ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS AND RADIUS PROPERTIES

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Let $\mathcal{S}$ denote the class of normalized univalent functions f in the unit disk Δ. One of the problems addressed in this paper is that of the $\mathcal{F}$-radius in $\mathcal{G}$ when $\mathcal{F}, \mathcal{G} \subset \mathcal{S}$, namely the maximum value of r_0 such that $r^{-1}f(rz) \in \mathcal{G}$ for all $f \in \mathcal{F}$ and $0 < r \leq r_0$. The investigations are concerned primarily with the classes $\mathcal{U}$ and $\mathcal{P}(2)$ consisting of univalent functions satisfying

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq 1 \quad \text{and} \quad \left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2,$$

respectively, for all $|z| < 1$. Similar radius properties are also obtained for a geometrically motivated subclass $\mathcal{S}_p \subset \mathcal{S}$. Several new sufficient conditions for f to be in the class $\mathcal{U}$ are also presented.

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1. INTRODUCTION AND PRELIMINARIES

Denote by $\mathcal{A}$ the class of all functions f, normalized by $f(0) = f'(0) - 1 = 0$, that are analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and by $\mathcal{S}$ the subclass of univalent functions in Δ. Denote by $\mathcal{S}^*$ the subclass consisting of functions f in $\mathcal{S}$ that are starlike (with respect to origin), i.e., $tw \in f(\Delta)$ whenever $t \in [0, 1]$ and $w \in f(\Delta)$. Analytically, $f \in \mathcal{S}^*$ if and only if $\operatorname{Re}(zf'(z)/f(z)) > 0$ in Δ. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{U}$ if

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| \leq 1, \quad z \in \Delta.$$

In [6], the authors introduced a subclass $\mathcal{P}(2)$ of $\mathcal{U}$, consisting of functions f for which

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad z \in \Delta.$$

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We have the strict inclusion $\mathcal{P}(2) \subsetneq \mathcal{U} \subsetneq \mathcal{S}$ (see [1, 6, 10] for a proof). An interesting fact is that each function in

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2} \right\}$$

belongs to $\mathcal{U}$. Also, it is well-known that these are the only functions in $\mathcal{S}$ having integral coefficients in the power series expansions of $f \in \mathcal{S}$ (see [2]). From the analytic characterization of starlike functions, it is a simple exercise to see that $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*$.

Further work on the classes $\mathcal{U}$ and $\mathcal{P}(2)$, including some interesting generalizations of these classes, may be found in [7, 9, 11]. A function $f \in \mathcal{S}^*$ is said to be in $\mathcal{T}^*$ if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,$$

where $a_k \geq 0$ for $k = 2, 3, \ldots$. Functions of this form are discussed in detail by Silverman [13, 14]. The work of Silverman led to a large number of investigations for univalent functions of the above form.

In this paper we shall be mainly concerned with functions $f \in \mathcal{A}$ of the form

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}, \quad z \in \Delta. \tag{1}$$

The class of functions f of this form for which $b_n \geq 0$ is especially interesting and deserves a separate discussion. We remark that if $f \in \mathcal{S}$ then $z/f(z)$ is nonvanishing in the unit disk Δ. Hence it can be represented as Taylor series of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \quad z \in \Delta.$$

The above representation is convenient for our investigation.

Now, we introduce a subclass $\mathcal{S}_p$ of starlike functions, namely,

$$\mathcal{S}_p = \left\{ f \in \mathcal{S}^* : \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \operatorname{Re} \frac{zf'(z)}{f(z)},\ z \in \Delta \right\}.$$

Geometrically, $f \in \mathcal{S}_p$ if and only if the domain values of $zf'(z)/f(z)$, $z \in \Delta$, is the parabolic region $(\operatorname{Im} w)^2 \leq 2 \operatorname{Re} w - 1$. It is well-known [12, Theorem 2] that $f(z) = z + a_n z^n$ is in $\mathcal{S}_p$ if and only if $(2n-1)|a_n| \leq 1$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two subclasses of $\mathcal{A}$. If for every $f \in \mathcal{F}$, $r^{-1} f(rz) \in \mathcal{G}$ for $r \leq r_0$, and r_0 is the maximum value for which this holds, then we say that

r_0 is the $\mathcal{G}$-radius in $\mathcal{F}$. There are many results of this type in the theory of univalent functions. For example, the $\mathcal{S}_p$-radius in $\mathcal{S}^*$ was found by Rønning in [12] to be 1/3. Moreover, the class $\mathcal{S}_p$ and its associated class of uniformly convex functions, introduced by Goodman [4, 5], have been investigated in [12]. We recall here the following result.

THEOREM A [12, Theorem 4]. *If* $f \in \mathcal{S}$ *then* $\frac{1}{r}f(rz) \in \mathcal{S}_p$ *for* $0 < r \leq 0.33217\ldots$.

The paper is organized as follows. We investigate the $\mathcal{P}(2)$-radius in $\mathcal{F}$, where $\mathcal{F}$ is the subclasses of $\mathcal{U}$ consisting of functions $f \in \mathcal{U}$ of the form (1) that satisfies either the condition $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$ (see Theorem 1) or $b_n \geq 0$ (see Corollary 1). In Theorem 2 we obtain a necessary coefficient condition for a function f of the form (1) with $b_n \geq 0$ to be in $\mathcal{S}_p$, while in Theorem 3 we obtain a sufficient coefficient condition for a nonvanishing analytic function $z/f(z)$ of the form (1) (where $b_n \in \mathbb{C}$) to be in $\mathcal{S}_p$. In Theorem 4 we derive the value of the $\mathcal{S}$-radius in $\mathcal{S}_p$. In Theorems 5 and 6 we establish new necessary and sufficient conditions for a function to belong to the class $\mathcal{U}$. Finally, in Corollary 2 we show that $\mathcal{T}^* \subset \mathcal{U}$, which is somewhat surprising.

2. LEMMAS

For the proof of our results we need the following result (see [3, Theorem 11 on p. 193 of Vol. 2]) which reveals the importance of the area theorem in the theory of univalent functions.

LEMMA 1. *Let* $\mu > 0$ *and* $f \in \mathcal{S}$ *be of the form* $(z/f(z))^{\mu} = 1 + \sum_{n=1}^{\infty} b_n z^n$. *Then we have* $\sum_{n=1}^{\infty}(n-\mu)|b_n|^2 \leq \mu$.

We also have

LEMMA 2 ([9]). *Let* $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ *be a non-vanishing analytic function in* Δ *and* $f(z) = z/\phi(z)$. *Then*

(a) $f \in \mathcal{U}$ *if* $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$;

(b) $f \in \mathcal{P}(2)$ *if* $\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2$.

3. MAIN RESULTS

It is easy to see that the rational function $f(z) = z/(1+Az^3)$ belongs to $\mathcal{U}$ if and only if $|A| \le 1/2$. Further, for $|A| = 1/2$ we have

$$|(z/f(z))''| = |6Az| \le 3|z| \le 2$$

provided $|z| \le 2/3$. It seems reasonable to expect that the $\mathcal{P}(2)$-radius in $\mathcal{U}$ is at least 2/3, and we formulate our first result.

THEOREM 1. *If $f \in \mathcal{U}$ is of the form (1) such that $\sum_{n=2}^{\infty}(n-1)|b_n| \le 1$, then $\frac{1}{r}f(rz) \in \mathcal{P}(2)$ for $0 < r \le 2/3$; and 2/3 is the largest number with this property, especially in the class for which $b_1 = b_2 = 0$.*

Proof. Let $f \in \mathcal{U}$ be of the form (1). We need to show that $\frac{1}{r}f(rz) \in \mathcal{P}(2)$ for $0 < r \le 2/3$. Using (1), for $0 < r \le 1$ we can write

$$\frac{z}{\frac{1}{r}f(rz)} = 1 + \sum_{n=1}^{\infty}(b_n r^n)z^n.$$

According to Lemma 2(b), it suffices to show that

$$\sum_{n=2}^{\infty} n(n-1)|b_n|r^n \le 2$$

for $0 < r \le 2/3$. It is easy to see by induction that $nr^n \le 3r$ for all $0 < r \le 2/3$ and for $n \ge 2$. In view of this observation, and the assumption that $\sum_{n=2}^{\infty}(n-1)|b_n| \le 1$, we obtain

$$\sum_{n=2}^{\infty} n(n-1)|b_n|r^n \le 3r \le 2 \quad \text{for } r \le 2/3.$$

Hence, by Lemma 2(b), $\frac{1}{r}f(rz) \in \mathcal{P}(2)$ for $0 < r \le 2/3$.

To prove the sharpness, we consider $f_\theta(z) = z/(1 + e^{i\theta}z^3/2)$. Then we observe that $f_\theta \in \mathcal{U}$, but it does not belongs to $\mathcal{P}(2)$. We see that $\frac{1}{r}f_\theta(rz) \in \mathcal{P}(2)$ for $0 < r \le 2/3$ and $r = 2/3$ is the largest value with the desired property. $\square$

An interesting consequence of Theorem 1 is stated later in Corollary 1.

THEOREM 2. *If a function f of the form (1) with $b_n \ge 0$ is in $\mathcal{S}_p$, then*

$$\sum_{n=1}^{\infty}(2n-1)b_n \le 1. \tag{2}$$

Proof. Let $f \in \mathcal{S}_p$. Then

$$z\left(\frac{z}{f(z)}\right)' = \frac{z}{f(z)} - \left(\frac{z}{f(z)}\right)^2 f'(z). \tag{3}$$

Therefore, as $f \in \mathcal{S}_p$ is of the form (1), we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \Leftrightarrow \left|\frac{-z\left(\frac{z}{f(z)}\right)'}{\frac{z}{f(z)}}\right| \le \operatorname{Re}\frac{\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'}{\frac{z}{f(z)}}$$

$$\Leftrightarrow \left|\frac{-\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \le \operatorname{Re}\left(1 - \frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right).$$

If $z \in \Delta$ is real and tends to 1^- through reals, then from the last inequality we have

$$\frac{\sum_{n=1}^{\infty} nb_n}{1+\sum_{n=1}^{\infty} b_n} \le \operatorname{Re}\left(1 - \frac{\sum_{n=1}^{\infty} nb_n}{1+\sum_{n=1}^{\infty} b_n}\right),$$

from which we obtain the desired inequality $\sum_{n=1}^{\infty}(2n-1)b_n \le 1$. $\square$

Remark 1. Condition (2) for functions of the form (1) with nonnegative coefficients b_n is not sufficient for the corresponding f to be in the class $\mathcal{S}_p$. As an example, consider the function $f(z) = z/(1+z)$. It is easy to see that the condition for the class $\mathcal{S}_p$, namely,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \operatorname{Re}\frac{zf'(z)}{f(z)},$$

does not hold for all $z \in \Delta$, for example at the boundary point $z = (-1+\mathrm{i})/\sqrt{2}$, hence at some points in Δ.

Remark 2. Let $0 < \lambda < 1$ and $f(z) = z - \lambda z^m$, where $m \ge 2$. Then

$$\frac{z}{f(z)} = \frac{1}{1-\lambda z^{m-1}} = 1 + \sum_{k=1}^{\infty} \lambda^k z^{k(m-1)},$$

which is nonvanishing in the unit disk. It follows from the previous theorem that if $f \in \mathcal{S}_p$, the coefficient must satisfy the condition

$$\sum_{k=1}^{\infty}[2k(m-1)-1]\lambda^k \le 1,$$

which simplifies to $\lambda(2m-1) \le 1$. Thus, a necessary condition for f to belong to $\mathcal{S}_p$ is $0 \le \lambda \le 1/(2m-1)$. It is a simple exercise to see that this condition also is a sufficient condition for $f \in \mathcal{S}_p$ (see also [12, Theorem 2]). Thus, the upper bound for λ cannot be improved. This observation shows that the constant 1 on the right hand side of inequality (2) cannot be replaced by a larger constant. In this sense, condition (2) is sharp.

THEOREM 3. *Let $f(z)$ be a nonvanishing analytic function in $0 < |z| < 1$ of the form* (1). *Then the condition*

$$\sum_{n=1}^{\infty}(2n+1)|b_n| \le 1 \tag{4}$$

is sufficient for f to belong to the class $\mathcal{S}_p$.

Proof. As in the proof of Theorem 2, we notice that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \Leftrightarrow \left|-\frac{\sum_{n=1}^{\infty} nb_nz^n}{1+\sum_{n=1}^{\infty} b_nz^n}\right| \le \operatorname{Re}\left(1-\frac{\sum_{n=1}^{\infty} nb_nz^n}{1+\sum_{n=1}^{\infty} b_nz^n}\right).$$

Thus, to show that f is in $\mathcal{S}_p$, it suffices to show that the quotient

$$-\frac{\sum_{n=1}^{\infty} nb_nz^n}{1+\sum_{n=1}^{\infty} b_nz^n}$$

lies in the parabolic region $(\operatorname{Im} w)^2 \le 1+2\operatorname{Re} w$. Geometric considerations show that this condition holds if

$$\left|\frac{\sum_{n=1}^{\infty} nb_nz^n}{1+\sum_{n=1}^{\infty} b_nz^n}\right| \le \frac{1}{2}, \quad z \in \Delta. \tag{5}$$

From condition (4) we obtain that $\sum_{n=1}^{\infty}(2n+1)|b_n|\,|z|^n \le 1$ and so

$$\sum_{n=1}^{\infty} n|b_n|\,|z|^n \le \frac{1}{2}\left(1-\sum_{n=1}^{\infty}|b_n|\,|z|^n\right).$$

Finally, we find that

$$\left|\frac{\sum_{n=1}^{\infty} nb_n z^n}{1+\sum_{n=1}^{\infty} b_n z^n}\right| \leq \frac{1}{2}\frac{1-\sum_{n=1}^{\infty}|b_n|\,|z|^n}{1-\sum_{n=1}^{\infty}|b_n|\,|z|^n} = \frac{1}{2}.$$

This means that inequality (5) holds and, therefore, $f \in \mathcal{S}_p$. □

THEOREM 4. *If $f \in \mathcal{S}$ is given by (1), then $\frac{1}{r}f(rz) \in \mathcal{S}_p$ for $0 < r \leq r_0$, where r_0, which depends on the second coefficient of f, is the root of the equation*

$$\frac{4}{(1-r^2)^2} + \frac{4}{1-r^2} - (8+12r^2) - 9r^2\ln(1-r^2) = (1-(3/2)|f''(0)|r)^2. \tag{6}$$

Proof. Let $f \in \mathcal{S}$ be given by (1). Then $z/f(z)$ is nonvanishing and for $0 < r \leq 1$ we have

$$\frac{z}{\frac{1}{r}f(rz)} = 1 + (b_1 r)z + (b_2 r^2)z^2 + \cdots \quad (b_1 = -f''(0)/2),$$

and if

$$\sum_{n=1}^{\infty}(2n+1)|b_n|r^n \leq 1 \tag{7}$$

for some r, then $\frac{1}{r}f(rz) \in \mathcal{S}_p$ by Theorem 3. By Lemma 1 with $\mu = 1$, we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1. \tag{8}$$

Now, by the Cauchy-Schwarz inequality and (8),

$$\begin{aligned}\sum_{n=2}^{\infty}(2n+1)|b_n|r^n &= \sum_{n=2}^{\infty}\sqrt{n-1}|b_n|\frac{2n+1}{\sqrt{n-1}}r^n \\ &\leq \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}\frac{(2n+1)^2}{n-1}r^{2n}\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=2}^{\infty}\frac{(2n+1)^2}{n-1}r^{2n}\right)^{\frac{1}{2}} = \left(\frac{16r^4-12r^6}{(1-r^2)^2} - 9r^2\ln(1-r^2)\right)^{\frac{1}{2}}.\end{aligned}$$

In particular, for $0 < r \leq r_0$, the last expression is less than or equal to $1 - 3|b_1|r$. Therefore, (7) holds, concluding the proof. □

Remark 3. One can easily show that equation (6) has a unique solution for $0 < r \leq 1$ and $|b_1| \leq 1/3$. Indeed, let

$$G(r) = \frac{4}{(1-r^2)^2} + \frac{4}{1-r^2} - (8+12r^2) - 9r^2 \ln(1-r^2) - (1-3|b_1|r)^2$$

and $1 - r^2 = x$. Now, for $0 \leq x < 1$ we consider the new function

$$H(x) = \frac{4}{x^2} + \frac{4}{x} + 12x - 20 - 9(1-x)\ln x - (1-3|b_1|\sqrt{1-x})^2.$$

For this function, we see that $H(x) \to +\infty$ when $x \to 0+$, $H(1) = -1$, and

$$H'(x) = -\frac{8}{x^3} - \frac{4}{x^2} + 12 - 9\left(-\ln x + \frac{1-x}{x}\right) - \left(1-3|b_1|\sqrt{1-x}\right)\frac{6|b_1|}{\sqrt{1-x}}$$
$$= -8\left(\frac{1-x^3}{x^3}\right) - 4\left(\frac{1-x^2}{x^2}\right) + 9\ln x - 9\left(\frac{1-x}{x}\right) - (1-3|b_1|\sqrt{1-x})\frac{6|b_1|}{\sqrt{1-x}},$$

which is negative for $0 < x < 1$ while $|b_1| \leq 1/3$, showing that equation (6) has a unique solution in the interval $(0,1)$.

Also, in Theorem 4, we have actually obtained $\mathcal{F}$-radius in $\mathcal{S}$, where $\mathcal{F}$ is the subclass of $\mathcal{S}_p$ consisting of functions f given by (1) with coefficients satisfying the condition $\sum_{n=1}^{\infty}(2n+1)|b_n| \leq 1$.

Remark 4. For $f''(0) = 0$ in Theorem 4, we have $r_0 = 0.30066\ldots$, and the result is the best possible, the extremal function being of the form

$$\frac{z}{f(z)} = 1 + \sum_{n=2}^{\infty} \frac{2n+1}{n-1} r_0^n z^n.$$

To see this, for $|\zeta| < 1$ we have

$$\frac{r_0\zeta}{f(r_0\zeta)} = 1 + \sum_{n=2}^{\infty} b_n \zeta^n,$$

where

$$b_n = \frac{2n+1}{n-1} r_0^{2n}$$

and

$$\sum_{n=1}^{\infty}(2n+1)|b_n| = \sum_{n=2}^{\infty} \frac{(2n+1)^2}{n-1} r_0^{2n} = 1,$$

by the definition of r_0 from (6). This means that $\frac{1}{r}f(rz)$ belongs to $\mathcal{S}_p$ for $0 < r \leq r_0$. Moreover, for $|z| = r > r_0$ we have $\sum_{n=1}^{\infty}(2n+1)|b_n| > 1$. Therefore,

f is extremal for the class $\mathcal{F}$ of functions f given by (1) with coefficients satisfying the condition $\sum_{n=1}^{\infty}(2n+1)|b_n| \leq 1$. In this sense, the result is sharp.

On the other hand, the function f is univalent because it can be easily seen that $f \in \mathcal{U}$. Indeed, we have

$$\sum_{n=2}^{\infty}(n-1)|b_n| - 1 = \sum_{n=2}^{\infty}(2n+1)r_0^n - 1 = \frac{2r_0}{(1-r_0)^2} - 2r_0 + \frac{r_0^2}{1-r_0} - 1$$
$$= -\frac{(1-3r_0)[r_0(1-r_0)+1]}{(1-r_0)^2} < 0.$$

According to Lemma 2(a), $f \in \mathcal{U}$, hence f is univalent. Finally, we only need to prove that the function

$$\frac{z}{f(z)} = 1 + \sum_{n=2}^{\infty}\frac{2n+1}{n-1}r_0^n z^n = 1 + 2\frac{(r_0 z)^2}{1-r_0 z} - 3r_0 z\log(1-r_0 z)$$

has no zeros in the unit disk. This is easy because

$$\mathrm{Re}\left(\frac{z}{f(z)}\right) = 1 - \sum_{n=2}^{\infty}\frac{2n+1}{n-1}r_0^n \geq 1 - \sum_{n=2}^{\infty}(2n+1)r_0^n > 0.$$

Thus, we have established that r_0 in Theorem 4 is the best possible radius when $f''(0) = 0$. In other words, if $\mathcal{F}$ is the subclass of functions $f \in \mathcal{S}$ of the form (1) such that $f''(0) = 0$, then $\frac{1}{r}f(rz)$ belongs to $\mathcal{S}_p$ for $0 < r \leq r_0$, where r_0 is the largest value with the desired property.

It is known that $\mathcal{U} \subsetneq \mathcal{S}$. In [8], the authors have shown that the $\mathcal{U}$-radius in the class $\mathcal{S}$ is $1/\sqrt{2}$. Our next result is simple but is surprising as it identifies an important subclass of $\mathcal{S}$ which lies in $\mathcal{U}$. We remark that a function $f \in \mathcal{U}$ does not necessarily imply that $\mathrm{Re}(f'(z)) > 0$ throughout $|z| < 1$, see [7].

THEOREM 5. *If f is given by* (1) *with $b_n \geq 0$ and such that* $\mathrm{Re}(f'(z)) > 0$ *for $z \in \Delta$, then $f \in \mathcal{U}$.*

Proof. Remark that if $f \in \mathcal{A}$ satisfies $\mathrm{Re}(f'(z)) > 0$ for $z \in \Delta$, then f must be univalent in Δ (see [3]). Also, notice that

$$\mathrm{Re}(f'(z)) > 0 \Leftrightarrow \mathrm{Re}\,\frac{\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'}{\left(\frac{z}{f(z)}\right)^2} > 0 \Leftrightarrow \mathrm{Re}\,\frac{1 - \sum_{n=2}^{\infty}(n-1)b_n z^n}{\left(1 + \sum_{n=1}^{\infty} b_n z^n\right)^2} > 0.$$

For $z \to 1^-$ along the positive real axis, the last inequality above becomes

$$\mathrm{Re}\, \frac{1-\sum\limits_{n=2}^{\infty}(n-1)b_n}{\left(1+\sum\limits_{n=1}^{\infty} b_n\right)^2} \geq 0,$$

which gives $\sum\limits_{n=2}^{\infty}(n-1)b_n \leq 1$ and so $f \in \mathcal{U}$, by Lemma 2(a). □

THEOREM 6. *A function f of the form* (1) *with $b_n \geq 0$ and $z/f(z) \neq 0$ in Δ, is in $\mathcal{U}$ if and only if*

$$\sum_{n=2}^{\infty}(n-1)b_n \leq 1. \tag{9}$$

Proof. On account of Lemma 2(a), it suffices to prove the necessary part. To do this, we let $f \in \mathcal{U}$ of the form (1). This means that

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| = \left|\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1\right| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| < 1.$$

Choosing values of z on the real axis and then letting $z \to 1^-$ through real values, we obtain the coefficient condition (9). □

For example, by (9), the functions

$$\frac{z}{(1+z)^2},\ \frac{z}{1+z},\ \frac{z}{1+z^2} \text{ and } \frac{z}{1+z+z^2}$$

are in $\mathcal{U}$.

As an immediate consequence of Theorems 1 and 6, we have the following result.

COROLLARY 1. *If $f \in \mathcal{U}$ is of the form* (1) *such that $b_n \geq 0$, then $\frac{1}{r}f(rz) \in \mathcal{P}(2)$ for $0 < r \leq 2/3$; and $2/3$ is the largest number with this property, at least when $b_1 = 0 = b_2$.*

We next show that a certain class of functions in $\mathcal{S}^*$ is in $\mathcal{U}$, which is again a surprising simple result. Using this result, we can generate functions in $\mathcal{S}^*$ that are also in $\mathcal{U}$.

THEOREM 7. *If $f \in \mathcal{S}^*$ is of the form* (1) *with $b_n \geq 0$, then the coefficient inequality* (9) *holds.*

Proof. Suppose that $f \in \mathcal{S}^*$ is of the form (1) with $b_n \geq 0$. We have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \Leftrightarrow \left|\frac{zf'(z)}{f(z)} - 1\right| < \left|\frac{zf'(z)}{f(z)} + 1\right|$$

$$\Leftrightarrow \left|\frac{-z\left(\frac{z}{f(z)}\right)'}{2\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'}\right| < 1 \Leftrightarrow \left|\frac{-\sum\limits_{n=1}^{\infty} nb_n z^n}{2 + b_1 z - \sum\limits_{n=3}^{\infty}(n-2)b_n z^n}\right| < 1.$$

For $z \to 1^-$ through real values, from the last inequality we obtain that

$$\frac{\sum\limits_{n=1}^{\infty} nb_n}{2 + b_1 - \sum\limits_{n=3}^{\infty}(n-2)b_n} \leq 1,$$

which is equivalent to (9). Therefore, $f \in \mathcal{U}$. □

Remark 5. Although condition (9) will be a useful necessary condition for a rational function f of the form (1) (with $b_n \geq 0$) to be starlike, it is not a sufficient condition for the starlikeness for functions $f \in \mathcal{U}$. To prove this, we consider the function

$$f_1(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{2}z^3}.$$

By Theorem 6, $f_1 \in \mathcal{U}$. On the other hand, it is easy to see that

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1 - z^3}{1 + \frac{1}{2}z + \frac{1}{2}z^3}$$

and at the boundary point $z_0 = (-1 + \mathrm{i})/\sqrt{2}$, we have

$$\frac{z_0 f_1'(z_0)}{f_1(z_0)} = \frac{2 - 2\sqrt{2}}{3} + \frac{1 - 2\sqrt{2}}{3}\mathrm{i},$$

which implies that $\operatorname{Re}\{z_0 f_1'(z_0)/f_1(z_0)\} < 0$. Consequently, there are points in the unit disk $|z| < 1$ for which $\operatorname{Re}\{zf_1'(z)/f_1(z)\} < 0$, which shows that the function f_1 is not starlike in Δ.

Corollary 2. *If* $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n$ *is in* $\mathcal{S}^*$, *where* $a_n \geq 0$ *for* $n \geq 2$, *then* $f \in \mathcal{U}$.

Proof. Let $f \in \mathcal{S}^*$. Then $z/f(z)$ is nonvanishing in the unit disk. So, $z/f(z)$ can be expressed as

$$\frac{z}{f(z)} = \frac{1}{1 - a_2 z - a_3 z^2 - \cdots} = 1 + b_1 z + b_2 z^2 + \cdots,$$

where $b_n \geq 0$ for all $n \in \mathbb{N}$. Then, by Theorem 7, the inequality

$$\sum_{n=2}^{\infty}(n-1)b_n \leq 1$$

holds. Hence, by Theorem 6, $f \in \mathcal{U}$. $\square$

Corollary 2 is especially helpful in obtaining functions that are both starlike as well as in $\mathcal{U}$, as there are numerous results concerning starlike functions with negative coefficients. For example, $f_m(z) = z - z^m/m$ is in $\mathcal{S}^*$, hence in $\mathcal{U}$. Since $f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n$ is in $\mathcal{S}^*$ if and only if $\sum_{n=2}^{\infty} n|a_n| \leq 1$ (see [13, Theorem 2]), this result can be used to generate functions $f \in \mathcal{U}$ that are not starlike.

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