8th International Congress of Serbian Society of Mechanics
Kragujevac, Serbia, June 28-30, 2021

SIMPLE ALGORITHM FOR COMPUTING THE STIFFNESS MATRIX OF COMPOSITE CROSS-SECTION

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Abstract:

The paper proposes an algorithm for computing the stiffness matrix of composite cross-sections made of two or more different materials. Irrespective of the cross-section geometry, the calculation puts forward a numerical solution applying the finite element method (FEM). The cross-section is discretised into fibers and integration over the section area is performed numerically. The required geometric characteristics and stiffness matrix of the entire cross-section are formed by the assembling procedure. Beam theory equations are used for calculating stresses and strains at characteristic points of the cross-section. Materially nonlinear behaviour of concrete and steel is considered. The advantage of the proposed numerical algorithm is that results of satisfying accuracy are obtained in only a few iterations. The algorithm is implemented in own computer program with a straightforward input data procedure.

Keywords: composite cross-section, stress and strain analysis, numerical simulations

1. Introduction

The computation of the stiffness matrix of composite cross-sections of two or more materials is complex by nature [1, 2]. It should be performed numerically, introducing the material nonlinearity as a function of load intensity [3]. The paper proposes the procedure for computing the stiffness matrix and calculating stresses and strains in composite cross-sections. The cross-section is discretised into fibres, depending on the required accuracy [4]. At the cross-section level, the governing equilibrium equations according to the Bernoulli beam theory are written.

The analysed cross-section consists of a hollow cylinder steel profile of outer diameter D and wall thickness t filled with concrete. We take the nonlinear behavior of concrete and structural steel materials into account proposed in EC2 [5] and EC3 [6], respectively. The stiffness, stress, and strain result from the numerical simulations. To verify the results, we compared stiffness with the values proposed by EC4 [7] according to first-order theory and second-order theory for different D / t ratios. The deviations vary in the range of about 10%, which confirms a satisfying match of the results.

2. Determination of stresses and strains at the composite cross-section

Figure 1 shows a composite cross-section of arbitrary shape consisting of two different materials (M_1 and M_2). The origin of coordinate system y_s and z_s is in the centroid of the cross-section. External loading consists of axial force and bending moments around two axes.

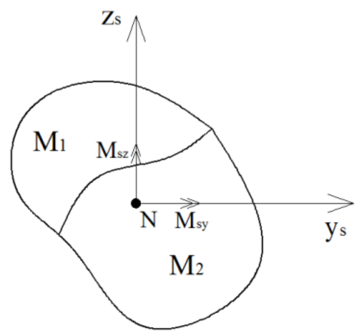

Fig. 1. Composite cross-section of arbitrary shape

Figure 2 shows the discretisation of the cross-section into finite elements [8, 9, 10, 11]. The discretisation is performed separately for material M_1 into *n* number of elements and M_2 into *m* number of elements. Those elementary cross-sectional areas of materials M_1 and M_2 are noted A_{1i} and A_{2j}, respectively.

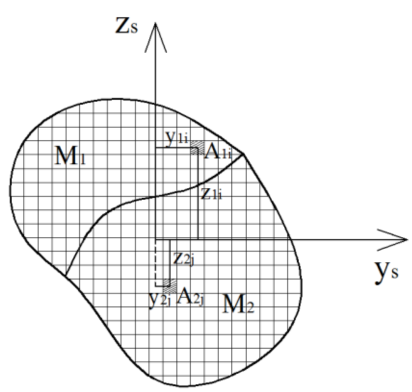

Fig. 2. The discretisation of the composite cross-section into finite fibre elements

If we analyze the composite rod segment of unit length with the cross-section given in Figure 2, then each segment has an axial stiffness equal to:

$$K_{1i} = A_{1i} \cdot E_1 \tag{1}$$

$$K_{2j} = A_{2j} \cdot E_2 \tag{2}$$

where E_1 and E_2 are the modulus of elasticity of materials M_1 and M_2, respectively.

To propose an efficient solution, certain simplifications and approximations of the actual behavior are introduced. We assume that a plane cross-section remains plane after deformation (Bernoulli's hypothesis). Figure 3 shows one segment of a composite beam divided into finite elements and connected to a rigid plate. The plate has an external load (N_s, M_{sy}, and M_{sz}). Therefore, calculating the stresses and strain in the composite cross-section reduces to calculating the displacements of a rigid plate that rests on a finite element system. In the static sense, these fibre elements in cross-section behave as truss elements, taking the load and deflections only in the directions of their longitudinal axis.

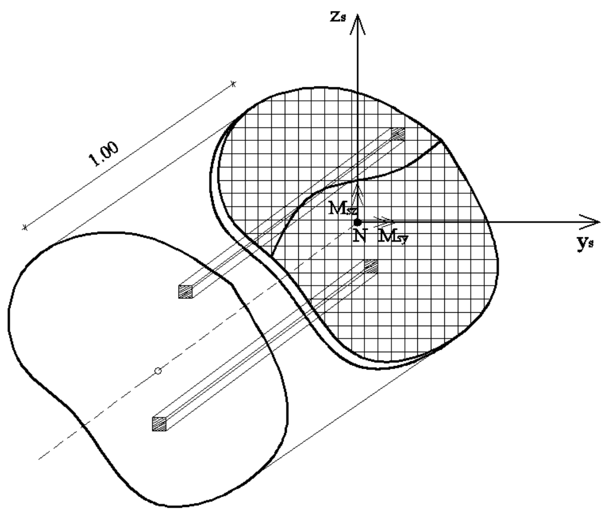

Fig. 3. A segment of a composite rod connected to a rigid plate

Figure 4 shows the rigid plate supported on a system of material elements M_1 and M_2 of axial stiffnesses K_{1i} and K_{2j}. All finite elements are defined in the coordinate system (y_0, O, z_0). The pole *O* has three degrees of freedom: one deflection u_0 perpendicular to the cross-section plane, and two rotations $\varnothing_{0y}$ about y_0 axis and $\varnothing_{0z}$ about z_0 axis. The deflection of each finite element centroid can be expressed using the displacement vector q_0 of pole *O*.

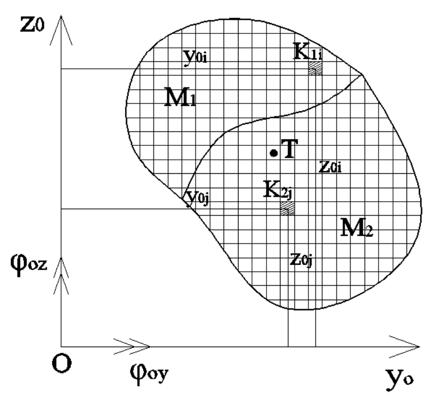

Fig. 4. Rigid plate supported by a system of material elements M_1 and M_2

The deflection $u(y_i, z_i)$ of the fibre element centroid in the longitudinal axis direction can be derivated as the function of its coordinates and the displacements of the pole *O* as follows:

$$u(y_i, z_i) = u_0 - \phi_{0y} \cdot z_i + \phi_{0z} \cdot y_i \quad (3)$$

Eq. (3) can be written in matrix form as:

$$u(y_i, z_i) = [T_{1i}] \cdot \begin{bmatrix} u_0 \\ \phi_{0y} \\ \phi_{0z} \end{bmatrix} \quad (4)$$

Where T_{1i} is the coordinate transformation matrix given as:

$$T_{1i} = [1 \quad -z_i \quad y_i] \quad (5)$$

As per finite element theory, the element stiffness matrix is obtained by the standard transformation of the coordinates:

$$[K_{1i}^*] = [T_{1i}^T] \cdot K_{1i} \cdot [T_{1i}] = K_{1i} \cdot \begin{bmatrix} 1 & -z_i & y_i \\ -z_i & z_i^2 & -y_i \cdot z_i \\ y_i & -y_i \cdot z_i & y_i^2 \end{bmatrix} \quad (6)$$

The global stiffness matrix for the material M_1 is obtained by the assembly procedure of all element stiffness matrices:

$$[K_1^*] = \sum_{i=1}^{n} [K_{1i}^*] \quad (7)$$

A similar procedure is repeated for the material M_2, etc.:

$$[K_2^*] = \sum_{j=1}^{m} [K_{2j}^*] \quad (8)$$

The global stiffness matrix of the whole composite cross-section K_0^* is computed by summing all stiffness matrices of materials in the cross-section (1, 2, ... s) in the following form:

$$[K_0^*] = [K_1^*] + [K_2^*] + \ldots + [K_s^*] \quad (9)$$

Equation 10 gives the standard equilibrium equation for the element.

$$K_0^* \cdot q_0 = Q_0 \quad (10)$$

The equivalent nodal force vector Q_0 of the node *O* consists of external load.

Further, the problem reduces to solving a simple system of three equations with three unknowns u_0, $\varnothing_{0y}$, $\varnothing_{0z}$. As defined in the standard displacement-based finite element method, these represent the displacement vector of pole *O*.

Once the displacement vector of pole *O* is obtained, using Eq. (3), it is possible to calculate the displacement vectors for each fibre element for all the materials within the cross-section.

Since the element segment has unit length, the calculated displacements are equal to the strain values in the centroids of all elements. Multiplying these strains with the modulus of elasticity of the material, we obtain stress values.

$$\sigma_i(y_i, z_i) = E_1 \cdot \varepsilon_i(y_i, z_i) \quad (11)$$

$$\sigma_j(y_j, z_j) = E_2 \cdot \varepsilon_j(y_j, z_j) \tag{12}$$

Finally, determining the stress and strain at the points of the composite cross-section is formally solved. If we assume that the displacement in the longitudinal axis direction equals u=0, the equation (3) equals the neutral axis of the cross-section:

$$0 = u_0 - \phi_{0y} \cdot z + \phi_{0z} \cdot y \tag{13}$$

2.1 Composite cross-sections subjected to bending moment about one axis

Often, composite cross-sections are subjected to bending moment about one axis only. Consequently, the discretisation of the cross-section is performed by dividing the cross-section into layers, as shown in Figure 5.

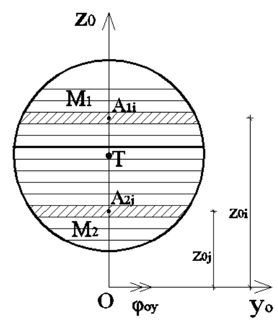

Fig. 5. The discretisation of composite cross-section subjected to bending moment about one axis

All elements of the stiffness matrix are calculated about an arbitrarily selected axis *O*, i.e. the origin of the coordinate system. Besides, one of the coordinate axes must be the symmetry axis of the cross-section.

$$\left[K_{1i}^*\right] = \left[T_{1i}^T\right] \cdot K_{1i} \cdot \left[T_{1i}\right] = K_{1i} \cdot \begin{bmatrix} 1 & -z_i \\ -z_i & z_i^2 \end{bmatrix} \tag{14}$$

Next, we assembly the transformed element stiffness matrices for materials M_1 and M_2 using equations (7) and (8).
As stated above, the conditional equations, ie. equilibrium equations of the whole system can be written in matrix form by the equation:

$$(K_1^* + K_2^*) \cdot u_0 = Q_0 \tag{15}$$

Expression (15) becomes the system of two equations with two unknowns: deflection of the point O in the longitudinal axis direction and rotation around axis O. After obtaining the unknown displacement vector of point O in Eq. 13, it is possible to determine the displacement vector of centroid in each layer. Since the rod segment itself has unit length, the displacements represent the strains for each layer. Finally, the equations (11) and (12) derivate stresses for each layer.
The proposed calculation is implemented in own computer algorithm with a straightforward input data procedure. Besides, it enables us to divide cross-section into as many layers and obtain the results with high accuracy.

3. Nonlinear behavior of materials

For linearly elastic materials, the algorithm derivates the stresses and strains in only one iteration. However, as for most building materials, linear behavior occurs only at lower load intensities. Design codes propose the uniaxial stress-strain relations for different materials: concrete, steel, wood, etc.

In our computer algorithm, we apply the stress-strain relations suggested by EC2 [5] and EC3 [6]. The idea is to apply the proposed numerical model even when the material does not behave ideally elastic. There are many incremental iterative procedures available for calculation using FEM. Here, we apply the full loading in one step and rich the equilibrium state after number of iterations.

In the first iteration, we assume that material behaves as linear elastic and we calculate unknown displacements and strains in all finite elements of the composite cross-section. Corresponding stress values result from adopted stress-strain relations. The ratio of calculated stresses and strains represents the value of the secant modulus of the material. We will use it in the next iteration for recalculating stiffness matrices, followed by calculating the displacements vector of the pole *O*. Finally, the displacements and the required strains in the cross-section are obtained.

The specified iterative procedure repeats until the deviations between two consecutive iterations satisfy the norm values. Usually, it takes up to ten iterations before achieving satisfactory calculation accuracy. Figure 6 schematically shows the flow of an iterative procedure for one of the materials with nonlinear behavior.

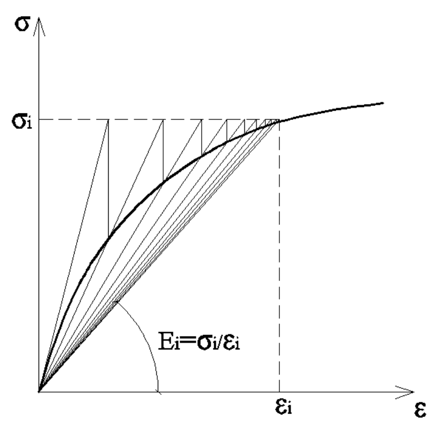

Fig. 6. Iteration procedure

4. Numerical simulations

To illustrate the presented numerical procedure, we use a composite cross-section consisting of a circular hollow steel profile filled with concrete. The required input data as shown in Figure 7 are f_{ck} - compressive strength of concrete, E_c - modulus of elasticity of concrete, f_y - structural steel yield strength, E_a - modulus of elasticity of structural steel.

Cross-section shape: circular cross section

Cross-section dimensions:	D/t=	101.6/2.7	[mm/mm]
Material characteristics:	Concrete: C25/30		
	fck=	25	MPa
	Ec=	31	GPa
	Steel: S355		
	fy=	355	MPa
	Ea=	210	GPa
Load:	N=	200,00	kN
	M=	2,00	kNm

Fig. 8. Algorithm input data

The cross-section consisting of concrete core and steel profile is divided into 20 layers, each with a centroid defined by Zc and Za coordinates. Figure 8 shows the strains and stresses at these points. Figure 9 shows the normal stresses diagram in the concrete core and steel profile.

CONCRETE CORE

n	Zc [m]	ε(z)	σc [MPa]
20	0,096495	0,000927	17,804205
19	0,091685	0,000894	17,349486
18	0,086875	0,000860	16,880836
17	0,082065	0,000827	16,398256
16	0,077255	0,000793	15,901746
15	0,072445	0,000760	15,391304
14	0,067635	0,000727	14,866933
13	0,062825	0,000693	14,328630
12	0,058015	0,000660	13,776397
11	0,053205	0,000627	13,210234
10	0,048395	0,000593	12,630140
9	0,043585	0,000560	12,036115
8	0,038775	0,000526	11,428160
7	0,033965	0,000493	10,806274
6	0,029155	0,000460	10,170458
5	0,024345	0,000426	9,520711
4	0,019535	0,000393	8,857033
3	0,014725	0,000359	8,179425
2	0,009915	0,000326	7,487886
1	0,005105	0,000293	6,782417

STEEL PROFILE

n	Za [m]	ε(z)	σa [MPa]
20	0,001350	0,000267	55,997836
19	0,003770	0,000283	59,525320
18	0,010794	0,000332	69,762476
17	0,021734	0,000408	85,707222
16	0,035519	0,000504	105,798773
15	0,050800	0,000610	128,070429
14	0,066081	0,000716	150,342085
13	0,079866	0,000812	170,433636
12	0,090806	0,000888	186,378381
11	0,097830	0,000936	196,615538
10	0,100250	0,000953	200,143022
9	0,097830	0,000936	196,615538
8	0,090806	0,000888	186,378381
7	0,079866	0,000812	170,433636
6	0,066081	0,000716	150,342085
5	0,050800	0,000610	128,070429
4	0,035519	0,000504	105,798773
3	0,021734	0,000408	85,707222
2	0,010794	0,000332	69,762476
1	0,003770	0,000283	59,525320

Fig. 8. Strain and stress results in the concrete core and steel profile

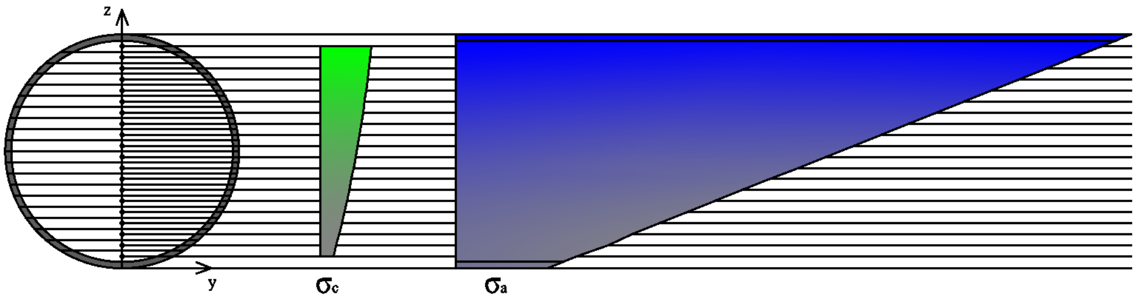

Fig. 9. Normal stresses diagram in the concrete core and steel profile

Calculated bending stiffness of the composite cross-section equals EI = 288.17 kNm^2. Figure 10 shows the stiffness EI – bending moment M relation.

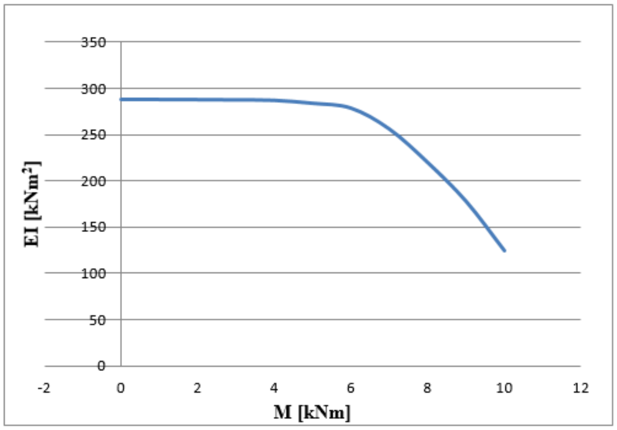

Fig. 10. EI-M relation diagram

We can conclude that at lower values of the bending moment, during linear-elastic behavior of the material, the stiffness of the composite cross-section has an approximately constant value. As the intensity of the bending moment increases, the stiffness of the composite cross-section decreases significantly.

With the increase of the load intensity, part of the composite cross-section will go through plastic deformation, as demonstrated in Figure 11. Also, we excluded part of the concrete core under tension in the calculation.

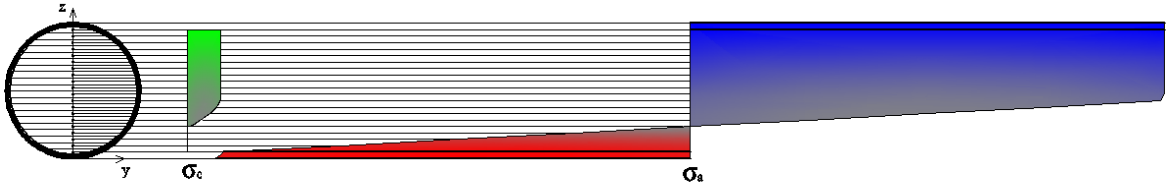

Fig. 11. Normal stresses diagram in the concrete core and steel profile after plastic deformations

Table 1 shows comparison of bending stiffness EI calculated by the proposed algorithm for different D/t ratios of composite cross-section with the bending stiffness proposed by EC4 according to first-order theory ($EI_{eff,EC4}$) and second-order theory ($EI_{eff,II,EC4}$) in EC4 [7]. The input data are: f_{ck} = 25 MPa, f_y = 355 MPa.

We notice that the bending stiffness EI for all D/t ratios is less than the $EI_{eff,EC4}$. Nevertheless, the bending stiffness EI for all D/t ratios is higher than $EI_{eff,II,EC4}$. These differences vary in the range of 1.3% to 11.4%.

D/t [mm/mm]	EI [kNm^2]	$EI_{eff,EC4}$ [kNm^2]	$EI_{eff,II,EC4}$ [kNm_2]	$EI/EI_{eff,EC4}$	$EI/EI_{eff,II,EC4}$
101.6/2.7	288.38	294.95	253.54	0.978	1.137
101.6/4.0	361.51	378.35	329.84	0.955	1.096
114.3/2.7	395.35	440.04	376.48	0.898.	1.050
114.3/4.0	515.53	561.60	487.69	0.918	1.057

Table 1. Bending stiffness of composite cross-sections for different D / t ratios

3. Conclusions

Presented numerical procedure and computer algorithm efficiently determine the bending stiffness of a composite cross-section of arbitrary shape for different load levels. Obtained stiffness corresponds to values proposed by EC4. In addition, it is possible to perform a detailed analysis of stresses and strains in a cross-section consisting of two or more different materials. The advantage of the proposed numerical algorithm is that results of satisfying accuracy are obtained in only few iterations which is convenient for structural design.

Acknowledgments

The first and the second authors thank the Ministry of Science of the Republic of Serbia for financial support under project number III42012.

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