M. OBRADOVIĆ AND S. PONNUSAMY
KODAI MATH. J.
34 (2011), 169–178

A CLASS OF UNIVALENT FUNCTIONS DEFINED BY A DIFFERENTIAL INEQUALITY

Milutin Obradović and Saminathan Ponnusamy*

Abstract

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathbf{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. For $\lambda > 0$, denote by $\mathcal{M}(\lambda)$ the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left|z^2\left(\frac{z}{f(z)}\right)'' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le \lambda, \quad z \in \mathbf{D}.$$

We show that functions in $\mathcal{M}(1)$ are univalent in $\mathbf{D}$ and we present one parameter family of functions in $\mathcal{M}(1)$ that are also starlike in $\mathbf{D}$. In addition to certain inclusion results, we also present characterization formula, necessary and sufficient coefficient conditions for functions in $\mathcal{M}(\lambda)$, and a radius property of $\mathcal{M}(1)$.

1. Introduction and main results

Let $\mathcal{H}$ be the class of analytic functions in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, mapping $\mathbf{D}$ into the complex plane $\mathbf{C}$ and $\mathcal{A}$ be the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ in $\mathcal{H}$. Let $\mathcal{S}$ denote the class of functions f in $\mathcal{A}$ such that f is univalent in $\mathbf{D}$. For $\lambda > 0$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{U}(\lambda)$ if

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le \lambda, \quad z \in \mathbf{D}.$$

Denote by $\mathcal{P}(\lambda)$, the subclass of $\mathcal{A}$, consisting of functions f for which

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le 2\lambda, \quad z \in \mathbf{D}.$$

2000 *Mathematics Subject Classification*. 30C45.

Key words and phrases. Coefficient inequality, analytic, Hadamard convolution, univalent and starlike functions.

*Corresponding author.

The work of the first author was supported by MNZZS Grant, No. ON174017, Serbia. The research of the second author was supported by National Board for Higher Mathematics, India.

Received March 5, 2010; revised October 20, 2010.

Set $\mathcal{U}(1) := \mathcal{U}$ and $\mathcal{P}(1) := \mathcal{P}$, see [4, 9]. We have the strict inclusion $\mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ (see [1, 4, 10] for a proof). Many properties of the classes $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ have been studied extensively in [5, 6, 7, 8, 9]. More generally

$$\mathcal{P}(\lambda) \subsetneq \mathcal{U}(\lambda) \subsetneq \mathcal{S} \quad \text{for } 0 < \lambda \le 1$$

and for a proof of this inclusion, we refer to [5]. Also, it is well-known that there are only nine functions in $\mathcal{S}$ having integral coefficients in the power series expansions of $f \in \mathcal{S}$ (see [3]). That is, if we set $\mathcal{S}_{\mathbf{Z}} = \{f \in \mathcal{S} : a_n \in \mathbf{Z}\}$, then

$$\mathcal{S}_{\mathbf{Z}} = \left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\}.$$

Further, it is easy to see that the corresponding $g \in \mathcal{S}_{\mathbf{Z}}$ have the property that

$$g'(z)\left(\frac{z}{g(z)}\right)^2 - 1 \in \{0, -z^2, 0, \mp z^2, -z^2\} \quad \text{and} \quad z^2\left(\frac{z}{g(z)}\right)'' \in \{0, 2z^2, 0, \mp 2z^2, 2z^2\},$$

respectively. Consequently, we obtain the interesting fact that each function in $\mathcal{S}_{\mathbf{Z}}$ belongs to $\mathcal{U} \cap \mathcal{P}$. Finally, we observe that

$$z^2\left(\frac{z}{g(z)}\right)'' + g'(z)\left(\frac{z}{g(z)}\right)^2 - 1 \in \{0, z^2, 0, \mp z^2, z^2\}.$$

In view of this observation, we introduce the following:

Definition 1. For $\lambda > 0$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{M}(\lambda)$ if $|M_f(z)| \le \lambda$ for $z \in \mathbf{D}$, where

$$M_f(z) = z^2\left(\frac{z}{f(z)}\right)'' + f'(z)\left(\frac{z}{f(z)}\right)^2 - 1. \tag{1}$$

Also, denote the class $\mathcal{M}(1)$ by $\mathcal{M}$.

Now, we state our main results and the proofs of these will be given in Section 3.

Theorem 1 (Inclusion property). *For $0 < \lambda \le 1$, we have the strict inclusion $\mathcal{M}(\lambda) \subsetneq \mathcal{U}(\lambda) \cap \mathcal{P}(\lambda) \subsetneq \mathcal{S}$. In particular, $\mathcal{M} \subsetneq \mathcal{U} \cap \mathcal{P} = \mathcal{P} \subsetneq \mathcal{S}$.*

From the earlier discussion and Theorem 1, we easily see that

$$\mathcal{S}_{\mathbf{Z}} \subsetneq \mathcal{M} \subsetneq \mathcal{P} \subsetneq \mathcal{U} \subsetneq \mathcal{S}$$

and it is worth recalling that the Koebe function belongs to $\mathcal{M}$.

Example 1. Consider the function f defined by

$$\frac{z}{f(z)} = 1 + \frac{1}{2}z + \frac{\lambda}{2}z^3$$

where $0 < \lambda \le 1$. Then

$$\left|\frac{z}{f(z)}\right| \ge 1 - \frac{1}{2}|z| - (\lambda/2)|z|^3 > \frac{1-\lambda}{2} \ge 0$$

and so $z/f(z) \neq 0$ in $\mathbf{D}$ whenever $0 < \lambda \le 1$. Further

$$f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 = -\lambda z^3 \quad \text{and} \quad M_f(z) = 2\lambda z^3$$

so that there exists a function $f \in \mathcal{U}(\lambda)$ such that $f \notin \mathcal{M}(\lambda)$. Also, for each μ with $|\mu| \le 1/2$, it is easy to see that the function f defined by

$$f(z) = \frac{z}{1 + \mu z + \frac{1}{2}z^3}$$

belongs $\mathcal{U} \backslash \mathcal{M}$.

Theorem 2 (Sufficiency coefficient condition). *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function in $\mathbf{D}$ that satisfy the coefficient condition*

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n| \le \lambda. \tag{2}$$

Then the function f defined by $f(z) = z/\phi(z)$ is in $\mathcal{M}(\lambda)$.

For example, according to (2) with $\lambda = 1$, each function in $\mathcal{S}_{\mathbf{Z}}$ belongs to $\mathcal{M}$.

Let $\mathcal{S}^*$ denote the class of univalent functions in $f \in \mathcal{S}$ such that the range $f(\mathbf{D})$ is a starlike domain (with respect to the origin). Analytically, $f \in \mathcal{S}^*$ if and only if $\mathrm{Re}(zf'(z)/f(z)) > 0$ in $\mathbf{D}$.

It is easy to see that each $g \in \mathcal{S}_{\mathbf{Z}}$ is starlike in $\mathbf{D}$. Also, it has been shown that for arbitrarily small values of λ we have $\mathcal{U}(\lambda) \not\subset \mathcal{S}^*$. Indeed, Fournier and Ponnusamy [2, Theorem 3] obtained that every function $f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n \in \mathcal{A}$ satisfying

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2}, \quad |z| < 1,$$

belongs to $\mathcal{S}^*$. Moreover, there exists a non-starlike function $f \in \mathcal{U}$ such that

$$0 < \frac{-|a_2(f)| + \sqrt{2 - |a_2(f)|^2}}{2} < \sup_{|z|<1}\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \le 1 - |a_2(f)|.$$

In particular, $\mathcal{U} \not\subset \mathcal{S}^*$. Moreover, Theorem 1 shows that $\mathcal{M} \subsetneq \mathcal{U}$ and therefore, it is natural to ask whether the class $\mathcal{M}$ is included in $\mathcal{S}^*$. This remains an open question.

If f and g are analytic functions on $\mathbf{D}$ with $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$, then the convolution (Hadamard product) of f and g, denoted by $f*g$, is an analytic function on $\mathbf{D}$ given by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n, \quad z\in\mathbf{D}.$$

Although $\mathscr{U}$ is not included in $\mathscr{M}$, in the following result, we show that the class $\mathscr{U}$ can be used to construct functions belonging to $\mathscr{M}$.

THEOREM 3 (Multiplier theorem). *Let $f\in\mathscr{U}(\lambda_1)$ and $g\in\mathscr{U}(\lambda_2)$ have the form*

$$\frac{z}{f(z)}=1+b_1z+b_2z^2+\cdots \quad \text{and} \quad \frac{z}{g(z)}=1+c_1z+c_2z^2+\cdots$$

and such that $\dfrac{z}{f(z)}*\dfrac{z}{g(z)}\neq 0$ on $\mathbf{D}$. Then the function H defined by

$$H(z)=\frac{z}{(z/f(z))*(z/g(z))}$$

is in the class $\mathscr{M}(\lambda)$, where $\lambda=\lambda_1\lambda_2$. In particular, if $f,g\in\mathscr{U}$ then $H\in\mathscr{M}$.

COROLLARY 1 (Necessary coefficient condition). *Let $f\in\mathscr{M}$ of the form*

$$\frac{z}{f(z)}=1+b_1z+b_2z^2+\cdots.$$

Then we have

$$\sum_{n=2}^{\infty}(n-1)^4|b_n|^2\leq 1.$$

At this place it is appropriate to remind the reader of the fact that the inequality $\sum_{n=2}^{\infty}(n-1)|b_n|^2\leq 1$ follows merely from the condition $f\in\mathscr{S}$ for the expansion $\dfrac{z}{f(z)}=1+b_1z+b_2z^2+\cdots$. This result is known as the Prawitz theorem which is indeed an immediate consequence of Gronwall's area theorem. Thus, the necessary condition given in Corollary 1 is much stronger than this result.

THEOREM 4 (Characterization theorem). *Every $f\in\mathscr{M}(\lambda)$ has the representation*

$$\frac{z}{f(z)}=1-\frac{f''(0)}{2}z+\lambda\int_0^1\frac{w(tz)}{t^2}\log(1/t)\,dt,$$

for some $w:\mathbf{D}\to\mathbf{D}$ with $w(0)=w'(0)=0$.

Let $\mathscr{F}$ and $\mathscr{G}$ be two subclasses of $\mathscr{A}$. If for every $f \in \mathscr{F}$, $r^{-1}f(rz) \in \mathscr{G}$ for $r \le r_0$, and r_0 is the maximum value for which this holds, then we say that r_0 is the $\mathscr{G}$-radius in $\mathscr{F}$. There are many results of this type in the theory of univalent functions, see [8] and the references therein.

Because $\mathscr{M} \subsetneq \mathscr{U}$, it is natural to investigate the $\mathscr{M}$-radius in $\mathscr{U}$.

THEOREM 5 (Radii property). *If* $f \in \mathscr{U}$ *and* $g(z) = \dfrac{1}{r}f(rz)$, *then* $g \in \mathscr{M}$ *for* $0 < r \le r_0$, *where* $r_0 \approx 0.62977$ *is the unique positive root of the equation* $2r^6 - 2r^4 + 3r^2 - 1 = 0$.

2. Preliminary lemmas

Let $\mathscr{P}_n$ denote the class of functions p in $\mathscr{H}$ such that $p^{(k)}(0) = 0$ for $k = 0, 1, 2, \dots, n$, where $p^{(0)}(0) = p(0)$. We set

$$\mathscr{B}_n = \{w \in \mathscr{H} : |w(z)| \le 1, w^{(k)}(0) = 0 \text{ for } k = 0, 1, \dots, n\}.$$

LEMMA 1. *Suppose that* $p \in \mathscr{P}_n$, $\lambda > 0$ *and* α *is a complex number such that* $\mathrm{Re}(1/(1-\alpha)) > -n$. *If* p *satisfies the condition*

$$|(1-\alpha)zp'(z) + \alpha p(z)| \le \lambda, \quad z \in \mathbf{D} \tag{3}$$

then

$$|p(z)| \le \frac{\lambda |z|^{n+1}}{|1-\alpha|(n + \mathrm{Re}(1/(1-\alpha)))}, \quad z \in \mathbf{D}.$$

Proof. First, we rewrite (3) as

$$(1-\alpha)zp'(z) + \alpha p(z) = \lambda w(z),$$

where $w \in \mathscr{B}_n$. Now, by integration, we get

$$p(z) = \frac{\lambda}{1-\alpha} \int_0^1 t^{(\alpha/(1-\alpha))-1} w(tz)\, dt.$$

Because $|w(z)| \le |z|^{n+1}$ for $z \in \mathbf{D}$ by Schwarz' lemma, we obtain that

$$|p(z)| \le \frac{\lambda}{|1-\alpha|} \left(\frac{|z|^{n+1}}{n + 1 + \mathrm{Re}(\alpha/(1-\alpha))} \right), \quad z \in \mathbf{D}$$

and the desired conclusion follows. □

COROLLARY 2. *Suppose that* $p \in \mathscr{P}_n$, $\lambda > 0$ *and* $\alpha \neq 1$ *is a real number such that* $n + 1/(1-\alpha) > 0$. *Then*

$$|(1-\alpha)zp'(z) + \alpha p(z)| \le \lambda \Rightarrow |p(z)| \le \frac{\lambda(1-\alpha)}{|1-\alpha|(n(1-\alpha)+1)} \tag{4}$$

for $z \in \mathbf{D}$.

Suppose that $0 \neq \alpha < 1$. Then (4) becomes

$$\left|\left(\frac{1}{\alpha} - 1\right) zp'(z) + p(z)\right| \leq \frac{\lambda}{|\alpha|} \Rightarrow |p(z)| \leq \frac{\lambda}{n(1-\alpha)+1)}, \quad z \in \mathbf{D}.$$

Now, if we allow $\alpha \to -\infty$, then the last relation gives that

$$|-zp'(z) + p(z)| \leq 0 \Rightarrow |p(z)| \leq 0, \quad z \in \mathbf{D}$$

so that $p(z) = 0$ is the only solution which satisfies the above implication. Now, we state an improved version of it.

Lemma 2. *Suppose that $p \in \mathscr{P}_n$ $(n \geq 1)$ satisfies the condition*

$$|-zp'(z) + p(z)| \leq \lambda, \quad z \in \mathbf{D} \tag{5}$$

for some $\lambda > 0$. Then we have

$$|p(z)| \leq \frac{\lambda |z|^{n+1}}{n} \quad \textit{and} \quad |zp'(z)| \leq \lambda |z|^{n+1}\left(1 + \frac{1}{n}\right), \quad z \in \mathbf{D}.$$

Proof. The condition (5) implies that

$$-zp'(z) + p(z) = \lambda w(z)$$

where $w \in \mathscr{B}_n$. It follows easily that

$$p(z) = -\lambda \int_0^1 t^{-2} w(tz)\, dt \quad \text{and} \quad -zp'(z) = \lambda w(z) + \lambda \int_0^1 t^{-2} w(tz)\, dt.$$

Because $|w(z)| \leq |z|^{n+1}$ for $z \in \mathbf{D}$, by Schwarz' lemma, the desired conclusion follows from the last two formulas. □

We state the above lemmas in a general form in order to apply them for functions with missing coefficients. However, for our application the case $n = 1$ suffices. Setting $n = 1$ in Lemma 2, we have

Corollary 3. *Suppose that p is analytic in $\mathbf{D}$, $p(0) = p'(0) = 0$ and satisfies the condition*

$$|-zp'(z) + p(z)| \leq \lambda, \quad z \in \mathbf{D} \tag{6}$$

for some $\lambda > 0$. Then we have

$$|p(z)| \leq \lambda |z|^2 \quad \textit{and} \quad |zp'(z)| \leq 2\lambda |z|^2, \quad z \in \mathbf{D}.$$

3. Proofs

Proof of Theorem 1. Set

$$p(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1. \tag{7}$$

Then p is analytic in $\mathbf{D}$, $p(0) = p'(0) = 0$,

$$-zp'(z) = z^2\left(\frac{z}{f(z)}\right)'' \quad \text{and} \quad -zp'(z) + p(z) = M_f(z), \tag{8}$$

where M_f is defined by (1). Now, suppose that $f \in \mathcal{M}(\lambda)$. Then, we obtain that

$$|-zp'(z) + p(z)| \le \lambda, \quad z \in \mathbf{D}.$$

By Corollary 3, it follows that

$$|p(z)| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| \le \lambda|z|^2 \quad \text{and} \quad |zp'(z)| = \left|z^2\left(\frac{z}{f(z)}\right)''\right| \le 2\lambda|z|^2, \quad z \in \mathbf{D}$$

and therefore, $f \in \mathcal{U}(\lambda) \cap \mathcal{P}(\lambda)$. $\square$

Proof of Theorem 2. Let f be given by $f(z) = z/\phi(z)$, where $\phi(z) \neq 0$ in $\mathbf{D}$. Then the power series representation of ϕ gives

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n.$$

By (7) and (8), it follows easily that

$$M_f(z) = \sum_{n=2}^{\infty}(n-1)^2 b_n z^n.$$

Thus, using the coefficient condition (2), we deduce that

$$|M_f(z)| \le \sum_{n=2}^{\infty}(n-1)^2|b_n|\,|z|^n \le \sum_{n=2}^{\infty}(n-1)^2|b_n| \le \lambda$$

and therefore, $f \in \mathcal{M}(\lambda)$. $\square$

Proof of Theorem 3. Suppose that $f \in \mathcal{U}(\lambda_1)$ and $g \in \mathcal{U}(\lambda_2)$. By hypotheses, $\dfrac{z}{H(z)} \neq 0$ for $z \in \mathbf{D}$, and f and g have the power series representation of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n \quad \text{and} \quad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

respectively. As $f \in \mathcal{U}(\lambda_1)$, we have

$$\left|-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1\right| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| \le \lambda_1.$$

Therefore, with $z = re^{i\theta}$ for $r \in (0,1)$ and $0 \le \theta \le 2\pi$, the last inequality gives

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right|^2 d\theta \le \lambda_1^2.$$

Allowing $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \le \lambda_1^2.$$

Similarly, as $g \in \mathcal{U}(\lambda_2)$, we have

$$\sum_{n=2}^{\infty}(n-1)^2|c_n|^2 \le \lambda_2^2.$$

Now, since

$$\frac{z}{f(z)} * \frac{z}{g(z)} = 1 + b_1c_1z + b_2c_2z^2 + \cdots$$

and

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|\,|c_n| \le \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)^2|c_n|^2\right)^{1/2} \le \lambda_1\lambda_2,$$

by (2), we conclude that $H \in \mathcal{M}(\lambda)$, $\lambda = \lambda_1\lambda_2$. □

Proof of Corollary 1. As in the proof of Theorems 2 and 3, we see that

$$M_f(z) = \sum_{n=2}^{\infty}(n-1)^2 b_n z^n$$

and therefore, we easily have the desired necessary condition. □

Proof of Theorem 4. Let $f \in \mathcal{M}(\lambda)$. By assumption,

$$M_f(z) = \lambda w(z)$$

for some $w \in \mathcal{B}_1$. Let $\phi(z) = 1 + b_1z + b_2z^z + \cdots$ denote $z/f(z)$. Then

$$M_f(z) = \sum_{n=2}^{\infty}(n-1)^2 b_n z^n = \lambda w(z)$$

which leads to

$$\phi(z) - 1 - b_1z = \sum_{n=2}^{\infty} b_n z^n = \lambda z\,\mathrm{Li}_2(z) * w(z),$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

is the dilogarithm. By using the well-known representation

$$\mathrm{Li}_2(z) = z \int_0^1 \frac{\log(1/t)}{1 - tz}\, dt,$$

we obtain

$$\begin{aligned}\phi(z) &= 1 + b_1 z + \lambda w(z) * z^2 \int_0^1 \frac{\log(1/t)}{1 - tz}\, dt \\ &= 1 + b_1 z + \lambda \int_0^1 w(z) * \frac{z^2}{1 - tz} \log(1/t)\, dt \\ &= 1 + b_1 z + \lambda \int_0^1 \frac{w(tz)}{t^2} \log(1/t)\, dt.\end{aligned}$$

Since $b_1 = -f''(0)/2$, the desired representation follows. $\square$

Proof of Theorem 5. Let $f \in \mathcal{U}$. Then, because f is univalent, f has the form

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}, \quad z \in \mathbf{D}. \tag{9}$$

Since $f \in \mathcal{U}$, we have (see the proof of Theorem 3)

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n|^2 \le 1. \tag{10}$$

We need to show that $\frac{1}{r} f(rz) \in \mathcal{M}$ for $0 < r \le r_0$ where $r_0 \approx 0.62977$ is the root of the equation $r^4(1 + r^2) = (1 - r^2)^3$ lying in the interval $(0, 1)$.

Using (9), for $0 < r \le 1$, we can write

$$\frac{z}{\frac{1}{r} f(rz)} = 1 + \sum_{n=1}^{\infty} (b_n r^n) z^n.$$

According to Theorem 2, it suffices to show that

$$\sum_{n=2}^{\infty} (n-1)^2 |b_n r^n| \le 1$$

for $0 < r \le r_0$.

Now, by the Cauchy-Schwarz inequality and (10),

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)^2r^{2n}\right)^{1/2}$$
$$\leq \left(\sum_{n=2}^{\infty}(n-1)^2r^{2n}\right)^{1/2} = \left(\frac{r^4(1+r^2)}{(1-r^2)^3}\right)^{1/2}.$$

In particular, for $0 < r \leq r_0$, the last expression is less than or equal to 1. The proof is complete. □

Acknowledgement. The authors thank the referee for his/her careful reading of the manuscript and for many useful comments.

References

[1] L. A. Aksentiev, Sufficient conditions for univalence of regular functions (in Russian), Izv. Vysš. Učebn. Zaved. Matematika **3** (4) (1958), 3–7.

[2] R. Fournier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex Var. Elliptic Equ. **52** (2007), 1–8.

[3] B. Friedman, Two theorems on schlicht functions, Duke Math. J. **13** (1946), 171–177.

[4] M. Nunokawa, M. Obradović and S. Owa, One criterion for univalency, Proc. Amer. Math. Soc. **106** (1989), 1035–1037.

[5] M. Obradović and S. Ponnusamy, New criteria and distortion theorems for univalent functions, Complex Variables Theory Appl. **44** (2001), 173–191.

[6] M. Obradović and S. Ponnusamy, Radius properties for subclasses of univalent functions, Analysis (Munich) **25** (2006), 183–188.

[7] M. Obradović and S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, J. Math. Anal. Appl. **336** (2007), 758–767.

[8] M. Obradović and S. Ponnusamy, Coefficient characterization for certain classes of univalent functions, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), 251–263.

[9] M. Obradović and S. Ponnusamy, On certain subclasses of univalent functions and radius properties, Rev. Roumaine Math. Pures Appl., **54** (2009), 317–329.

[10] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Amer. Math. Soc. **33** (1972), 392–394.

Milutin Obradović
Department of Mathematics
Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73
11000 Belgrade
Serbia
E-mail: obrad@grf.bg.ac.rs

Saminathan Ponnusamy
Department of Mathematics
Indian Institute of Technology Madras
Chennai–600 036
India
E-mail: samy@iitm.ac.in