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Product of univalent functions

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ABSTRACT

Let $\mathcal{S}$ denote the class of functions f analytic and univalent in the unit disk $|z| < 1$ normalized such that $f(0) = 0 = f'(0) - 1$. In this article the authors discuss the radius of univalence of $F(z) = g(z)h(z)/z$ when g and h belong to certain subsets of $\mathcal{S}$. The paper concludes with the following conjecture. If $g, h \in \mathcal{S}$, then F is univalent for $|z| < 1/3$ and the number $1/3$ cannot improved. The conjecture is shown to be true for some subclasses of $\mathcal{S}$, e.g. the class of starlike functions, and the class $\mathcal{U}$ consisting of functions $f \in \mathcal{A}$ satisfying the functional inequality

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| < 1, \quad |z| < 1.$$

Some other related results are also presented.

1. Introduction and main results

In what follows, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$. We denote by $\mathcal{H}$ the space of all functions which are analytic in $\mathbb{D}$. Here we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the family of all functions $f \in \mathcal{H}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$, and set

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}.$$

A function $f \in \mathcal{S}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps the unit disk $\mathbb{D}$ onto a convex domain is called a convex function. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{S}$ that are convex. A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^*(\alpha)$, called *starlike functions of order* α, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{D},$$

for some α with $0 \le \alpha < 1$. It is well-known that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$. Quite a number of results are known for functions from the class $\mathcal{S}$ and its subclasses such as $\mathcal{S}^*(\alpha)$ and $\mathcal{K}$ (see [1,2]). Let $\mathcal{U}(\lambda)$ denote the set of all $f \in \mathcal{A}$ in $\mathbb{D}$ satisfying the condition [3,4]

$$|U_f(z)| < \lambda, \qquad U_f(z) = f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \quad \text{for } z \in \mathbb{D}, \tag{1}$$

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for some $\lambda \in (0, 1]$. It is well-known that $\mathcal{U} := \mathcal{U}(1)$ is included in $\mathcal{S}$, see [5]. It is interesting to observe that the Koebe function belongs to $\mathcal{U}$ although functions in $\mathcal{U}$ are not necessarily starlike in $\mathbb{D}$ (see for example [4,6]). Moreover, since $\mathcal{U}(\lambda) \subset \mathcal{U}$ for $\lambda \in (0, 1]$, functions in $\mathcal{U}(\lambda)$ are univalent in $\mathbb{D}$ whenever $\lambda \in (0, 1]$. Set

$$\mathcal{U}_2(\lambda) = \{f \in \mathcal{U}(\lambda) : f''(0) = 0\}.$$

For convenience, we let $\mathcal{U}_2 = \mathcal{U}_2(1)$. It is known that functions in $\mathcal{U}_2$ are included in the class $\mathcal{P}(1/2)$, where

$$\mathcal{P}(1/2) = \{f \in \mathcal{A} : \operatorname{Re}(f(z)/z) > 1/2 \text{ for } z \in \mathbb{D}\}.$$

We remark that $\mathcal{K} \subset \mathcal{P}(1/2)$.

In the foregoing discussion, we say that $f \in \mathcal{U}(\lambda)$ in the disk $|z| < r$ if the inequality in (1) holds for $|z| < r$ instead of the whole unit disk $\mathbb{D}$. In other words, this is equivalent to saying that g defined by $g(z) = r^{-1}f(rz)$ belongs to $\mathcal{U}$, when f belongs to $\mathcal{U}$ in the disk $|z| < r$. A similar convention will be followed when we say $f \in \mathcal{U}_2(\lambda)$ (resp. $f \in \mathcal{S}^*(\alpha)$ or $f \in \mathcal{S}$) in the disk $|z| < r$. In recent years, the class $\mathcal{U}$ and its association with a number of subclasses of $\mathcal{S}$ together with certain integral transformations have been studied in detail (see [3,4,7–9]).

In this paper the following problem is considered: For $g \in \mathcal{F}_1 \subset \mathcal{S}$ and $h \in \mathcal{F}_2 \subset \mathcal{S}$, consider the function F defined by

$$F(z) = \frac{g(z)h(z)}{z}, \quad z \in \mathbb{D}. \tag{2}$$

For suitable choices of $\mathcal{F}_1$ and $\mathcal{F}_2$, we determine r so that F is *starlike of order* γ (resp. $F \in \mathcal{U}$ and $F \in \mathcal{S}$) in the disk $|z| < r$. Two sharp results are proved (see Theorems 1 and 2). For a non-sharp case (see Theorem 3), we propose a conjecture at the end.

Theorem 1. *Let $g \in \mathcal{S}^*(\alpha)$ and $h \in \mathcal{S}^*(\beta)$, where $0 \le \alpha + \beta < 1$. Then the function F defined by (2) is starlike of order γ in the disk $|z| < r_\gamma^* = \frac{1-\gamma}{\gamma+3-2(\alpha+\beta)}$. The result is sharp.*

Proof. Assume that $g \in \mathcal{S}^*(\alpha)$ and $h \in \mathcal{S}^*(\beta)$. Then

$$\frac{zg'(z)}{g(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \quad z \in \mathbb{D}.$$

By the subordination principle, it follows that

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \ge \frac{1-(1-2\alpha)r}{1+r}, \quad |z| = r.$$

A similar inequality holds for h. By the assumptions on g and h, we deduce that $F(z)/z \neq 0$ in $\mathbb{D}$. From (2), we have

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} - 1$$

and so, for $|z| = r$

$$\begin{aligned}
\operatorname{Re}\frac{zF'(z)}{F(z)} &= \operatorname{Re}\frac{zg'(z)}{g(z)} + \operatorname{Re}\frac{zh'(z)}{h(z)} - 1 \\
&\ge \frac{1-(1-2\alpha)r}{1+r} + \frac{1-(1-2\beta)r}{1+r} - 1 \\
&= \frac{1-(3-2(\alpha+\beta))r}{1+r} \\
&> \gamma \quad \text{for } 0 < r = |z| < \frac{1-\gamma}{\gamma+3-2(\alpha+\beta)}.
\end{aligned}$$

To prove sharpness, we consider

$$g(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad \text{and} \quad h(z) = \frac{z}{(1-z)^{2(1-\beta)}}.$$

Then

$$F(z) = \frac{z}{(1-z)^{4-2(\alpha+\beta)}} \quad \text{and} \quad F'(z) = \frac{1+(3-2(\alpha+\beta))z}{(1-z)^{5-2(\alpha+\beta)}}$$

so that $F'(z) = 0$ at $z = -1/(3-2(\alpha+\beta))$ and $z/F(z) \neq 0$ in $\mathbb{D}$. Hence F is locally univalent in $|z| < r_0^* = 1/(3-2(\alpha+\beta))$ and not in any larger disk. Moreover,

$$\frac{zF'(z)}{F(z)} = \frac{1+(3-2(\alpha+\beta))z}{1-z}$$

showing that

$$\left.\frac{zF'(z)}{F(z)}\right|_{z=-r} = \frac{1-(3-2(\alpha+\beta))r}{1+r} \le \gamma,$$

if $r_\gamma^* \le r < 1$. Thus, F is starlike of order γ in $|z| < r_\gamma^*$, but not in a larger disk. Hence the radius of starlikeness of order γ is sharp. □

Corollary 1. *Let $g \in \mathcal{S}^*(\alpha)$ and $h \in \mathcal{S}^*(\beta)$. Then the function F defined by (2) belongs to $\mathcal{S}^*(\gamma)$, where $\gamma = \alpha+\beta-1$ with $0 \le \gamma < 1$. In particular, $F \in \mathcal{S}^*$ whenever $g \in \mathcal{S}^*(\alpha)$ and $h \in \mathcal{S}^*(1-\alpha)$. The implication is sharp.*

The case $\alpha = \beta = 1/2$ in Corollary 1 gives that $F \in \mathcal{S}^*$ whenever $g, h \in \mathcal{S}^*(1/2)$. Moreover the case $\alpha = \beta = \gamma = 0$ in Theorem 1 gives the following

Corollary 2. *Let $g, h \in \mathcal{S}^*$. Then the function F defined by (2) is starlike in the disk $|z| < \frac{1}{3}$. The result is sharp.*

We recall that $\mathcal{U} \subsetneq \mathcal{S}$. Using the power series method, the present authors in [4] considered the following question: *Given a univalent function f, is it possible to generate functions in $\mathcal{U}$ or in $\mathcal{S}^*$?* Usually the method of convolution provides an affirmative answer to such problems. In our next result and corollaries, we actually provide another multiplier method to obtain functions in $\mathcal{U}$. These results may be considered as a counterpart of Corollary 2 for the class $\mathcal{U}$.

Theorem 2. *Suppose that $g, h \in \mathcal{U}$. Then the function F defined by (2) belongs to $\mathcal{U}$ in the disk $|z| < \frac{1}{3}$. The result is sharp.*

Proof. Suppose that $g \in \mathcal{U}$. Then, using the notation of (1), we can write

$$-z\left(\frac{z}{g(z)}\right)' + \frac{z}{g(z)} - 1 = U_g(z) = w(z) \tag{3}$$

where $w : \mathbb{D} \to \mathbb{D}$ is analytic in $\mathbb{D}$, $w(0) = w'(0) = 0$. We observe from the classical Schwarz lemma that $|w(z)| \le |z|^2$. From (3), it follows easily that

$$\frac{z}{g(z)} = 1 - b_2 z - \int_0^1 \frac{w(tz)}{t^2}dt, \quad b_2 = \frac{g''(0)}{2!},$$

so that

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - 1\right| \le |z|^2 \quad \text{and} \quad \left|\frac{z}{g(z)} - 1\right| \le |b_2|\,|z| + |z|^2. \tag{4}$$

A similar conclusion holds when $h \in \mathcal{U}$. That is,

$$\left|\left(\frac{z}{h(z)}\right)^2 h'(z) - 1\right| \le |z|^2 \quad \text{and} \quad \left|\frac{z}{h(z)} - 1\right| \le |c_2|\,|z| + |z|^2 \tag{5}$$

where $c_2 = h''(0)/2$. Since the functions $g, h \in \mathcal{U}$ are univalent, from the definition of F, $F(z)/z \neq 0$ in $\mathbb{D}$ and

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)} - 1$$

and so, we obtain

$$\left(\frac{z}{F(z)}\right)^2 F'(z) - 1 = \frac{zg'(z)}{g(z)}\frac{z^2}{g(z)h(z)} + \frac{zh'(z)}{h(z)}\frac{z^2}{g(z)h(z)} - \frac{z^2}{g(z)h(z)} - 1$$

and thus, the last expression can be rewritten as

$$U_F(z) = \left(\left(\frac{z}{g(z)}\right)^2 g'(z) - 1\right)\frac{z}{h(z)} + \left(\left(\frac{z}{h(z)}\right)^2 h'(z) - 1\right)\frac{z}{g(z)} - \left(\frac{z}{g(z)} - 1\right)\left(\frac{z}{h(z)} - 1\right).$$

We want to determine the disk $|z| < r$ on which the condition $|U_F(z)| \le 1$ holds. Now, we see that $|U_F(z)| \le 1$ holds in the disk $|z| < r$ if the inequality

$$\left|\left(\frac{z}{g(z)}\right)^2 g'(z) - 1\right|\left|\frac{z}{h(z)}\right| + \left|\left(\frac{z}{h(z)}\right)^2 h'(z) - 1\right|\left|\frac{z}{g(z)}\right| + \left|\frac{z}{g(z)} - 1\right|\left|\frac{z}{h(z)} - 1\right| \le 1 \tag{6}$$

holds in the disk $|z| < r$. As functions in $\mathcal{U}$ are univalent, the Bieberbach estimate for the second coefficient of the univalent function g gives that $|b_2| \leq 2$ (cf. [1,2]). Similarly, $|c_2| \leq 2$ as $h \in \mathcal{U}$. Using these conditions and (4) and (5), we see that the inequality (6) holds, if

$$3|z|^4 + 8|z|^3 + 6|z|^2 = (1+|z|)^3(3|z|-1) + 1 \leq 1.$$

Thus, the function F is in the class $\mathcal{U}$ in the disk $|z| < 1/3$.

In order to prove sharpness, we consider $g(z) = h(z) = z/(1-z)^2$. Then $g, h \in \mathcal{U}$ and the corresponding F gives that

$$\left| \left(\frac{z}{F(z)}\right)^2 F'(z) - 1 \right| = |z|^2 |3z^2 - 8z + 6|.$$

It follows that

$$\left| \left(\frac{z}{F(z)}\right)^2 F'(z) - 1 \right|_{z=-r} = (1+r)^3(3r-1) + 1 \geq 1,$$

if $\frac{1}{3} \leq r < 1$. It is important to point out that $F(z) = g(z)h(z)/z$ is not even univalent in the disk of radius more than 1/3. Thus, the number 1/3 is also sharp for the univalence of F. $\square$

In the proof of Theorem 2, we have used the estimate $|b_2| \leq 2$ and $|c_2| \leq 2$. However, there are many interesting situations where $|b_2|$ and $|c_2|$ are smaller than 2. In such cases, Theorem 3 may be stated in an improved form. In fact, in this case F defined by (2) belongs to $\mathcal{U}$ in $|z| < r_0$ if r_0 is the smallest positive root of the equation

$$3|z|^4 + 2(|b_2| + |c_2|)|z|^3 + (2 + |b_2|\,|c_2|)|z|^2 - 1 = 0$$

in the unit interval (0, 1). In particular, if $g, h \in \mathcal{U}_2$, then we have $b_2 = c_2 = 0$ and so we get $r_0 = \sqrt{3}/3 \approx 0.57735$ and thus, we obtain that $F \in \mathcal{U}_2$ in the disk $|z| < \sqrt{3}/3$. More precisely, we have

Corollary 3. *Suppose that $g, h \in \mathcal{U}_2$. Then the function F defined by (2) belongs to $\mathcal{U}_2$ in the disk $|z| < \sqrt{3}/3$.*

In fact a slightly general result may now be stated without proof as it follows easily.

Corollary 4. *Suppose that $g \in \mathcal{U}_2(\lambda)$ and $h \in \mathcal{U}_2(\lambda')$. Then F defined by (2) belongs to $\mathcal{U}_2(\mu)$ in the disk $|z| < r$, where*

$$r = \sqrt{\frac{2\mu}{\lambda + \lambda' + \sqrt{(\lambda+\lambda')^2 + 12\mu\lambda\lambda'}}}.$$

In particular, by a proper choice of λ' in this corollary, we can easily obtain the following

Corollary 5. *If $g \in \mathcal{U}_2(\lambda)$ and $h \in \mathcal{U}_2((1-\lambda)/(1+3\lambda))$, then F defined by (2) belongs to $\mathcal{U}_2$. In particular, if $g, h \in \mathcal{U}_2(1/3)$, then $F \in \mathcal{U}_2$ and hence F is univalent in $\mathbb{D}$.*

For the proof of the next result, we need the following lemma.

Lemma A. *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function on $\mathbb{D}$ and let f be of the form $f(z) = z/\phi(z)$.*

(a) *If the condition*

$$\sum_{n=2}^{\infty} (n-1)|b_n| \leq \lambda$$

holds for some $\lambda \in (0, 1]$, then $f \in \mathcal{U}(\lambda)$.

(b) *If the condition*

$$\sum_{n=2}^{\infty} (n-1)|b_n| \leq 1 - |b_1|$$

holds, then $f \in \mathcal{S}^*$.

The conclusion (a) in Lemma A is from [3,8] whereas (b) is due to Reade et al. [10, Theorem 1].

Theorem 3. *Let $g, h \in \mathcal{S}$. Then the function F defined by (2) belongs to the class $\mathcal{U}$ in the disk $|z| < r_0$, where $r_0 \approx 0.30294$ is the smallest positive root of the equation*

$$6r^2 + 2\left(\sqrt{2}+4\right) r^3 + \frac{2r^4\sqrt{3-2r^2}}{1-r^2} + 4r^2 \left(\frac{r^2(6r^2-1-4r^4)}{(1-r^2)^2} + \log\left(\frac{1}{1-r^2}\right)\right)^{\frac{1}{2}} + \frac{r^4(3-2r)}{(1-r)^2} - 1 = 0$$

in the interval (0, 1).

Proof. The proof relies on the Area theorem. Let $g, h \in \mathcal{S}$. Then, z/g and z/h can be expressed as

$$\frac{z}{g(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{and} \quad \frac{z}{h(z)} = 1 + c_1 z + c_2 z^2 + \cdots, \quad z \in \mathbb{D}.$$

First, we observe that $b_1 = -g''(0)/2$ and $c_1 = -h''(0)/2$. By the Bieberbach theorem, it follows that $|b_1| \leq 2$ and $|c_1| \leq 2$. Moreover, since $g, h \in \mathcal{S}$, the well-known Area theorem (see [2, Theorem 11 on p. 193 of Vol. 2]) due to Gronwall gives

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty}(n-1)|c_n|^2 \leq 1. \tag{7}$$

In particular,

$$\sum_{n=2}^{\infty}|b_n|^2 \leq 1 \quad \text{and} \quad \sum_{n=2}^{\infty}|c_n|^2 \leq 1. \tag{8}$$

From the definition of F and the power series representations of z/g and z/h, we have

$$\begin{aligned}\frac{z}{F(z)} &= \left(1 + b_1 z + b_2 z^2 + \cdots\right)\left(1 + c_1 z + c_2 z^2 + \cdots\right)\\ &= 1 + \sum_{n=1}^{\infty} B_n z^n.\end{aligned} \tag{9}$$

Comparison of the coefficients z^n on both sides of the last equations gives

$$B_n = \sum_{k=0}^{n} b_k c_{n-k}$$

where $b_0 = c_0 = 1$. From the last relation and (8), we obtain

$$|B_2| \leq |b_2 + c_2| + |b_1|\,|c_1| \leq |b_2| + |c_2| + |b_1|\,|c_1| \leq 2 + |b_1|\,|c_1|,$$

and, since $|b_3| \leq 1/\sqrt{2}$ and $|c_3| \leq 1/\sqrt{2}$ by (7), it follows that

$$|B_3| \leq |b_3 + c_3| + |b_1 c_2 + b_2 c_1| \leq \sqrt{2} + (|b_1| + |c_1|).$$

Finally, for $n \geq 4$ we see that

$$\begin{aligned}|B_n| &\leq |b_0|\,|c_n| + |b_1|\,|c_{n-1}| + |b_{n-1}|\,|c_1| + |b_n|\,|c_0| + \sum_{k=2}^{n-2}|b_k|\,|c_{n-k}|\\ &\leq |b_n| + |c_n| + |c_1|\,|b_{n-1}| + |b_1|\,|c_{n-1}| + \left(\sum_{k=2}^{n-2}|b_k|^2\right)^{1/2}\left(\sum_{k=2}^{n-2}|c_k|^2\right)^{1/2}\\ &\leq |b_n| + |c_n| + |c_1|\,|b_{n-1}| + |b_1|\,|c_{n-1}| + 1, \quad (\text{by } (8)).\end{aligned} \tag{10}$$

Here the second inequality is a consequence of Cauchy–Schwarz inequality.

Now, we consider G defined by $G(z) = r^{-1}F(rz)$ $(0 < r \leq 1)$ so that, by (9),

$$\frac{z}{G(z)} = 1 + \sum_{n=1}^{\infty} B_n r^n z^n.$$

Now we apply Lemma A and show that $G \in \mathcal{U}$. Thus, to complete the proof, it suffices to show that

$$S := \sum_{n=2}^{\infty}(n-1)|B_n|r^n = |B_2|r^2 + 2|B_3|r^3 + T \leq 1 \quad \text{for } 0 < r \leq r_0.$$

In view of the inequality (10), we see that

$$T := \sum_{n=4}^{\infty}(n-1)|B_n|r^n \leq T_1 + T_2 + |c_1|T_3 + |b_1|T_4 + T_5 = R$$

with

$$T_1 = \sum_{n=4}^{\infty}(n-1)|b_n|r^n, \qquad T_2 = \sum_{n=4}^{\infty}(n-1)|c_n|r^n, \qquad T_3 = \sum_{n=4}^{\infty}(n-1)|b_{n-1}|r^n,$$
$$T_4 = \sum_{n=4}^{\infty}(n-1)|c_{n-1}|r^n, \quad \text{and} \quad T_5 = \sum_{n=4}^{\infty}(n-1)r^n = \frac{r^4(3-2r)}{(1-r)^2}.$$

Then an appropriate good upper bound for the sum S is required to complete our investigation. Since

$$|B_2|r^2 + 2|B_3|r^3 \le (2+|b_1c_1|)r^2 + 2\left(\sqrt{2}+|b_1|+|c_1|\right)r^3,$$

it follows that

$$S \le (2+|b_1c_1|)r^2 + 2\left(\sqrt{2}+|b_1|+|c_1|\right)r^3 + R,$$

where R is as above. The proof will be completed once we get an upper bound for the sum R. Using the Cauchy–Schwarz inequality and (7), we see that

$$\begin{aligned} T_1 &\le \left(\sum_{n=4}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=4}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=4}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} = \frac{r^4}{1-r^2}\sqrt{3-2r^2} \quad \text{(by using the sum for } T_5\text{)}. \end{aligned}$$

Again, by the Cauchy–Schwarz inequality,

$$\begin{aligned} T_3 &= \sum_{n=4}^{\infty}\sqrt{n-2}\,|b_{n-1}|\frac{(n-1)r^n}{\sqrt{n-2}} \\ &\le \left(\sum_{n=4}^{\infty}(n-2)|b_{n-1}|^2\right)^{\frac{1}{2}}\left(\sum_{n=4}^{\infty}\frac{(n-1)^2r^{2n}}{n-2}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=4}^{\infty}\frac{(n-1)^2r^{2n}}{n-2}\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=4}^{\infty}\left(n+\frac{1}{n-2}\right)r^{2n}\right)^{\frac{1}{2}} \\ &= r\left(\frac{1}{(1-r^2)^2} - 1 - 2r^2 - 4r^4 + r^2\log\frac{1}{1-r^2}\right)^{\frac{1}{2}} \\ &= r^2\left(\frac{r^2(6r^2-1-4r^4)}{(1-r^2)^2} + \log\left(\frac{1}{1-r^2}\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Because of the symmetry in the expression, similar inequalities hold for the sums T_2 and T_4. From the above computations, it follows that $S \le 1$ if

$$(2+|b_1c_1|)r^2 + 2\left(\sqrt{2}+|b_1|+|c_1|\right)r^3 + R \le 1.$$

The inequality clearly holds whenever

$$\begin{aligned} T(|b_1|,|c_1|) := (2+|b_1c_1|)r^2 + 2\left(\sqrt{2}+|b_1|+|c_1|\right)r^3 + \frac{2r^4\sqrt{3-2r^2}}{1-r^2} \\ + (|b_1|+|c_1|)r^2\left(\frac{r^2(6r^2-1-4r^4)}{(1-r^2)^2} + \log\left(\frac{1}{1-r^2}\right)\right)^{\frac{1}{2}} + \frac{r^4(3-2r)}{(1-r)^2} \le 1. \end{aligned}$$

Recall that $|b_1| \le 2$ and $|c_1| \le 2$ and therefore, for $S \le 1$, it is clearly sufficient to show that $T(2,2) \le 1$. Thus, $S \le 1$ for $0 < r \le r_0$, where $r_0 \approx 0.30294$ is the smallest positive root of the equation $T(2,2) = 1$ as in the statement. □

Corollary 6. *Let $g, h \in \mathcal{S}$ such that $g''(0) = 0$. Then the function F defined by (2) belongs to the class $\mathcal{U}$ in the disk $|z| < r_0$, where $r_0 \approx 0.384622$ is the smallest positive root of the equation*

$$2r^2 + 2\left(\sqrt{2}+2\right)r^3 + \frac{2r^4\sqrt{3-2r^2}}{1-r^2} + 2r^2\left(\frac{r^2(6r^2-1-4r^4)}{(1-r^2)^2} + \log\left(\frac{1}{1-r^2}\right)\right)^{\frac{1}{2}} + \frac{r^4(3-2r)}{(1-r)^2} - 1 = 0$$

in the interval $(0, 1)$.

Proof. Following the proof of Theorem 3 and the notation, $S \leq 1$ whenever $T(0, 2) \leq 1$. We see that $r_0 \approx 0.384622$ is the smallest positive root of the equation $T(0, 2) = 1$ and the proof is complete. □

Corollary 7. *Let $g, h \in \mathcal{S}$ such that $g''(0) = h''(0) = 0$. Then the function F defined by (2) belongs to the class $\mathcal{U}$ in the disk $|z| < r_0$, where $r_0 \approx 0.435895$ is the smallest positive root of the equation*

$$2r^2 + 2\sqrt{2}r^3 + \frac{2r^4\sqrt{3-2r^2}}{1-r^2} + \frac{r^4(3-2r)}{(1-r)^2} - 1 = 0$$

in the interval $(0, 1)$. *Moreover, F is starlike in the disk $|z| < r_0$.*

Proof. Again, the proof of Theorem 3 shows that $S \leq 1$ whenever $T(0, 0) \leq 1$. It follows that $r_0 \approx 0.43589$ is the smallest positive root of the equation $T(0, 0) = 1$ and the proof of the first part is complete. Because $g''(0) = h''(0) = 0$, we have $F''(0) = 0$ and therefore, the starlikeness of F in the disk $|z| < r_0$ is a consequence of Lemma A(b). □

As in the case of Corollary 2, the choice $g(z) = h(z) = z/(1-z)^2$ supports the following

Conjecture. *If $g, h \in \mathcal{S}$, then the function F defined by (2) is univalent in the disk $|z| < \frac{1}{3}$. The number $1/3$ cannot be improved since it is attained when both g and h represent the Koebe function $k(z) = z/(1-z)^2$.*

Also sharp versions of the last two corollaries remain open.

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