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Some properties of certain expressions of analytic functions

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ABSTRACT

Let $\mathcal{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk and normalized by $f(0) = f'(0) - 1 = 0$. In this paper the expression $\frac{f'(z)-1}{f(z)/z}$ is studied using differential subordinations and different properties of $\frac{f(z)}{z}$, as well as sufficient conditions for starlikeness and univalence of $f(z) \in \mathcal{A}$, are obtained. Also, several open problems are posed.

1. Introduction and preliminaries

Let $\mathcal{H}(\mathbb{D})$ be the class of functions that are analytic in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}(\mathbb{D})$ that are normalized such that $f(0) = f'(0) - 1 = 0$.

For a function $f \in \mathcal{A}$, we say that it is *strongly starlike of order* α, $0 < \alpha \le 1$, if

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\right]\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

The corresponding class is denoted by $\widetilde{S}^*(\alpha)$. In particular, $S^* \equiv \widetilde{S}^*(1)$ is the class of *starlike functions*. These classes are subclasses of the class of univalent functions [1]. The geometric characterization of a starlike function f is that it maps the unit disk onto a starlike region, i.e. $\omega \in f(\mathbb{D})$ implies $t\omega \in f(\mathbb{D})$ for all $t \in [0, 1]$.

Expressions

$$f'(z) - 1 \quad \text{and} \quad \frac{f(z)}{z}$$

often appear in criteria for starlikeness (univalence), either in the condition, or in the conclusion. Two such results are given below and more details can be found in [2,3].

Theorem A ([4]). *Let $b \in \mathcal{H}(\mathbb{D}) \cap C^0(\overline{\mathbb{D}})$, $b(0) = 0$, $\sup_{z\in\mathbb{D}} |b(z)| = 1$ and $c = \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|\mathrm{d}t$. For $0 < \alpha \le 1$ let*

$$\lambda(\alpha) = \frac{\sin(\alpha\pi/2)}{\sqrt{1 + 2c\cos(\alpha\pi/2) + c^2}}.$$

If $f \in \mathcal{A}$ and

$$|f'(z) - 1| \le \lambda(\alpha) \cdot |b(z)|, \quad z \in \mathbb{D},$$

then $f(z) \in \widetilde{S}^*(\alpha)$. Additionally, if

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$$b(t) = \max_{0\le\varphi\le 2\pi} |b(te^{i\varphi})|, \quad 0 \le t \le 1,$$

then the constant $\lambda(\alpha)$ cannot be replaced by any larger number without violating the conclusion.

Theorem A, without the sharpness part, was previously obtained by Ponnusamy and Singh in [5]. For $\alpha = 1$ and $b(z) = z$, using the Schwartz lemma, we obtain: if $f \in \mathcal{A}$ and $|f'(z) - 1| \le 2/\sqrt{5}$, $z \in \mathbb{D}$, then f is a starlike function. The same result, only with "$<$" instead of "$\le$", was proven by Mocanu in [6].

Theorem B ([7]). *If $f \in \mathcal{A}$ and*

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}, \quad z \in \mathbb{D},$$

then

$$\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}, \quad z \in \mathbb{D}.$$

In this paper we study the quotient

$$\frac{f'(z) - 1}{f(z)/z}, \tag{1}$$

its modulus and real part, and obtain conditions over them that lead to some properties of $f'(z) - 1$ and $f(z)/z$, as well as to criteria for univalence, starlikeness and strong starlikeness of order α.

For that purpose we will use a method from the theory of differential subordinations. A valuable reference on this topic is [3].

First we introduce subordination. Let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathbb{D}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. In particular, if $g(z)$ is univalent in $\mathbb{D}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For obtaining the main result, we will use the method of differential subordinations. The general theory of differential subordinations, as well as the theory of first-order differential subordinations, was introduced by Miller and Mocanu in [8, 9]. Namely, if $\phi : \mathbb{C}^2 \to \mathbb{C}$ (where $\mathbb{C}$ is the complex plane) is analytic in a domain D, if $h(z)$ is univalent in $\mathbb{D}$, and if $p(z)$ is analytic in $\mathbb{D}$ with $(p(z), zp'(z)) \in D$ when $z \in \mathbb{D}$, then $p(z)$ is said to satisfy a first-order differential subordination if

$$\phi(p(z), zp'(z)) \prec h(z). \tag{2}$$

The univalent function $q(z)$ is said to be a *dominant* of the differential subordination (2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2). If $\widetilde{q}(z)$ is a dominant of (2) and $\widetilde{q}(z) \prec q(z)$ for all dominants of (2), then we say that $\widetilde{q}(z)$ is the *best dominant* of the differential subordination (2).

From the theory of first-order differential subordinations we will make use of the following lemma.

Lemma 1 ([9]). *Let $q(z)$ be univalent in the unit disk $\mathbb{D}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathbb{D})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that:*

(i) *$Q(z)$ is starlike in the unit disk $\mathbb{D}$; and*
(ii) $\operatorname{Re} \frac{zh'(z)}{Q(z)} = \operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)} \right\} > 0$, $z \in \mathbb{D}$.

If $p(z)$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{3}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (3).

Using Lemma 1 we will prove the following result that will be used in later sections for studying the modulus and the real part of (1).

Lemma 2. *Let $q(z)$ be univalent in the unit disk $\mathbb{D}$, $q(0) = 0$ and $q(z) \neq -1$, $z \in \mathbb{D}$. Also, let:*

(i) $\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{1+q(z)} \right] > 0$, $z \in \mathbb{D}$; *and*
(ii) $\operatorname{Re} \left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)-1}{1+q(z)} \right] > 0$, $z \in \mathbb{D}$.

If $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and

$$\frac{f'(z) - 1}{f(z)/z} \prec \frac{zq'(z) + q(z)}{1 + q(z)} \tag{4}$$

then $\frac{f(z)}{z} - 1 \prec q(z)$, and $q(z)$ is the best dominant of (4).

Proof. We choose $\theta(\omega)=\frac{\omega}{1+\omega}$ and $\phi(\omega)=\frac{1}{1+\omega}$. Then $\theta(\omega)$ and $\phi(\omega)$ are analytic in a domain $D=\mathbb{C}\setminus\{-1\}$ which contains $q(\mathbb{D})$ and $\phi(\omega)\neq 0$ when $\omega\in q(\mathbb{D})$. Further,

$$Q(z)=zq'(z)\phi(q(z))=\frac{zq'(z)}{1+q(z)}$$

is starlike since

$$\operatorname{Re}\frac{zQ'(z)}{Q(z)}=\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{1+q(z)}\right]>0,\quad z\in\mathbb{D},$$

and for the function $h(z)=\theta(q(z))+Q(z)=\frac{zq'(z)+q(z)}{1+q(z)}$ we have

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)-1}{1+q(z)}\right]>0,\quad z\in\mathbb{D}.$$

Now we choose $p(z)=\frac{f(z)}{z}-1$ which is analytic in $\mathbb{D}$, $p(0)=0$ and $p(z)\neq -1$ for all $z\in\mathbb{D}$ (equivalently $p(\mathbb{D})\subseteq D$). Therefore, the conditions of Lemma 1 are satisfied and, considering that subordinations (3) and (4) are equivalent, we obtain the conclusion of Lemma 2. □

2. Results over the modulus of (1)

In this section we will study the modulus of the expression (1) and obtain conclusions over $f(z)/z$ and $f'(z)-1$ that will lead to sufficient conditions for starlikeness and univalence. Using Lemma 2 we obtain:

Theorem 1. *Let $f\in\mathcal{A}$, $\frac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$ and $0<\mu\leq 1$. If*

$$\frac{f'(z)-1}{f(z)/z}\prec\frac{2\mu z}{1+\mu z}\equiv h_1(z) \tag{5}$$

then

$$\frac{f(z)}{z}-1\prec\mu z \tag{6}$$

and μz is the best dominant of (5). Furthermore,

$$\left|\frac{f(z)}{z}-1\right|<\mu,\quad z\in\mathbb{D}, \tag{7}$$

and this conclusion is sharp, i.e., in the inequality (7), μ cannot be replaced by a smaller number such that the implication holds.

Proof. Let us note that function $q(z)=\mu z$ satisfies all conditions from Lemma 2 and that subordinations (4) and (5) are equivalent. Therefore, (6) follows directly from Lemma 2. As for the sharpness, let us assume that (5) and $|f(z)/z-1|<\mu_1$, $z\in\mathbb{D}$, i.e., $\frac{f(z)}{z}-1\prec\mu_1 z$ hold. But μz is the best dominant of (5), meaning that $\mu z\prec\mu_1 z$, i.e., $\mu\leq\mu_1$. □

It is easy to verify that when $0<\mu<1$, $h_1(\mathbb{D})$ (h_1 is defined in (5)) is an open disk with center $c=\frac{h_1(1)+h(-1)}{2}=-\frac{2\mu^2}{1-\mu^2}$ and radius $r=h_1(1)-c=\frac{2\mu}{1-\mu^2}$; and for $\mu=1$,

$$h_1(\mathbb{D})=\{x+iy: x<1, y\in\mathbb{R}\}.$$

Therefore, Theorem 1 can be written in the following, equivalent form.

Theorem 1′. *Let $f\in\mathcal{A}$ and $\frac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$.*

(i) *If $0<\mu<1$ and*

$$\left|\frac{f'(z)-1}{f(z)/z}+\frac{2\mu^2}{1-\mu^2}\right|<\frac{2\mu}{1-\mu^2},\quad z\in\mathbb{D},$$

then $\left|\frac{f(z)}{z}-1\right|<\mu$, $z\in\mathbb{D}$.

(ii) *If*

$$\operatorname{Re}\left[\frac{f'(z)-1}{f(z)/z}\right]<1,\quad z\in\mathbb{D},$$

then $\left|\frac{f(z)}{z}-1\right|<1$, $z\in\mathbb{D}$.

These implications are sharp, i.e., in both cases the radius of the open disk from the conclusion is the smallest possible so the corresponding implication holds.

Theorem 1, together with the properties of the image in which $\frac{2\mu z}{1+\mu z}$ maps to the unit disk, yields the next corollary.

Corollary 1. *Let $f \in \mathcal{A}$ and $0 < \lambda \le 1$. If*

$$|f'(z) - 1| < \lambda \cdot \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D}, \tag{8}$$

then

$$\left|\frac{f(z)}{z} - 1\right| < \frac{\lambda}{2-\lambda} \equiv \mu, \quad z \in \mathbb{D}.$$

Proof. At the beginning let us note that condition (8) implies $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$, and further

$$\left|\frac{f'(z)-1}{f(z)/z}\right| < \lambda = \frac{2\mu}{1+\mu}, \quad z \in \mathbb{D}.$$

In the case when $0 < \lambda < 1$, i.e., $0 < \mu < 1$, this leads to

$$\left|\frac{f'(z)-1}{f(z)/z} + \frac{2\mu^2}{1-\mu^2} - \frac{2\mu^2}{1-\mu^2}\right| < \frac{2\mu}{1+\mu}, \quad z \in \mathbb{D},$$

i.e.,

$$\left|\frac{f'(z)-1}{f(z)/z} + \frac{2\mu^2}{1-\mu^2}\right| < \frac{2\mu^2}{1-\mu^2} + \frac{2\mu}{1+\mu} = \frac{2\mu}{1-\mu^2}, \quad z \in \mathbb{D}.$$

Now, the conclusion follows from Theorem 1′(i).

In the case when $\lambda = \mu = 1$ we have

$$\left|\frac{f'(z)-1}{f(z)/z}\right| < 1, \quad \text{i.e.,} \quad \mathrm{Re}\left[\frac{f'(z)-1}{f(z)/z}\right] < 1, \quad z \in \mathbb{D}$$

and the rest follows from Theorem 1′(ii). □

Remark 1. For the function $f(z) = \frac{z}{1+az}$, $0 < a \le \frac{3-\sqrt{5}}{2} = 0.381966\ldots$, we obtain

$$\max_{|z|=1}\left|\frac{f'(z)-1}{f(z)/z}\right| = \max_{|z|=1}\left|\frac{az(2+az)}{1+az}\right| = \frac{a(2-a)}{1-a} \equiv \lambda \in (0, 1]$$

and

$$\max_{|z|=1}\left|\frac{f(z)}{z} - 1\right| = \max_{|z|=1}\left|\frac{-az}{1+az}\right| = \frac{a}{1-a} < \mu \equiv \frac{\lambda}{2-\lambda} = \frac{a(2-a)}{a^2-4a+2}.$$

This example raises the question of whether the result from Corollary 1 is sharp or not, i.e., does there exist $\mu < \frac{\lambda}{2-\lambda}$ such that the implication from the corollary holds? This is still an *open problem*.

Using Corollary 1 we obtain the following implications.

Corollary 2. *Let $f \in \mathcal{A}$ and $0 < \lambda \le 1$. If*

$$|f'(z) - 1| < \lambda \cdot \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D},$$

then

$$|f'(z) - 1| < \frac{2\lambda}{2-\lambda}, \quad z \in \mathbb{D},$$

and

$$\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] > 1 - \frac{3\lambda}{2}, \quad z \in \mathbb{D}.$$

Proof. The conditions of Corollary 1 are satisfied, and so

$$\left|\frac{f(z)}{z} - 1\right| < \frac{\lambda}{2-\lambda} \equiv \mu, \quad z \in \mathbb{D},$$

i.e.,

$$\left|\frac{f(z)}{z}\right| < 1+\mu \quad z \in \mathbb{D},$$

$$0 \le 1-\mu < \operatorname{Re}\left[\frac{f(z)}{z}\right] < 1+\mu, \quad z \in \mathbb{D},$$

and

$$\operatorname{Re}\left[\frac{z}{f(z)}\right] > \frac{1}{1+\mu}, \quad z \in \mathbb{D}.$$

From here,

$$|f'(z)-1| = \left|\frac{f'(z)-1}{f(z)/z}\right| \cdot \left|\frac{f(z)}{z}\right| < \lambda \cdot (1+\mu) = \frac{2\lambda}{2-\lambda}, \quad z \in \mathbb{D},$$

and

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] = \operatorname{Re}\left[\frac{f'(z)-1}{f(z)/z}\right] + \operatorname{Re}\left[\frac{z}{f(z)}\right] > -\lambda + \frac{1}{1+\mu} = 1-\frac{3\lambda}{2}, \quad z \in \mathbb{D}. \quad \square$$

Combining Theorem A and Corollary 2 we obtain:

Corollary 3. *Let $f \in \mathcal{A}$, $0 < \alpha \le 1$ and*

$$\lambda(\alpha) = \frac{2\sin(\alpha\pi/2)}{\sqrt{5+4\cos(\alpha\pi/2)}}.$$

If

$$|f'(z)-1| < \frac{2\lambda(\alpha)}{2+\lambda(\alpha)} \cdot \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D},$$

then $f(z) \in \widetilde{S}^*(\alpha)$.

Proof. From Corollary 2, using $\lambda = \frac{2\lambda(\alpha)}{2+\lambda(\alpha)}$, we have

$$|f'(z)-1| < \frac{2\lambda}{2-\lambda} = \lambda(\alpha), \quad z \in \mathbb{D},$$

which, according to the Schwartz lemma, leads to

$$|f'(z)-1| \le \lambda(\alpha) \cdot |z|, \quad z \in \mathbb{D}.$$

Now, choosing $b(z) = z$ in Theorem A yields

$$c = \sup_{z\in\mathbb{D}} \int_0^1 |b(tz)|\mathrm{d}t = \sup_{z\in\mathbb{D}} \frac{|z|}{2} = \frac{1}{2}$$

and $f(z) \in \widetilde{S}^*(\alpha)$. $\square$

Remark 2. In Remark 1 we concluded that Corollary 1 is not sharp, which implies that Corollary 2 and Corollary 3 are not sharp, too. Finding their sharp versions is still an *open problem*.

The following example gives some concrete conclusions that can be obtained from the previous results by specifying the values λ and α.

Example 1. Let $f \in \mathcal{A}$.

(i) If $|f'(z)-1| < \frac{2}{3} \cdot \left|\frac{f(z)}{z}\right|$, $z \in \mathbb{D}$, then:
  (a) $|f'(z)-1| < 1$ and $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$ (this implies univalence of f).
  (b) $\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0$, $z \in \mathbb{D}$ (this implies starlikeness of f).
  ($\lambda = \frac{2}{3}$ in Corollary 1.)

(ii) If $|f'(z)-1| < \left|\frac{f(z)}{z}\right|$, $z \in \mathbb{D}$, then $\left|\frac{f(z)}{z} - 1\right| < 1$, $z \in \mathbb{D}$ ($\lambda = 1$ in Corollary 1).

(iii) If $|zf''(z) + f'(z) - 1| < |f'(z)|$, $z \in \mathbb{D}$, then $\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > -\frac{1}{2}$, $z \in \mathbb{D}$, which implies univalence of f ($zf'(z)$ instead of $f(z)$ and $\lambda = 1$ in Corollary 2).

(iv) (a) If $\left|\frac{f'(z)-1}{f(z)/z}+1\right|<\sqrt{3}, z\in\mathbb{D}$, then $\left|\frac{f(z)}{z}-1\right|<\frac{\sqrt{3}}{3}, z\in\mathbb{D}$.
(b) If $\left|\frac{f'(z)-1}{f(z)/z}+\sqrt{2}-1\right|<1, z\in\mathbb{D}$, then $\left|\frac{f(z)}{z}-1\right|<\sqrt{2}-1, z\in\mathbb{D}$.
($\mu=\frac{\sqrt{3}}{3}$ and $\mu=\sqrt{2}-1$ in Theorem 1′, respectively.)

(v) If $|f'(z)-1|<\frac{2}{1+\sqrt{5}}\cdot\left|\frac{f(z)}{z}\right|, z\in\mathbb{D}$, then f is a starlike function ($\alpha=1$ in Corollary 3). This result is weaker than Example 1(i)(c) since $\frac{2}{1+\sqrt{5}}<\frac{2}{3}$.

Remark 3. It remains an *open problem* whether $\lambda=2/3$ is the largest number such that $|f'(z)-1|<\lambda\cdot|f(z)/z|, z\in\mathbb{D}$, implies starlikeness (univalence) of f.

3. Results over the real part of (1)

Choosing $q(z)=\frac{2\alpha z}{1-z}$ in Lemma 2 we obtain:

Theorem 2. *Let $f\in\mathcal{A}$, $\frac{f(z)}{z}\neq 0$ for all $z\in\mathbb{D}$ and $0<\alpha\le 1$. If*

$$\frac{f'(z)-1}{f(z)/z}\prec 1+\frac{1}{1-z}-\frac{2-z}{1-(1-2\alpha)z}\equiv h_2(z) \tag{9}$$

then

$$\frac{f(z)}{z}-1\prec\frac{2\alpha z}{1-z} \tag{10}$$

and $\frac{2\alpha z}{1-z}$ is the best dominant of (9). Furthermore,

$$\mathrm{Re}\left[\frac{f(z)}{z}\right]>1-\alpha,\quad z\in\mathbb{D}, \tag{11}$$

and this conclusion is sharp, i.e., in the inequality (11), $1-\alpha$ cannot be replaced by a larger number such that the implication holds.

Proof. Indeed, $q(z)=\frac{2\alpha z}{1-z}$ is univalent in the unit disk, $q(0)=0$ and $q(z)\neq -1$ for all $z\in\mathbb{D}$. Now, for $z\in\mathbb{D}$ and $a=1-2\alpha\in[-1,1)$ we have

$$\begin{aligned}\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{1+q(z)}\right]&=\mathrm{Re}\left[\frac{z}{1-z}+\frac{1}{1-az}\right]>-\frac{1}{2}+\frac{1}{1+|a|}\\&=\frac{1-|a|}{2(1+|a|)}\ge 0,\end{aligned}$$

meaning that condition (i) from Lemma 2 is satisfied. Condition (ii) from Lemma 2 is also satisfied because of

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)-1}{1+q(z)}\right]>\mathrm{Re}\,\frac{1}{1+q(z)}>0,\quad z\in\mathbb{D}.$$

Therefore, all conditions from Lemma 2 are fulfilled and (10) follows from the fact that

$$\frac{zq'(z)+q(z)}{1+q(z)}=h_2(z).$$

Further, (11) follows from $q(\mathbb{D})=\{x+iy: x>-\alpha, y\in\mathbb{R}\}$.

Now, let us assume that (9) and $\mathrm{Re}\left[\frac{f(z)}{z}\right]>1-\alpha_1, z\in\mathbb{D}$, i.e., $\frac{f(z)}{z}-1\prec\frac{2\alpha_1 z}{1-z}$ hold. But $\frac{2\alpha z}{1-z}$ is the best dominant of (9), meaning that $\frac{2\alpha z}{1-z}\prec\frac{2\alpha_1 z}{1-z}$, i.e., $-\alpha_1\le-\alpha$ and $1-\alpha_1\le 1-\alpha$. □

Let us note that for the function $h_2(z)$ defined within expression (9) we have

$$h_2(z)=\begin{cases}1+\dfrac{1}{1-z}-\dfrac{1}{a}+\left(\dfrac{1}{a}-2\right)\cdot\dfrac{1}{1-az}, & \alpha\neq\dfrac{1}{2},\ \text{i.e., } a\neq 0\\ \dfrac{1}{1-z}-1+z, & \alpha=\dfrac{1}{2},\ \text{i.e., } a=0\end{cases}, \tag{12}$$

where $a=1-2\alpha$.

Now, the definition of subordination and the properties of $h_2(\mathbb{D})$ and $q(\mathbb{D})$ yield the results over the real part of (1). First we will study the case $\alpha\in(0,1/3)$.

Corollary 4. *Let $f \in \mathcal{A}$, $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$ and $0 < \alpha < 1/3$. If*

$$\operatorname{Re}\left[\frac{f'(z)-1}{f(z)/z}\right] > \Delta \equiv \frac{3}{2} \cdot \begin{cases} 1 - \dfrac{1}{1-\alpha}, & 0 < \alpha \leq 1/4 \\ 1 - \dfrac{1}{3\alpha}, & 1/4 \leq \alpha < 1/3, \end{cases} \qquad z \in \mathbb{D}, \tag{13}$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 - \alpha, \quad z \in \mathbb{D}. \tag{14}$$

Proof. First we will show that inequality (13) implies subordination (9). For the function $h_2(z)$ defined by (12) we have $h_2(0) = 0$. Also, $h_2(z)$ is close-to-convex univalent in the unit disk because $Q(z) = \frac{zq'(z)}{1+q(z)}$ is starlike and $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$ for all $z \in \mathbb{D}$ (see (i) and (ii) in Lemmas 1 and 2). Therefore subordination (9) is equivalent to

$$\frac{f'(z)-1}{f(z)/z} \in h_2(\mathbb{D}), \quad z \in \mathbb{D}.$$

Now we will analyze $h_2(\mathbb{D})$. For $a = 1 - 2\alpha \in (1/3, 1)$ and $t = \cos\theta$ we have that

$$\operatorname{Re} h_2(e^{i\theta}) = \frac{3}{2} - \frac{1}{a} + \left(\frac{1}{a} - 2\right) \cdot \operatorname{Re} \frac{1}{1 - ae^{i\theta}}$$

is a continuous function on $(0, 2\pi)$, monotone on $(0, \pi)$ and $(\pi, 2\pi)$ and bounded by

$$w_1 = \frac{3}{2} - \frac{3}{1+a} = \frac{3}{2} \cdot \left(1 - \frac{1}{1-\alpha}\right)$$

and

$$w_2 = \frac{3}{2} - \frac{1}{1-a} = \frac{3}{2} \cdot \left(1 - \frac{1}{3\alpha}\right).$$

Also,

$$\operatorname{Im} h_2(e^{i\theta}) = \frac{1}{2} \operatorname{ctg} \frac{\theta}{2} + \left(\frac{1}{a} - 2\right) \cdot \frac{a \sin\theta}{1 + a^2 - 2a\cos\theta}$$

is a continuous function on $(0, 2\pi)$ that takes values within the whole set of real numbers.

The previous analysis over the real and imaginary parts of $h_2(e^{i\theta})$ justifies the following inclusion:

$$\{x + iy : x > \Delta_1, y \in \mathbb{R}\} \subset h_2(\mathbb{D}),$$

where $\Delta_1 = \max\{w_1, w_2\}$. It is easy to check that $\Delta = \Delta_1 < 0$ for $0 < \alpha < 1/3$, which proves that (13) implies (9).

Further, it is easy to verify that $q(z) = \frac{2\alpha z}{1-z}$ is starlike univalent in the unit disk, $q(0) = 0$ and $q(\mathbb{D}) = \{x + iy : x > -\alpha, y \in \mathbb{R}\}$. Thus, subordination (10) is equivalent to (14).

Now, the implication from this corollary follows directly from Theorem 2. $\square$

From Corollary 4 we easily obtain:

Corollary 5. *Let $f \in \mathcal{A}$, $0 < \alpha < 1/3$ and Δ be defined as in (13). If*

$$|f'(z) - 1| < |\Delta| \cdot \left|\frac{f(z)}{z}\right|, \quad z \in \mathbb{D},$$

then

$$\operatorname{Re}\left[\frac{f(z)}{z}\right] > 1 - \alpha, \quad z \in \mathbb{D}.$$

Choosing $\alpha = \frac{1}{4}$ in Corollaries 4 and 5 we obtain $\Delta = -\frac{1}{2}$, i.e., we obtain:

Example 2. Let $f \in \mathcal{A}$.

(i) If $\frac{f(z)}{z} \neq 0$ and $\operatorname{Re}\left[\frac{f'(z)-1}{f(z)/z}\right] > -\frac{1}{2}$, $z \in \mathbb{D}$, then $\operatorname{Re}\left[\frac{f(z)}{z}\right] > \frac{3}{4}$, $z \in \mathbb{D}$.

(ii) If $|f'(z) - 1| < \frac{1}{2} \cdot \left|\frac{f(z)}{z}\right|$, $z \in \mathbb{D}$, then $\operatorname{Re}\left[\frac{f(z)}{z}\right] > \frac{3}{4}$, $z \in \mathbb{D}$.

Remark 4. Similarly to before, the question about the sharpness of Corollaries 4 and 5 is still an *open problem*.

Now we will show that in the case where $\alpha = 1$, Theorem 2 can be written in the following equivalent form.

Corollary 6. *Let $f \in \mathcal{A}$ and $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. If*

$$\frac{f'(z)-1}{f(z)/z} \in \mathbb{C} \setminus \{1 + \mathrm{i}y : y \in \mathbb{R}, |y| \geq \sqrt{3}\} \equiv \Omega, \quad z \in \mathbb{D},$$

then

$$\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0, \quad z \in \mathbb{D}.$$

This result is sharp, i.e., the zero in the conclusion cannot be replaced by a larger number such that the implication holds.

Proof. It is enough to show that $h_2(\mathbb{D}) = \Omega$. In the same way as in the proof of Corollary 4, for $\alpha = 1$, i.e. $a = -1$, we have

$$h_2(z) = 2 + \frac{1}{1-z} - \frac{3}{1+z},$$
$$\mathrm{Re}\, h_2(e^{\mathrm{i}\theta}) = 2 + \frac{1}{2} - 3 \cdot \frac{1}{2} = 1,$$
$$\mathrm{Im}\, h_2(e^{\mathrm{i}\theta}) = \frac{1}{2}\,\mathrm{ctg}\,\frac{\theta}{2} + \frac{3}{2}\,\mathrm{tg}\,\frac{\theta}{2}.$$

Simple calculations show that $|\mathrm{Im}\, h_2(e^{\mathrm{i}\theta})|$ attains all real values from $[\sqrt{3}, +\infty)$. So, for $\alpha = 1$, $h_2(\mathbb{D}) = \Omega$. This proves the implication from this corollary and its sharpness follows from the sharpness of Theorem 2. $\square$

From the previous corollary, having in mind the properties of the set Ω we obtain:

Example 3. Let $f \in \mathcal{A}$.

(i) If $\left|\mathrm{Im}\left[\frac{f'(z)-1}{f(z)/z}\right]\right| < \sqrt{3}$, $z \in \mathbb{D}$, then $\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0$, $z \in \mathbb{D}$.

(ii) If $\mathrm{Re}\left[\frac{f'(z)-1}{f(z)/z}\right] < 1$, $z \in \mathbb{D}$, then $\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0$, $z \in \mathbb{D}$.

(iii) If $|f'(z) - 1| < 2\left|\frac{f(z)}{z}\right|$, $z \in \mathbb{D}$, then $\mathrm{Re}\left[\frac{f(z)}{z}\right] > 0$, $z \in \mathbb{D}$.

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