Turing patterns (Source: see reference 10), and patterns from nature: c) “brain coral” Diploria labyrinthiformis (Source: see reference 12), d) sea urchin shell (Source: see reference 13) and e) Puffer fish, Tetraodon mbu (Source: see reference 11)

5. CONCLUSIONS

Based on all the above, the following conclusions can be drawn:

1. “Diamond” and “star” shapes, formed of isosceles triangles and rectangles originating from the ratio of the side and the radius of the circumscribed circle around the regular pentagon, can tile the whole Euclidean plane without overlapping and gaps.
2. These shapes, as well as the third, "pineapple" shape formed by combining them (two diamonds and one star), can tile the plane in several different ways, including periodic and non-periodic tiling.
3. Among the non-periodic tilings, a special variety of patterns can be obtained by applying radial and rotational arrangement of tiles.
4. Applying the rule of substitution, we can get larger aggregations of tiles and also use them in tiling the plane. We can always reduce the obtained solutions to more elementary forms.
5. The obtained solutions, in addition to their geometric curiosity, can also have a decorative role in design.
6. By reducing the presented solutions to triangles and rectangles, we obtain patterns similar to those we find in nature, especially in those where 5-fold symmetry appears.

From all the presented we conclude that, although it is probable that there are infinitely many different solutions, the way of forming them can still be examined and researched in terms of understanding the rules of their formation and thus a better knowledge of the world around us.

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