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Radius of univalence of certain class of analytic functions

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Abstract. Let $\mathcal{A}$ denote be the class of analytic functions in the unit disk $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. For $z/f(z) \neq 0$ in $\mathbb{D}$, consider

$$\mathcal{U}_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) \text{ and } B(z) = \frac{f(z)}{z}.$$

Under a suitable condition on Ω we determine the radius of univalence of f whenever $\mathcal{U}_f(z) \in \Omega$ or $B(z) \in \Omega$ for $z \in \mathbb{D}$.

1. Introduction

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ be the open disk in the complex plane $\mathbb{C}$ so that $\mathbb{D}_1 = \mathbb{D}$. We denote by $\mathcal{A}$ the set of all analytic function f defined on $\mathbb{D}$ normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. The radius of univalence of a subset $\mathcal{F}$ of $\mathcal{A}$ is the largest number $r \in (0, 1]$ such that every $f \in \mathcal{F}$ is univalent in $\mathbb{D}_r$. There is a long history in determining radius of univalence of various subsets $\mathcal{F}$ (see for example [3]).

The class of Bazilevič functions has been studied by many mathematicians as an interesting subclass of $\mathcal{S}$ introduced by Bazilevič in [2]. As a special case, we consider those functions f in $\mathcal{A}$ satisfying the condition

$$\text{Re}\left(\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right) > 0, \quad z \in \mathbb{D}, \tag{1}$$

for some $\mu \leq 0$. The class of functions f defined by (1), denoted simply by $f \in \mathcal{B}(-\mu)$, has been studied extensively. In particular, functions in $\mathcal{B}(-\mu)$ is known to be in $\mathcal{S}$ whenever $\mu \leq 0$. On the other hand, functions in $\mathcal{B}(-1) := \mathcal{B}$ (i.e. the case $\mu = 1$) is not necessarily univalent in $\mathbb{D}$. See for example, [8] and the references therein for a detailed information on the importance of the class of Bazilevič functions and some of it subclasses. Thus, it is natural to identify Ω so that $\{\mathcal{U}_f(z) : z \in \mathbb{D}\} \subset \Omega$ implies that f is univalent in $\mathbb{D}$, where

$$\mathcal{U}_f(z) := \left(\frac{z}{f(z)}\right)^2 f'(z).$$

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For instance, if $\Omega = \{w : |w-1| < 1\}$ then $\mathcal{U}_f(z) \in \Omega$ implies that $f \in \mathcal{S}$ (see Aksentiev [1, 4]). We denote by $\mathcal{V}(\alpha)$ the class of all functions $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}\,\mathcal{U}_f(z) > \alpha$, for some a fixed $\alpha < 1$ and for all $z \in \mathbb{D}$.

Finally, we introduce the class

$$\mathcal{U}(\lambda) := \left\{ f \in \mathcal{A} : \left|\mathcal{U}_f(z) - 1\right| < \lambda, \text{ for } z \in \mathbb{D} \right\}$$

and let $\mathcal{U} := \mathcal{U}(1)$. We emphasize that $\mathcal{U}$ is a particular subclass of $\mathcal{S}$ as demonstrated by Aksentiev [1] (see also [4]). Thus, functions in $\mathcal{U}(\lambda)$ are univalent if $0 < \lambda \leq 1$ but not necessarily univalent if $\lambda > 1$.

2. Preliminary Lemmas

For the proofs of our results, we need the following lemmas.

Lemma A. *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function on $\mathbb{D}$ and let $f(z) = z/\phi(z)$. Then, we have the following:*

- **(a)** *If $\sum_{n=2}^{\infty}(n-1)|b_n| \leq \lambda$, then $f \in \mathcal{U}(\lambda)$.*
- **(b)** *If $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1 - |b_1|$, then $f \in \mathcal{S}^*$.*
- **(c)** *If $f \in \mathcal{U}(\lambda)$, then $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq \lambda^2$.*
- **(d)** *If $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$ and $b_1 = -f''(0)/2 = 0$, then $f \in \mathcal{S}^* \cap \mathcal{U}$.*

The conclusion **(a)** in Lemma A is from [5, 7] whereas the **(b)** is due to Reade et. al. [9, Theorem 1]. Finally, as $f \in \mathcal{U}(\lambda)$, we have

$$\left|\mathcal{U}_f(z) - 1\right| = \left| -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 \right| = \left| \sum_{n=2}^{\infty} (n-1) b_n z^n \right| \leq \lambda$$

and so **(c)** follows from Prawitz' theorem which is an immediate consequence of Gronwall's area theorem. The case **(d)** may be obtained by combining **(a)** and **(b)**.

Next we recall the following result due to Obradović and Ponnusamy [6] which provides equivalent conditions for univalent functions.

Lemma B. *Let $f \in \mathcal{A}$ have the form*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \quad \text{with } b_n \geq 0 \text{ for all } n \geq 2 \tag{2}$$

and for all z in a neighborhood of $z = 0$. Then we have the following equivalence:

- *(a)* $f \in \mathcal{S}$
- *(b)* $\dfrac{f(z)f'(z)}{z} \neq 0$ *for* $z \in \mathbb{D}$
- *(c)* $\displaystyle\sum_{n=2}^{\infty}(n-1)b_n \leq 1$
- *(d)* $f \in \mathcal{U}$.

We believe that the following lemma might be known in the literature. Since we do not have the source of it even this were known, we include its proof as it is required in the sequel.

Lemma 2.1. *Let p be analytic in* $\mathbb{D}$*,* $p(0) = 1$*,* $p'(0) = b$ *for some* $b \in [0, 2]$ *and* $\operatorname{Re} p(z) > \alpha$ *in* $\mathbb{D}$ *for some* $\alpha < 1$*. Then, we have*

$$|p(z) - 1| \leq |z| \left(\frac{2(1-\alpha)|z| + b}{1 - |z|^2} \right), \ z \in \mathbb{D}.$$

The result is sharp.

Proof. Set

$$p(z) = \frac{1 + (1-2\alpha)\omega(z)}{1 - \omega(z)}, \ z \in \mathbb{D}.$$

Then, by hypothesis, ω is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. If $p(z) = 1 + bz + p_2 z^2 + \cdots$, then we may rewrite the last equation as

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1 - 2\alpha} = \frac{b}{2(1-\alpha)} z + \left(p_2 - \frac{b^2}{2(1-\alpha)} \right) \frac{1}{2(1-\alpha)} z^2 + \cdots, \ z \in \mathbb{D}..$$

This gives $\omega'(0) = b/(2(1-\alpha))$, which by the Schwartz-Pick lemma implies that

$$|\omega(z)| \leq |z| \left(\frac{|z| + \frac{b}{2(1-\alpha)}}{1 + \frac{b}{2(1-\alpha)}|z|} \right) = |z| \left(\frac{2(1-\alpha)|z| + b}{2(1-\alpha) + b|z|} \right), \ z \in \mathbb{D}.$$

In view of this inequality, we see that

$$|p(z) - 1| \leq \left| \frac{2(1-\alpha)\omega(z)}{1 - \omega(z)} \right| \leq \frac{2(1-\alpha)|\omega(z)|}{1 - |\omega(z)|} \leq |z| \left(\frac{2(1-\alpha)|z| + b}{1 - |z|^2} \right), \ z \in \mathbb{D}.$$

The result is sharp for each value of b, $b \in [0, 2]$, as the function

$$p_b(z) = \frac{1 + bz + (1-2\alpha)z^2}{1 - z^2}$$

shows. □

Corollary 2.2. *Let p be analytic in* $\mathbb{D}$*,* $p(0) = 1$*,* $p'(0) = 0$ *and* $\operatorname{Re} p(z) > \alpha$ *in* $\mathbb{D}$ *for some* $\alpha < 1$*. Then, we have*

$$|p(z) - 1| \leq \frac{2(1-\alpha)|z|^2}{1 - |z|^2}, \ z \in \mathbb{D}.$$

The result is sharp. In particular,

$$|p(z) - 1| < \lambda \ \text{ for } \ |z| < \sqrt{\frac{\lambda}{\lambda + 2(1-\alpha)}} = r_{\alpha,\lambda}, \tag{3}$$

or equivalently,

$$|p(rz) - 1| < \lambda \ \text{ for } z \in \mathbb{D}, \text{ for each } 0 < r \leq r_{\alpha,\lambda}.$$

Proof. Set $b = 0$ in Lemma 2.1. □

3. Main Results

Theorem 3.1. *If* $f \in \mathcal{V}(\alpha)$*, then g defined by* $g(z) = r^{-1} f(rz)$ *belongs to* $\mathcal{U}(\lambda)$ *whenever* $0 < r \leq r_{\alpha,\lambda}$*, where* $r_{\alpha,\lambda}$ *is defined by* (3)*. The result is best possible.*

Proof. Let $f \in \mathcal{V}(\alpha)$, and define $p(z)$ by

$$p(z) = \mathcal{U}_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) = -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)}.$$

Then p is analytic in $\mathbb{D}$ such that $p(0) = 1$, $p'(0) = 0$ and $\mathrm{Re}\, p(z) > \alpha$. Now, for $0 < r < 1$, we set $g(z) = r^{-1} f(rz)$. Then, we see that

$$\mathcal{U}_g(z) = \mathcal{U}_f(rz).$$

By Corollary 2.2, it follows that for $0 < r \leq r_{\alpha,\lambda} = \sqrt{\frac{\lambda}{\lambda+2(1-\alpha)}}$,

$$\left|\mathcal{U}_g(z) - 1\right| = |p(rz) - 1| < \lambda \ \text{ for } z \in \mathbb{D}.$$

The sharpness function f is obtained by solving

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \frac{1 + (1-2\alpha)z^2}{1-z^2},\ z \in \mathbb{D}$$

and we complete the proof. □

For $\lambda = 1$ in Theorem 3.1 we have the following

Corollary 3.2. *If $f \in \mathcal{V}(\alpha)$, then $f \in \mathcal{U}$ in the disk $|z| < r_\alpha = 1/\sqrt{3-2\alpha}$. The radius r_α is best possible.*

Theorem 3.3. *Let $f \in \mathcal{A}$ and satisfy the condition $\mathrm{Re}\,\mathcal{U}_f(z) < \beta$ for all $z \in \mathbb{D}$, and for some $\beta > 1$. Then the function g defined by $g(z) = \rho^{-1} f(\rho z)$ belongs to $\mathcal{U}(\lambda)$ whenever $0 < \rho \leq \rho_{\beta,\lambda}$, where*

$$\rho_{\beta,\lambda} = \sqrt{\frac{\lambda}{\lambda + 2(\beta-1)}}.$$

The result is sharp.

Proof. The condition on f implies that F defined by

$$F(z) = 2 - \mathcal{U}_f(z)$$

belongs to $\mathcal{V}(\alpha)$ with α equals $2-\beta$. By a computation, the result follows easily from Theorem 3.1. So, we skip the details. □

If we let $\lambda = 1$ in Theorem 3.3, then we have

Corollary 3.4. *If $f \in \mathcal{A}$ satisfies the condition $\mathrm{Re}\,\mathcal{U}_f(z) < \beta$ for all $z \in \mathbb{D}$, and for some $\beta > 1$, then $f \in \mathcal{U}$ in the disk $|z| < r_\beta = 1/\sqrt{2\beta-1}$.*

Example 3.5. *From the last two corollaries, it can be easily seen that if $f \in \mathcal{A}$ satisfies either the condition*

$$\mathrm{Re}\,\mathcal{U}_f(z) > 0,\ z \in \mathbb{D},$$

or the condition

$$\mathrm{Re}\,\mathcal{U}_f(z) < 2,\ z \in \mathbb{D},$$

then $f \in \mathcal{U}$ in the disk $|z| < 1/\sqrt{3}$ and, in particular, f is univalent in $|z| < 1/\sqrt{3}$.

Theorem 3.6. *Let $f \in \mathcal{A}$ such that $\left|z/f(z)\right| \leq c$ for all $z \in \mathbb{D}$ and for some constant $c > \sqrt{1+a^2}$, where $a = |f''(0)/2|$. Then $f \in \mathcal{U}$ for $|z| < r_c$, where r_c is the root of the equation*

$$(c^2 - 1 - a^2)r^4(1 + r^2) - (1 - r^2)^3 = 0 \tag{4}$$

in the interval $(0, 1)$.

Proof. We may write f in the form

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \ z \in \mathbb{D}. \tag{5}$$

Therefore, with $z = re^{i\theta}$ for $r \in (0, 1)$ and $0 \leq \theta \leq 2\pi$, the last equation and the inequality $\left|z/f(z)\right| \leq c$ gives

$$1 + \sum_{n=1}^{\infty} |b_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left|\frac{z}{f(z)}\right|^2 d\theta \leq c^2.$$

Allowing $r \to 1^-$, we obtain the inequality

$$\sum_{n=2}^{\infty} |b_n|^2 \leq c^2 - 1 - |b_1|^2 = c^2 - 1 - a^2$$

since $b_1 = -f''(0)/2$ with $|b_1| = a$. Now, for $0 < r < 1$, we introduce the function g defined by $g(z) = r^{-1}f(rz)$ so that

$$\frac{z}{g(z)} = \frac{rz}{f(rz)} = 1 + \sum_{n=1}^{\infty} b_n r^n z^n$$

We need to show that $g \in \mathcal{U}$ for $0 < r \leq r_c$. For this, according to Lemma A**(a)**, it suffices to show that

$$S := \sum_{n=2}^{\infty} (n-1)|b_n| r^n \leq 1.$$

Now, by means of the Cauchy-Schwarz inequality, we have

$$S \leq \left(\sum_{n=2}^{\infty} |b_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n-1)^2 r^{2n}\right)^{\frac{1}{2}} \leq \left(c^2 - 1 - a^2\right)^{\frac{1}{2}} \left(\frac{r^4(1+r^2)}{(1-r^2)^3}\right)^{\frac{1}{2}}$$

which is less than or equal to 1 provided $\phi(r) \leq 0$, where

$$\phi(r) = (c^2 - 1 - a^2)r^4(1 + r^2) - (1 - r^2)^3.$$

Finally, it is easy to observe that the function $\phi(r)$ has only one solution in the interval $(0, 1)$. In view of this observation, it follows that $g \in \mathcal{U}$ for each r with $0 < r \leq r_c$, where $\phi(r_c) = 0$. The proof is complete. □

From the statement of Theorem 3.6, it is clear either we can fix c and then determine r_c, or can fix r_c first and then determine c satisfying the equation (4). For example, if we wish to have $r_c = 1/2$, then let $r_c = 1/2$ in (4) which gives $c = \sqrt{(32/5) + a^2}$. In particular, when $a = 0$, then the corresponding value of c is $4\sqrt{2/5}$ and therefore, we have the following

Corollary 3.7. *If $f \in \mathcal{A}$ with $f''(0) = 0$ and if*

$$\left|\frac{z}{f(z)}\right| \leq 4\sqrt{\frac{2}{5}} \approx 2.5298 \ \text{in } \mathbb{D},$$

or equivalently,

$$\left|f(z)/z\right| \geq \frac{1}{4}\sqrt{\frac{5}{2}} \approx 0.3952847 \ \ in\ \mathbb{D},$$

then $f \in \mathcal{U}$ *for* $|z| < 1/2$.

Theorem 3.8. *Let* $f \in \mathcal{A}$ *such that* $\left|\mathcal{U}_f(z)\right| \leq c$ *for all* $z \in \mathbb{D}$*, and for some* $c > 1$*. Then* $f \in \mathcal{U}$ *for* $|z| < r_c$*, where*

$$r_c = \sqrt{\frac{2}{\sqrt{4c^2-3}+1}}. \tag{6}$$

Proof. As in the proof of Theorem 3.6, we may write f in the form (5). Then we see that

$$|\mathcal{U}_f(z)| = \left|\left(\frac{z}{f(z)}\right)^2 f'(z)\right| = \left|-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)}\right| = \left|1-\sum_{n=2}^{\infty}(n-1)b_n z^n\right| \leq c$$

and therefore, we obtain that

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq c^2-1.$$

Again it suffices to show that the function g defined by $g(z) = r^{-1}f(rz)$ belongs to $\mathcal{U}$ for $0 < r \leq r_c$. Now, as before, we obtain that

$$S := \sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}r^{2n}\right)^{\frac{1}{2}} \leq \left(c^2-1\right)^{\frac{1}{2}}\left(\frac{r^4}{1-r^2}\right)^{\frac{1}{2}}$$

and so, $S \leq 1$ for $0 < r \leq r_c$, where r_c, given by (6), is the root of the equation

$$(c^2-1)r^4 + r^2 - 1 = 0$$

in the interval $(0,1)$. By Lemma A**(a)**, $g \in \mathcal{U}$ for $0 < r \leq r_c$. The desired conclusion follows. □

Setting $r_c = 1/2$ in (6) gives $c = \sqrt{13} \approx 3.60555$ and therefore, we obtain

Corollary 3.9. *Suppose that* $f \in \mathcal{A}$ *such that* $\left|\mathcal{U}_f(z)\right| < \sqrt{13}$ *for all* $z \in \mathbb{D}$*. Then* $f \in \mathcal{U}$ *for* $|z| < 1/2$.

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