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UNIVALENCY AND CONVOLUTION RESULTS ASSOCIATED WITH CONFLUENT HYPERGEOMETRIC FUNCTIONS

M. OBRADOVIĆ AND S. PONNUSAMY

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Abstract. Given the confluent hypergeometric functions $\Phi(a;c;z)$, we place conditions on a and c to guarantee that $z\Phi(a;c;z)$ will be in two subclasses of univalent functions. In addition, we obtain conditions to obtain some convolution results.

1. Introduction and Statement of Results

Given complex numbers a, b, and c with $c \neq 0, 1, 2, \ldots$, let

$$ {}_1F_1(a;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!} $$

denote the confluent (or Kummer) hypergeometric function. The series on the right defines an entire function for a, $c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$. In the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, this function is related to the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$ through the limit which exists uniformly on compact subsets of $\mathbb{D}$ (see [14]):

$$ {}_1F_1(a;c;z) = \lim_{|b|\to\infty} {}_2F_1(a,b;c;z/b) $$

where ${}_2F_1(a,b;c;z)$ is the analytic continuation to the slit plane $\mathbb{C}\backslash[1,\infty)$ of

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \quad z \in \mathbb{D}. $$

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For notational convenience, we use the shorter notation ${}_1F_1(a;c;z) = \Phi(a;c;z)$ in the sequel. It is well-known that the function $w(z) = \Phi(a;c;z)$ satisfies the following differential equation

$$zw''(z) + (c-z)w'(z) - aw(z) = 0 \tag{1.1}$$

and we have the following derivative formula

$$\Phi'(a;c;z) = \frac{a}{c}\Phi(a+1;c+1;z), \quad z \in \mathbb{C}. \tag{1.2}$$

Further if $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, we have the integral representation

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{tz}\, dt.$$

We make use of the standard notation $\mathcal{S}$ for the class of univalent functions f that are analytic in $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$. The starlike and the convex subclasses of $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. It is well-known that every $f \in \mathcal{S}^*$ is characterized by the inequality

$$\mathrm{Re}\,\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$. Next we recall the class of functions that has been studied in the recent years ([10, 11, 12]):

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| < \lambda \ \text{ for } z \in \mathbb{D} \right\},$$

where $\mathcal{A}$ denotes the class of all analytic functions f in $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. We have the strict inclusion $\mathcal{U}(1) := \mathcal{U} \subset \mathcal{S}$ (see Aksentév [1] and [2, p. 11]). Later, Krzyż [6] gave quasiconformal extensions of the extended complex plane, using the transformation $f(z) = 1/F(\zeta),\ \zeta = 1/z$ so that

$$F'(\zeta) = \left(\frac{z}{f(z)}\right)^2 f'(z).$$

At this juncture it is important to remark that functions in $\mathcal{U}$ need not be starlike (see [11]). Also functions in $\mathcal{S}^*$ need not be in $\mathcal{U}$ (see [4]). Extremal functions of many subclasses of $\mathcal{S}$ are in $\mathcal{U}$ (see [11]). For instance if

$$L = \left\{ z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2} \right\},$$

then each function in this collection is in $\mathcal{U} \cap \mathcal{S}^*$.

In this paper we shall establish condition so that $\Phi(a;c;z)$ satisfies the inequality

$$|\Phi'(a;c;z) - (a/c)| < |a/c|, \quad z \in \mathbb{D}$$

(and thus, $\Phi(a;c;z)$ is univalent in $\mathbb{D}$). We also obtain condition under which $f \in \mathcal{S}$ produces a class of functions $H_f(z)$ to be in $\mathcal{U}$ or in $\mathcal{S}^*$. Finally, we also obtain a sufficient condition on the parameters a, c for the normalized function $z\Phi(a;c;z)$ to belong to $\mathcal{U}$.

To obtain our results we employ two different methods: the first one makes use of a result due to Miller and Mocanu [8] on differential subordination. The second method uses a convolution result and a result due to Fejér [3] about convex decreasing sequences. In the rest of this section, we state our results and present some of their consequences. In Section 2, we recall relevant lemmas and in addition, we state and prove some new lemmas. The proofs of the main results will be given in Section 3.

Theorem 1.1. *Let $\rho > 0$, $0 \neq a \in \mathbb{C}$ and $c \in \mathbb{C}$ be such that $\operatorname{Re} c > \rho(|a| + 1)$. Then*

$$|\Phi(a;c;\rho z) - 1| < \frac{|a|\rho}{\operatorname{Re} c - \rho(|a|+1)}, \quad z \in \mathbb{D}.$$

Choosing $\rho = 1$, replacing a and c respectively by $a+1$ and $c+1$, we have

Corollary 1.2. *Let $-1 \neq a \in \mathbb{C}$ and $c \in \mathbb{C}$ be such that $\operatorname{Re} c > |a+1|$. Then*

$$|\Phi(a+1;c+1;z) - 1| < \frac{|a+1|}{\operatorname{Re} c - |a+1|}, \quad z \in \mathbb{D}. \tag{1.3}$$

In view of the derivative formula (1.2), the inequality (1.3) is equivalent to

$$|(c/a)\Phi'(a;c;z) - 1| < \frac{|a+1|}{\operatorname{Re} c - |a+1|}, \quad z \in \mathbb{D} \tag{1.4}$$

so that, for $\operatorname{Re} c \geq 2|a+1|$, one has

$$|(c/a)\Phi'(a;c;z) - 1| < 1, \quad z \in \mathbb{D}$$

and hence, $\Phi(a;c;z)$ is univalent in $\mathbb{D}$. In [9] sufficient conditions on the real numbers a and c are established in order to prove that $\operatorname{Re}((c/a)\Phi'(a;c;z)) > 0$ for $z \in \mathbb{D}$. Note that $1+z$ maps $\mathbb{D}$ onto a convex domain whereas $z(1+z)$ is not even univalent in $|z| < r$ if $r > 1/2$. Thus, in view of the different normalization at the origin, the convexity (resp. the starlikeness or the univalency) conditions for $\Phi(a;c;z)$ and $z\Phi(a;c;z)$ will be different.

We recall that the Hadamard product $f\star g$ of two convergent power series $f(z):=\sum_{n=0}^{\infty}a_n(f)z^n$ and $g(z):=\sum_{n=0}^{\infty}a_n(g)z^n$ in $\mathbb{D}$ is the power series defined by

$$(f\star g)(z):=f(z)\star g(z)=\sum_{n=0}^{\infty}a_n(f)a_n(g)z^n.$$

It is clear that $f\star g$ is analytic in $\mathbb{D}$.

Theorem 1.3. *Let $h_b(z)=1+bz+\sum_{n=2}^{\infty}b_nz^n$ be an analytic function on $\mathbb{D}$, where*

$$b_n=\sqrt{\frac{(a)_{n-1}}{(c)_{n-1}(n-1)!}}\qquad (n\geq 2)$$

with $a>0$ and $c\geq 2a+1$. Then for each $f\in\mathcal{S}$ for which $\frac{z}{f(z)}\star h_b(z)\neq 0$ in $\mathbb{D}$, the function H_f defined by

$$H_f(z)=\frac{z}{(z/f(z))\star h_b(z)}$$

belongs to $\mathcal{U}$ (and thus to $\mathcal{S}$) whenever $a>0$ and $c\geq 2a+1$. Moreover, $H_f(z)$ even belongs $\mathcal{S}^*$ if $|f''(0)|<2$, $a>0$ with $c\geq a+(a+1)(2-|f''(0)|)^2/4$.

The most important part in Theorem 1.3 is to obtain condition under which one actually has $\frac{z}{f(z)}\star h_b(z)\neq 0$ in $\mathbb{D}$. Often this is difficult to establish. However there are cases where one can provide a proof for this part so that the assumption that $\frac{z}{f(z)}\star h_b(z)\neq 0$ in $\mathbb{D}$ may be dropped from the hypothesis. For instance, we have

Corollary 1.4. *Let $f\in\mathcal{U}$ with $f''(0)=0$, and $h_b(z)$ be defined as in Theorem 1.3. Then the function H_f defined by*

$$H_f(z)=\frac{z}{(z/f(z))\star h_b(z)}$$

belongs to $\mathcal{U}$ (and thus to $\mathcal{S}$) whenever $a>0$, $b\in(1/2,1]$, and c satisfies the condition

$$\max\left\{2a+1,\frac{a}{b^2}\left[1+\sqrt{1-\sqrt{\frac{(a+1)b^2}{2(a+b^2)}}}\right]^2\right\}\leq c\leq\frac{a}{(2b-1)^2}. \tag{1.5}$$

In the limiting case $b\to(1/2)^+$, we obtain the following example:

Example 1.5. Let $f(z)$, $h_b(z)$ and $H_f(z)$ be as in Corollary 1.4 with $b = 1/2$. Then H_f belongs to $\mathcal{U}$ whenever a and c are related by a single sided inequality

$$c \geq \max\left\{2a+1, 4a\left[1+\sqrt{1-\sqrt{\frac{(a+1)}{2(1+4a)}}}\right]^2\right\}.$$

Remark 1.6. From the proof of Corollary 1.4, it is clear that the conclusion of Corollary 1.4 continues to hold if the hypothesis "$f \in \mathcal{U}$ with $f''(0) = 0$" is replaced simply by "$\operatorname{Re}(f(z)/z) > 1/2$ in $\mathbb{D}$". For instance, every convex function $f \in \mathcal{C}$ satisfies the later condition. Also, every normalized analytic function f such that $|f'(z) - 1| < 1$ in $\mathbb{D}$ satisfies the condition $\operatorname{Re}(f(z)/z) > 1/2$ in $\mathbb{D}$ although such functions need not even be starlike in $\mathbb{D}$. $\square$

There are number of papers dealing with univalency, starlikeness and convexity of the confluent hypergeometric function $z\Phi(a;c;z)$ (see [13, 9]). Our next result deals with a sufficient condition for $z\Phi(a;c;z)$ to belong to $\mathcal{U}$.

Theorem 1.7. *Assume that a and c are complex numbers such that* $\operatorname{Re} c > 2|a| + 1$. *If, in addition, a and c satisfy the condition*

$$(|c| - 2|a| - 1)(\operatorname{Re} c - 5 - |a|) - (|c-2| + |1-a|)|a| \geq 0, \tag{1.6}$$

then the function $z\Phi(a;c;z)$ belongs to the class $\mathcal{U}$.

In case a and c are real, this theorem takes the following simple form.

Corollary 1.8. *For $a > 0$ and*

$$c \geq C_a = \begin{cases} 2a+3+\sqrt{a^2+4} & \text{if } 0 < a \leq 1 \\ 2a+3+\sqrt{3a^2-2a+4} & \text{if } a \geq 1, \end{cases}$$

the function $z\Phi(a;c;z)$ belongs to $\mathcal{U}$ and thus univalent in $\mathbb{D}$.

Proof. The condition on c clearly implies that $c > 2a+1$. Further, when a and c are real, the condition (1.6) reduces to the inequality

$$c^2 - (4a+6)c + 2a^2 + 13a - a|1-a| + 5 \geq 0.$$

Solving this gives the desired condition $c \geq C_a$. The conclusion now follows from Theorem 1.7. $\square$

We remark that even a convex function need not belong to the class $\mathcal{U}$ and hence, Theorem 1.7 has its own merit. However, as an application of the last corollary, we set $a = 1$ and $c = 1 + \delta$ and obtain the following.

Example 1.9. For $\delta \geq 4+\sqrt{5}$, the function

$$g_\delta(z) = z\Phi(1;\delta+1;z) = \delta z \int_0^1 (1-t)^{\delta-1} e^{tz}\, dt$$

is in $\mathcal{U}$.

To state our final result and a lemma, we need the notion of subordination. Suppose that f and F are two analytic functions in $\mathbb{D}$, and F is univalent in $\mathbb{D}$. We say that f is *subordinate* to F, written $f(z) \prec F(z)$ or $f \prec F$, if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

Theorem 1.10. *Let $b \in (1/2, 1]$, $a, c > 0$ such that*

$$\max\left\{2a+1, \frac{a}{b^2}\left[1+\sqrt{1-\sqrt{\frac{(a+1)b^2}{2(a+b^2)}}}\right]^2, \left(1+\sqrt{1-\frac{1}{\sqrt{2}}}\right)^2 a\right\} \leq c \leq \frac{a}{(2b-1)^2}.$$

Then we have

$$z(b-1+\Phi(a;c;z)) \prec \frac{2z}{1-z}, \quad z \in \mathbb{D}.$$

As an example, we consider the limiting case $b \to (1/2)^+$ and obtain the following: If $a > 0$ and c satisfies the condition

$$c \geq \max\left\{2a+1, 4a\left[1+\sqrt{1-\sqrt{\frac{(a+1)}{2(1+4a)}}}\right]^2, \left(1+\sqrt{1-\frac{1}{\sqrt{2}}}\right)^2 a\right\},$$

then we have

$$2z\Phi(a;c;z) - z \prec \frac{4z}{1-z}, \quad z \in \mathbb{D}.$$

We conjecture that the constant 4 could be replaced by a smaller number.

2. Lemmas

Our results rely on a number of lemmas.

Lemma 2.1. [7, 8] *Let $\Omega \subset \mathbb{C}$ and let q be analytic and univalent on $\overline{\mathbb{D}}$ except for those $\zeta \in \partial\mathbb{D}$ for which $\lim_{z\to\zeta} q(z) = \infty$. Suppose that $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition*

$$\psi(q(\zeta), m\zeta q'(\zeta), \zeta^2 q''(\zeta); z) \notin \Omega \tag{2.1}$$

when $q(z)$ is finite, $m \geq n \geq 1$ and $|\zeta| = 1$. If p and q are analytic in $\mathbb{D}$, $p(z) = p(0) + p_n z^n + \cdots$, $p(0) = q(0)$, and further if

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$$

then $p(z) \prec q(z)$ *in* $\mathbb{D}$.

Suppose that $p(z)$ is analytic in $\mathbb{D}$ with $p(z) = p_n z^n + p_{n+1} z^{n+1} + \cdots$, and $q(z) = Mz$. Then the condition (2.1) reduces to a simple form

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega$$

whenever $K \geq nM$, $\mathrm{Re}\,\{Le^{-i\theta}\} \geq (n-1)K$, $z \in \mathbb{D}$ and $\theta \in \mathbb{R}$. In the proof of Theorem 1.1, we consider the situation $n = 1$ whereas in the proof of Theorem 1.7, we deal with the case $n = 2$. We remark that when ψ does not involve with $z^2 p''(z)$, then the condition on L may be dropped in the investigation.

Lemma 2.2. *Let* $\phi(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$ *be a non-vanishing analytic function on* $\mathbb{D}$ *and let* f *be of the form*

$$f(z) = \frac{z}{\phi(z)} = \frac{z}{1 + \sum_{n=1}^{\infty} d_n z^n}. \tag{2.2}$$

Then, we have the following:

(1) *If* $\sum_{n=2}^{\infty} (n-1)|d_n| \leq \lambda$, *then* $f \in \mathcal{U}(\lambda)$.
(2) *If* $\sum_{n=2}^{\infty} (n-1)|d_n| \leq 1 - |d_1|$, *then* $f \in \mathcal{S}^*$.

The first part of Lemma 2.2 is from [10, 12] whereas the second part is obtained from [15, Theorem 1]. Also, it would be appropriate to present an example to demonstrate Lemma 2.2. In [11], the present authors used the lemma to generate a class of functions in $\mathcal{U}(\lambda)$.

Example 2.3. Let α be a complex number, $a > -2$ and

$$d_n = \frac{\alpha}{(n-1)(n+a)(n+a+1)}.$$

Define $f_b(z)$ by

$$\frac{z}{f_b(z)} = 1 + bz + \sum_{n=2}^{\infty} d_n z^n.$$

Then it is easy to see that if $|\alpha| \leq (1 - |b|)(2 + a)$ then $\mathrm{Re}\,(z/f_b(z)) > 0$ in $\mathbb{D}$ so that $z/f_b(z) \neq 0$ in $\mathbb{D}$. Moreover,

$$\sum_{n=2}^{\infty} (n-1)|d_n| = |\alpha| \sum_{n=2}^{\infty} \frac{1}{(n+a)(n+a+1)} = \frac{|\alpha|}{2+a}.$$

By Lemma 2.2, it follows that f_b belongs to $\mathcal{U}(1 - |b|)$ whenever

$$|\alpha| \leq (1 - |b|)(2 + a).$$

In particular, f_0 belongs $\mathcal{U} \cup \mathcal{S}^*$ whenever $|\alpha| \leq 2 + a$.

The basic tool for our applications is the following widely used lemma.

Lemma 2.4. *If p is analytic in $\mathbb{D}$, $p(0) = 1$, and $\operatorname{Re} p(z) > 1/2$ in $\mathbb{D}$ then for any function F, analytic in $\mathbb{D}$, the function $p \star F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

The assertion of Lemma 2.4 readily follows by using Herglotz' representation theorem for analytic functions with positive real part (see also [17]). An important subclass of $\mathcal{A}$ is described in the following classical result of Fejér [3] which we state as a lemma.

Lemma 2.5. [3, 16] *Assume $B_0 = 1$, and $B_n \geq 0$ for $n \geq 1$, such that $\{B_n\}$ is a convex decreasing sequence, i.e., $0 \geq B_{n+1} - B_n \geq B_n - B_{n-1}$ for all $n \geq 1$. Then $\operatorname{Re} \{\sum_{n=0}^{\infty} B_n z^n\} > 1/2$ for all $z \in \mathbb{D}$.*

Using this lemma, we prove the following result.

Lemma 2.6. *Let $h_b(z) = 1 + bz + \sum_{n=2}^{\infty} b_n z^n$ be an analytic function on $\mathbb{D}$, where $b \in (1/2, 1]$ and b_n for $n \geq 2$ is defined as in Theorem 1.3. If $a > 0$ and c are related by the condition (1.5) then $\operatorname{Re} h_b(z) > 1/2$ in $\mathbb{D}$.*

Proof. Observe that $b_0 = 1$, $b_1 = b$, and $b_n > 0$ for all $n \geq 2$. By Lemma 2.5, it suffices to show that $\{b_n\}$ is a convex decreasing sequence; that is, we need to show that

(i) $b_n \geq b_{n+1}$ for all $n \geq 1$
(ii) $b_0 - b \geq b - b_2 \geq 0$
(iii) $b_3 - 2b_2 + b_1 \geq 0$
(iv) $b_{n+1} - 2b_n + b_{n-1} \geq 0$ for $n \geq 3$.

The condition (ii) holds if and only if $2b - 1 \leq b_2 \leq b$. This gives the inequality

$$a/b^2 \leq c \leq a/(2b-1)^2$$

which holds by (1.5). The condition (i) is equivalent to

$$\frac{(a)_{n-1}}{(c)_{n-1}(n-1)!} \geq \frac{(a)_n}{(c)_n n!}, \quad \text{i.e.} \quad n \geq \frac{a+n-1}{c+n-1} \quad \text{for } n \geq 1$$

which is clearly true for any $a > 0$ and $c \geq a$. Thus, we see that (i) and (ii) hold under the hypothesis, namely the condition (1.5). The condition (iii) is equivalent to

$$\sqrt{\frac{a(a+1)}{2c(c+1)}} - 2\sqrt{\frac{a}{c}} + b \geq 0. \tag{2.3}$$

Because $c \geq a/b^2$, it follows easily that

$$\frac{a+1}{c+1} \geq \alpha \frac{a}{c}, \quad \alpha = \frac{a+1}{b^2+a}.$$

In view of this simple observation, the inequality (2.3) holds whenever

$$\sqrt{\frac{a}{c}}\sqrt{\frac{\alpha}{2}} + b\sqrt{\frac{c}{a}} \geq 2.$$

Setting $x = \sqrt{c/a}$, and then solving the resulting equation one obtains

$$c \geq \frac{a}{b^2}\left[1+\sqrt{1-\sqrt{\frac{(a+1)b^2}{2(a+b^2)}}}\right]^2$$

which holds by the hypothesis.

Finally, it remains to verify the condition (iv). A simplification shows that the inequality (iv) is equivalent to

$$\sqrt{\frac{(a+n-1)(a+n)}{(c+n-1)(c+n)n(n+1)}} - 2\sqrt{\frac{a+n-1}{(c+n-1)n}} + 1 \geq 0 \quad \text{for } n \geq 2$$

which is same as

$$\begin{aligned} &\sqrt{(c+n-1)(c+n)n(n+1)} - 2\sqrt{(a+n-1)(c+n)(n+1)} \\ &\qquad + \sqrt{(a+n)(a+n-1)} \geq 0 \quad \text{for } n \geq 2. \end{aligned} \tag{2.4}$$

For $n \geq 2$ (as the hypothesis implies that $c \geq 2a+1$) the inequality (2.4) trivially holds because

$$n(c+n-1) \geq n(2a+n) = (n-2)^2 + 2a(n-2) + 4(a+n-1) \geq 4(a+n-1)$$

is satisfied for all $n \geq 2$. The desired conclusion follows. □

Remark 2.7. Clearly, we could relax the second condition on c, namely $c \geq 2a+1$. A relaxed condition is not so important for the purpose of deriving Corollary 1.4 and so we do not pay attention to this. □

If we apply Lemma 2.5 with $B_0 = 1 = B_1$, then the $\{B_n\}$ is a convex decreasing sequence if and only if $B_n = 1$ for all n. This means that the corresponding function in the conclusion of the lemma turned out to be $z/(1-z)$. Thus, by relaxing the hypothesis, we obtain the following lemma which can be used to derive some other result.

Lemma 2.8. *Let $h(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be an analytic function on $\mathbb{D}$, where b_n is defined as in Theorem* 1.3. *If $a > 0$ and*

$$c \geq \max\left\{2a+1, \left(1+\sqrt{1-\frac{1}{\sqrt{2}}}\right)^2 a\right\} \tag{2.5}$$

then $\mathrm{Re}\, h(z) > 0$ *in* $\mathbb{D}$.

Proof. Observe that $b_1 = b_0 = 1$ and $b_n > 0$ for all n, and so $\mathrm{Re}\, h(z) > 0$ holds in $\mathbb{D}$ if and only if

$$\mathrm{Re}\left(1 + \sum_{n=1}^{\infty} \frac{b_n}{2} z^n\right) > \frac{1}{2} \quad \text{for all } z \in \mathbb{D}.$$

By Lemma 2.5, it suffices to show that $\{B_n\}$ ($B_0 = 1$, $B_n = b_n/2$) is a convex decreasing sequences; that is, we need to show that

(i) $b_n \geq b_{n+1}$ for all $n \geq 1$
(ii) $b_0 - \dfrac{b_1}{2} \geq \dfrac{b_1}{2} - \dfrac{b_2}{2} \geq 0$
(iii) $b_{n+1} - 2b_n + b_{n-1} \geq 0$ for $n \geq 2$.

The condition (ii) holds trivially, because $b_2 = \sqrt{a/c} \leq 1$. As in the proof of Lemma 2.6, the condition (i) clearly holds as the condition on c and a implies that $c > a$. Thus, it remains to verify the condition (iii). However, from the proof of Lemma 2.6, it suffices to verify the condition (iii) only for $n = 2$, as the same has already been verified for all $n \geq 3$ (because $c \geq 2a + 1$). Thus, it suffices to check this inequality only for the case $n = 2$. For $n = 2$, the inequality (iii) reduces to (2.3) with $b = 1$ and so we get the condition

$$c \geq \left(1+\sqrt{1-\frac{1}{\sqrt{2}}}\right)^2 a.$$

The desired conclusion follows. $\square$

3. Proofs of Theorems 1.1, 1.3, 1.7 and 1.10

3.1. Proof of Theorem 1.1 . Let $p(z) = \Phi(a; c; \rho z) - 1$, $\rho > 0$. Then $p(z)$ is analytic in $\mathbb{D}$ with $p(0) = 0$. Since the function $w(z) = \Phi(a; c; z)$ satisfies the differential equation (1.1) and $w(\rho z) = p(z) + 1$, it can be easily seen that $p(z)$ satisfies the second order differential equation

$$z^2 p''(z) + (c - \rho z) z p'(z) - a\rho z(p(z) + 1) = 0.$$

If we let $\psi(r,s,t;z) = t + (c - \rho z)s - a\rho z(r+1)$ and $\Omega = \{0\}$, then the last equation may be written as

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

We will use Lemma 2.1 to prove that $|p(z)| < M$, where

$$M = \rho|a|/(\mathrm{Re}\, c - \rho(|a|+1)).$$

For this, it suffices to show that $\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega$, whenever $K \geq M$, $\mathrm{Re}\,\{Le^{-i\theta}\} \geq 0$, $z \in \mathbb{D}$ and θ is real. We have that

$$\begin{aligned}
|\psi(Me^{i\theta}, Ke^{i\theta}, L; z)| &= |Le^{-i\theta} + (c - \rho z)K - a\rho z M - a\rho z e^{-i\theta}| \\
&\geq \mathrm{Re}\,\{Le^{-i\theta}\} + \mathrm{Re}\,(Kc - \rho(K + aM)z) - \rho \mathrm{Re}\,(aze^{-i\theta}) \\
&> \mathrm{Re}\,\{Le^{-i\theta}\} + K\mathrm{Re}\, c - \rho K(1 + |a|) - \rho|a| \\
&\geq K(\mathrm{Re}\, c - \rho(1 + |a|)) - \rho|a| \\
&\geq M(\mathrm{Re}\, c - \rho(1 + |a|)) - \rho|a| = 0,
\end{aligned}$$

because $-\mathrm{Re}\,((K + aM)z) > -|K + aM| \geq -K - |a|M \geq -K - |a|K$ and $\mathrm{Re}\,(aze^{-i\theta}) < |a|$. In the last stage of the chain of inequalities, we have used the definition of M. Thus, we have shown that

$$\psi(Me^{i\theta}, Ke^{i\theta}, L; z) \notin \Omega = \{0\},$$

whenever $K \geq M$, $\mathrm{Re}\,\{Le^{-i\theta}\} \geq 0$, $z \in \mathbb{D}$ and θ is real. ☐

3.2. Proof of Theorem 1.3 . Let $f \in \mathcal{S}$. Then we have $z/f(z) \neq 0$ in $\mathbb{D}$ and hence has the form

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} d_n z^n}, \quad z \in \mathbb{D}. \tag{3.3}$$

Next, we recall the well-known Area Theorem [5, Theorem 11 on p.193 of Vol. 2]

$$\sum_{n=2}^{\infty} (n-1)|d_n|^2 \leq 1. \tag{3.4}$$

Observe that $b_1 = b$, $b_n > 0$ for all $n \geq 2$. Also, we note that $f''(0)/2 = -d_1$ and

$$\frac{z}{H_f(z)} = 1 + \sum_{n=1}^{\infty} b_n d_n z^n.$$

Now, we apply Lemma 2.2(1) and for this we need to show that

$$\sum_{n=2}^{\infty} (n-1)|b_n d_n| \leq 1.$$

By Theorem 1.1, we have (because by hypothesis a and c are positive, and also, $c \geq 2a+1\,(> a+1)$)

$$1 < \Phi(a+1; c+1; 1) \leq 1 + \frac{a+1}{c-a-1} = \frac{c}{c-a-1} \tag{3.5}$$

and using this, we see that

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-1)|b_n|^2 &= \sum_{n=2}^{\infty} (n-1)\frac{(a)_{n-1}}{(c)_{n-1}(n-1)!} \\
&= \sum_{n=1}^{\infty} n\frac{(a)_n}{(c)_n n!} \\
&= \frac{a}{c}\sum_{n=1}^{\infty} \frac{(a+1)_{n-1}}{(c+1)_{n-1}(n-1)!} \\
&= \frac{a}{c}\Phi(a+1; c+1; 1) \\
&\leq \frac{a}{c-a-1}, \quad \text{by (3.5).}
\end{aligned}$$

In view of this and (3.4), it follows that

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-1)|b_n d_n| &= \sum_{n=2}^{\infty} \left(\sqrt{n-1}|b_n|\right)\left(\sqrt{n-1}|d_n|\right) \\
&\leq \left(\sum_{n=2}^{\infty} (n-1)|b_n|^2\right)^{1/2} \left(\sum_{n=2}^{\infty} (n-1)|d_n|^2\right)^{1/2} \\
&\leq \sqrt{\frac{a}{c-a-1}} =: \lambda.
\end{aligned}$$

Thus, $H_f \in \mathcal{U}(\lambda)$ by Lemma 2.2(1). Since $\lambda \leq 1$ whenever $a > 0$ and $c \geq 2a+1$, it follows that H_f belongs to $\mathcal{U}$ and hence, to $\mathcal{S}$. For the starlikeness condition, by Lemma 2.2(1), it is enough to observe that $\lambda \leq 1 - |d_1|$ holds by hypothesis. □

3.6. Proof of Corollary 1.4 . Let $f \in \mathcal{U}$ with $f''(0) = 0$. Then, it is well-known that (see [12]) $\mathrm{Re}\,(f(z)/z) > 1/2$ in $\mathbb{D}$ so that

$$\left|\frac{z}{f(z)} - 1\right| < 1 \quad \text{for } \mathbb{D}.$$

By Lemma 2.2 and the hypothesis, we obtain that $\mathrm{Re}\, h_b(z) > 1/2$ in $\mathbb{D}$. In view of this observation, Lemma 2.4 implies that

$$\left|\frac{z}{f(z)} \star h_b(z) - 1\right| < 1 \quad \text{for } \mathbb{D}$$

and so $(z/f(z)) \star h_b(z) \neq 0$ in $\mathbb{D}$. Finally, the desired conclusion follows from Theorem 1.3. □

3.7. Proof of Theorem 1.7 . Set $\Phi(z) = \Phi(a;c;z)$. Then $\rho = 1$ in Theorem 1.1 shows that if $\mathrm{Re}\, c \geq 2|a| + 1$ then one has

$$|\Phi(z) - 1| < \frac{|a|}{\mathrm{Re}\, c - (|a| + 1)} \leq 1, \quad z \in \mathbb{D} \tag{3.8}$$

and therefore, $\mathrm{Re}\, \Phi(z) > 0$ in $\mathbb{D}$, by the hypothesis.

We want to prove that

$$\left|(z\Phi(z))' \left(\frac{z}{z\Phi(z)}\right)^2 - 1\right| < 1 \ \text{ for every } z \in \mathbb{D}.$$

For this, we begin by setting

$$(z\Phi(z))' \left(\frac{1}{\Phi(z)}\right)^2 - 1 = p(z).$$

Then $p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 0 = p'(0)$. Also,

$$\begin{aligned} z\Phi'(z) &= (p(z) + 1)\Phi^2(z) - \Phi(z) \\ z^2\Phi''(z) &= zp'(z)\Phi^2(z) + 2(p(z) + 1)^2\Phi^3(z) - 4(p(z) + 1)\Phi^2(z) + 2\Phi(z). \end{aligned}$$

Since the function Φ satisfies the differential equation (1.1), in terms of $p(z)$, we see that $p(z)$ satisfies the equation

$$\Psi(p(z), zp'(z); z) = 0$$

where

$$\Psi(r, s; z) = s + 2(r + 1)^2\Phi(z) + (c - 4 - z)(r + 1) + (2 - c + z - az)\frac{1}{\Phi(z)}.$$

We claim that $|p(z)| < 1$. In order to prove this, we apply Lemma 2.1 with $\Omega = \{0\}$, $n = 2$ and $q(z) = z$. Thus, by Lemma 2.1, it suffices to show that

$$\Psi(e^{i\theta}, Ke^{i\theta}; z) \notin \Omega,$$

whenever $K \geq 2$, $z \in \mathbb{D}$ and θ is real. It is a simple exercise to see that

$$\Psi(e^{i\theta}, Ke^{i\theta}; z) = e^{i\theta}A - B$$

where

$$\begin{aligned} A &= K + 8\cos^2(\theta/2)\Phi(z) + (c - 4 - z) \\ B &= 2 + az + ((c-2) - (1-a)z)\left(\frac{1}{\Phi(z)} - 1\right). \end{aligned}$$

Next, as $\operatorname{Re}\Phi(z) > 0$ in $\mathbb{D}$, we observe that

$$\begin{aligned} |A| \geq \operatorname{Re} A &= K + 8\cos^2(\theta/2)\operatorname{Re}\Phi(z) + (\operatorname{Re} c - 4 - \operatorname{Re} z) \\ &> 2 + (\operatorname{Re} c - 4 - 1) = \operatorname{Re} c - 3. \end{aligned}$$

Thus, if we can prove that $|B| < \operatorname{Re} c - 3$ under the hypotheses of the theorem, then we get $\Psi(e^{i\theta}, Ke^{i\theta}; z) \neq 0$, whenever $K \geq 2$, $z \in \mathbb{D}$ and θ is real, and so the proof will be completed, that is $|p(z)| < 1$ in $\mathbb{D}$. In order to prove that $|B| < \operatorname{Re} c - 3$, we need the following observation. It is easy to see that for $r < 1$, $|w - 1| < r$ holds if and only if

$$\left|\frac{1}{w} - \frac{1}{1-r^2}\right| < \frac{r}{1-r^2}.$$

In particular, $|w-1| < r$ implies that

$$\left|\frac{1}{w} - 1\right| < \frac{r}{1-r}.$$

With $r = |a|/(\operatorname{Re} c - (|a|+1))$, (3.8) takes the form $|\Phi(z) - 1| < r$ and so, it follows that

$$\left|\frac{1}{\Phi(z)} - 1\right| < \frac{r}{1-r} = \frac{|a|}{\operatorname{Re} c - (2|a|+1)}.$$

We observe that by the hypothesis $\operatorname{Re} c > 2|a| + 1$. Using the last inequality, we obtain that

$$\begin{aligned} |B| &\leq |2 + az| + |c - 2 + (1-a)z| \left|\frac{1}{\Phi(z)} - 1\right| \\ &< |a| + 2 + (|c-2| + |1-a|)\frac{|a|}{\operatorname{Re} c - (2|a|+1)} \end{aligned}$$

and so $|B| < \operatorname{Re} c - 3$ holds provided

$$|a| + 2 + (|c-2| + |1-a|)\frac{|a|}{\operatorname{Re} c - (2|a|+1)} \leq \operatorname{Re} c - 3.$$

This gives exactly the condition (1.6) stated in the theorem. Thus, $|p(z)| < 1$ in $\mathbb{D}$ and hence, $z\Phi(a; c; z)$ belongs to $\mathcal{U}$. □

3.9. Proof of Theorem 1.10 . Consider $h_b(z) = 1 + bz + \sum_{n=2}^{\infty} b_n z^n$ which is defined as in Theorem 1.3, where

$$b_n = \sqrt{\frac{(a)_{n-1}}{(c)_{n-1}(n-1)!}} \qquad (n \geq 2).$$

By the hypothesis, Lemma 2.6 gives $\operatorname{Re} h_b(z) > 1/2$ whereas Lemma 2.8 shows that $\operatorname{Re} h_1(z) > 0$ in $\mathbb{D}$. By a little manipulation we then get

$$(h_b \star h_1)(z) = 1 + (b-1)z + z\Phi(a;c;z).$$

As an application of Lemma 2.4, it follows that $\operatorname{Re}(h_b \star h_1)(z) > 0$ in $\mathbb{D}$. In terms of the subordination, the last fact may be reformulated as

$$1 + (b-1)z + z\Phi(a;c;z) \prec \frac{1+z}{1-z}, \quad z \in \mathbb{D}$$

and desired result follows. □

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M. Obradović, Department of Mathematics, Faculty of Civil Engineering, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia.

E-mail address: obrad@grf.bg.ac.yu

S. Ponnusamy, Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India.

E-mail address: samy@iitm.ac.in