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Milutin Obradović, Saminathan Ponnusamy & Karl-Joachim Wirths

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Abstract For an analytic function g defined on the unit disk $|z| < 1$, let $\Delta(r, g)$ denote the area of the image of the sub-disk $|z| < r$ under g, where $0 < r \leq 1$. In the case of the family $\mathcal{C}$ consisting of normalized analytic univalent functions f such that $f(\mathbb{D})$ convex, Yamashita conjectured that

$$\max_{f\in\mathcal{C}} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2, \text{ for } 0 < r \leq 1.$$

In this paper, we prove a more general version of this conjecture for the class $\mathcal{S}^\star(\beta)$ of starlike functions of order β. As a consequence, Yamashita's conjecture is true.

Keywords Analytic · Univalent · Convex · Starlike functions · Dirichlet-finite · Area integral · Gaussian hypergeometric functions

Mathematics Subject Classification (2000) Primary 30C45

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M. Obradović
Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia
e-mail: obrad@grf.bg.ac.rs

S. Ponnusamy (✉)
Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600113, India
e-mail: samy@isichennai.res.in

K.-J. Wirths
Institut für Analysis und Algebra, TU Braunschweig, 38106 Braunschweig, Germany
e-mail: kjwirths@tu-bs.de

1 Introduction and the Main Results

For an analytic function g defined on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, we denote the area of the image of the disk $|z| < r$ under g by $\Delta(r, g)$, where $0 < r \leq 1$. Thus

$$\Delta(r, g) = \int\int_{|z|<r} |g'(z)|^2 \, \mathrm{d}x\mathrm{d}y \quad (z = x + iy)$$

and so, if $g(z) = \sum_{n=0}^{\infty} b_n z^n$ then $g'(z) = \sum_{n=1}^{\infty} n b_n z^{n-1}$ so that

$$\Delta(r, g) = \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}. \tag{1}$$

We call g a Dirichlet-finite function if $\Delta(1, g)$, the area covered by the mapping $z \to g(z)$ for $|z| < 1$, is finite. Let $\mathcal{A}$ denote the family of all functions f which are analytic in $\mathbb{D}$ and normalized so that $f(0) = 0 = f'(0) - 1$. The class $\mathcal{S}$ defined by

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$$

has been the central object in the study of geometric function theory. If $f \in \mathcal{S}$ and

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \tag{2}$$

$(a_1 = 1)$, then we may apply the above formula for

$$\frac{f(z)}{z} = \sum_{n=1}^{\infty} a_n z^{n-1}$$

(instead of g) and, as a consequence of the de Branges theorem ($|a_n| \leq n$ for $n \geq 2$), one quickly gets that (see Yamashita [16])

$$\max_{f \in \mathcal{S}} \Delta\left(r, \frac{f(z)}{z}\right) = \pi \sum_{n=1}^{\infty} n(n+1)^2 r^{2n} = \frac{2\pi r^2(r^2+2)}{(1-r^2)^4}.$$

For each r, $0 < r < 1$, the maximum is attained only for the Koebe function $k(z) = z/(1-z)^2$ or its rotations $e^{-i\theta}k(e^{i\theta}z)$. Further with the aid of Gronwall's area theorem applied to $F(z) = 1/f(1/z)$ for $|z| > 1$ and the fact that $|a_2| \leq 2$, Yamashita [16] proved the following theorem.

Theorem A *We have*

$$\max_{f \in \mathcal{S}} \Delta\left(r, \frac{z}{f(z)}\right) = 2\pi r^2(r^2+2) \quad \textit{for } 0 < r \leq 1.$$

For each r, $0 < r \leq 1$, the maximum is attained only by the rotations of the Koebe function $k(z)$.

From Theorem A, it follows that $\Delta(1, z/f(z)) \leq 6\pi$. This shows that each $f \in \mathcal{S}$ is the quotient of z and $z/f(z)$, each of which is bounded and Dirichlet-finite in $\mathbb{D}$. We may now compare with the known bound

$$|z|\,(1+|z|)^{-2} \leq |f(z)| \leq |z|\,(1-|z|)^{-2}$$

from which we obtain $|z/f(z)| \leq 4$ in $\mathbb{D}$.

Let $\mathcal{C}$ denote the family of convex (univalent) functions $f \in \mathcal{A}$, i.e. $f \in \mathcal{S}$ such that $f(\mathbb{D})$ convex. For $\beta \in [0,1)$, let $\mathcal{S}^\star(\beta)$ denote the usual normalized class of all (univalent) starlike functions of order β. Analytically, a function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{S}^\star(\beta)$ if f satisfies the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in \mathbb{D}.$$

It is wellknown that $\mathcal{C} \subsetneq \mathcal{S}^\star(1/2)$, and $\mathcal{S}^\star := \mathcal{S}^\star(0)$ is the usual class of starlike functions, i.e. $f \in \mathcal{S}$ such that $f(\mathbb{D})$ is starlike with respect to the origin. At this point it is interesting to note that a function belonging to $\mathcal{S}^\star(1/2)$ may not be convex and univalent in $|z| < R$ for any $R > \sqrt{2\sqrt{3}-3} = 0.68\ldots$

Yamashita [16, p. 439] remarked that

$$\max_{f\in\mathcal{C}} \Delta\left(r, \frac{f(z)}{z}\right) = \frac{2\pi r^2}{(1-r)^2} \quad \text{for } 0 < r \leq 1.$$

For each r, $0 < r < 1$, the maximum is attained only by the rotations of the function $j(z) = z/(1-z)$. As a counterpart of Theorem A for the family $\mathcal{C}$ of convex functions, Yamashita [16, p. 439] conjectured that the following result holds.

Conjecture 1 *We have*

$$\max_{f\in\mathcal{C}} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2, \quad \textit{for } 0 < r \leq 1,$$

where the maximum is attained only by the rotations of the function $j(z) = z/(1-z)$.

We show that this conjecture is true for the larger class consisting of starlike functions of the order $1/2$. The proof uses a special case of the following lemma which is proved in [9]. However, for the convenience of the reader, we present an independent proof here.

Lemma 1 *Let $f \in \mathcal{S}^\star(\beta)$ and let $z/f(z)$ have the following expansion near $z = 0$,*

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots.$$

Then the following coefficient inequality holds:

$$\sum_{n=1}^{\infty}(n-(1-\beta))|b_n|^2 \leq 1-\beta. \tag{3}$$

Proof Our proof here is indeed a method of Clunie [1] (see also [2,12,13]). Let $g(z)=z/f(z)$ and $f\in\mathcal{S}^\star(\beta)$. Since

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z\in\mathbb{D},$$

and

$$\frac{zg'(z)}{g(z)} = 1-\frac{zf'(z)}{f(z)}, \quad z\in\mathbb{D},$$

there exists a function $\omega:\mathbb{D}\to\overline{\mathbb{D}}$ analytic in the unit disk such that

$$\frac{zg'(z)}{g(z)} = \frac{2(1-\beta)z\omega(z)}{1+z\omega(z)}, \quad z\in\mathbb{D},$$

or, equivalently

$$g'(z) = \big(2(1-\beta)g(z) - zg'(z)\big)\,\omega(z), \quad z\in\mathbb{D}.$$

By Clunie's method we derive for any $n\in\mathbb{N}$ the inequality

$$\sum_{k=1}^{n-1}|b_k|^2r^{2k-2}\left(k^2-(k-2(1-\beta))^2r^2\right) + |b_n|^2r^{2n-2}n^2 \leq 4(1-\beta)^2.$$

In this formula, we can take $r=1$ and allow $n\to\infty$. This gives the desired inequality (3). □

We remark that the case $\beta=0$ of Lemma 1 is the well-known Area Theorem (see [4, Vol. 2, Thm. 11, p. 193]) for functions $f\in\mathcal{S}$.

After this paper was accepted, we have observed that Lemma 1 can be obtained from the work of Pommerenke [10]. Indeed for $f\in\mathcal{S}$, we can define the function $F(\zeta)$ for $|\zeta|>1$ by

$$F(\zeta)=\frac{1}{f(1/\zeta)}=\zeta+\sum_{n=0}^{\infty}b_{n+1}\zeta^{-n}$$

so that $f \in \mathcal{S}^\star(\beta)$ if and only if $\operatorname{Re}(\zeta F'(\zeta)/F(\zeta)) > \beta$ for $|\zeta| > 1$ and $F(|\zeta| > 1)$ does not contain the origin. But then the following sharp inequality holds from [10]:

$$\sum_{n=0}^{\infty}(n+\beta)|b_{n+1}|^2 \le 1-\beta$$

which is same as (3).

Theorem 2 *We have*

$$\max_{f\in\mathcal{S}^\star(1/2)} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2, \quad \textit{for} \quad 0 < r \le 1,$$

where the maximum is attained by the rotations of the function $j(z) = z/(1-z)$.

Proof Let $f \in \mathcal{S}^\star(1/2)$. Since f is analytic and $f(z) \ne 0$ for $z \ne 0$, we can write f in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{4}$$

Now, by (1) and Lemma 1 with $\beta = 1/2$, it follows that

$$\begin{aligned}\pi^{-1}\Delta\left(r, \frac{z}{f(z)}\right) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\\ &\le r^2\left(\sum_{n=1}^{\infty}(2n-1)|b_n|^2 - \sum_{n=2}^{\infty}(n-1)|b_n|^2\right)\\ &\le r^2\left(1-\sum_{n=2}^{\infty}(n-1)|b_n|^2\right) \le r^2.\end{aligned}$$

The equality holds clearly for $j(z) = z/(1-z)$ and for the rotations of $j(z)$. □

This settles the conjecture of Yamashita. In order to state and prove a generalization of Theorem 2, we consider

$$f_\beta(z) = \frac{z}{(1-z)^{2(1-\beta)}},$$

where $0 \le \beta < 1$. It is easy to see that $f_\beta \in \mathcal{S}^\star(\beta)$ and f_β is extremal for many extremal problems for the full class $\mathcal{S}^\star(\beta)$. We see that

$$\frac{z}{f_\beta(z)} = (1-z)^{2(1-\beta)} = F(1, \delta; 1; z), \quad \delta = -2(1-\beta).$$

Here $F(a,b;c;z)$ denotes the Gaussian hypergeometric function

$$F(a,b;c;z) := 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad z \in \mathbb{D},$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k := a(a+1)\cdots(a+k-1)$ for $k \in \mathbb{N}$. In general, a, b and c are complex numbers with $c \neq 0, -1, -2, \ldots$. Note that if either $a = 0$ or $b = 0$, then $F(a,b;c;z) = 1$. Thus, we see that

$$\begin{aligned}\Delta\left(r, \frac{z}{f_\beta(z)}\right) &= \pi \sum_{n=1}^{\infty} n \left(\frac{(\delta)_n}{(1)_n}\right)^2 r^{2n} \\ &= \pi \delta^2 r^2 \sum_{n=0}^{\infty} \frac{(\delta+1)_n (\delta+1)_n}{(2)_n (1)_n} r^{2n} \\ &= \pi \delta^2 r^2 F\left(\delta+1, \delta+1; 2, r^2\right) \\ &= 4\pi(1-\beta)^2 r^2 F\left(2\beta-1, 2\beta-1; 2, r^2\right) =: A_\beta(r).\end{aligned}$$

Thus, we expect that the following general result is true.

Theorem 3 *Let $f \in \mathcal{S}^\star(\beta)$ for some $0 \leq \beta < 1$. Then we have*

$$\max_{f \in \mathcal{S}^\star(\beta)} \Delta\left(r, \frac{z}{f(z)}\right) = A_\beta(r), \quad \textit{for} \quad 0 < r \leq 1, \tag{5}$$

where the maximum is attained by the rotations of $f_\beta(z) = z/(1-z)^{2(1-\beta)}$.

In Sect. 2, we prove Theorem 3. Since $A_{1/2}(r) = \pi r^2$, Theorem 2 follows and hence, another proof of Yamashita's conjecture. Before we proceed to prove Theorem 3, it is worth mentioning certain basic properties of the functional given by $A_\beta(r)$ in (5), where $A_\beta(r) = 4\pi(1-\beta)^2 r^2 F(2\beta-1, 2\beta-1; 2, r^2)$. Note that for $r \in (0,1]$, we may write

$$A_\beta(r) = 4\pi(1-\beta)^2 \sum_{n=1}^{\infty} \left(\frac{(2\beta-1)_{n-1}}{(1)_{n-1}}\right)^2 \frac{r^{2n}}{n}.$$

Because the series on the right has positive coefficients [except for the case $\beta = 1/2$ for which $A_{1/2}(r) = \pi r^2$], $A_\beta(r)$ is an increasing and convex function of the real variable r, $0 < r \leq 1$. Thus, $A_\beta(r) \leq A_\beta(1)$. According to the well-known Gauss formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty \quad \text{for} \quad c > a+b,$$

Table 1 Values of $A_\beta(1)$ for various values of $\beta \in (0, 1)$

S.No	Values of β	Values of $A_\beta(1)$
1	0	$6\pi \approx 18.8496$
2	1/4	8
3	1/2	$\pi \approx 3.14159$
4	2/3	1.52995
5	3/4	1
6	4/5	0.743367
7	49/60	0.666206
8	5/6	0.592763

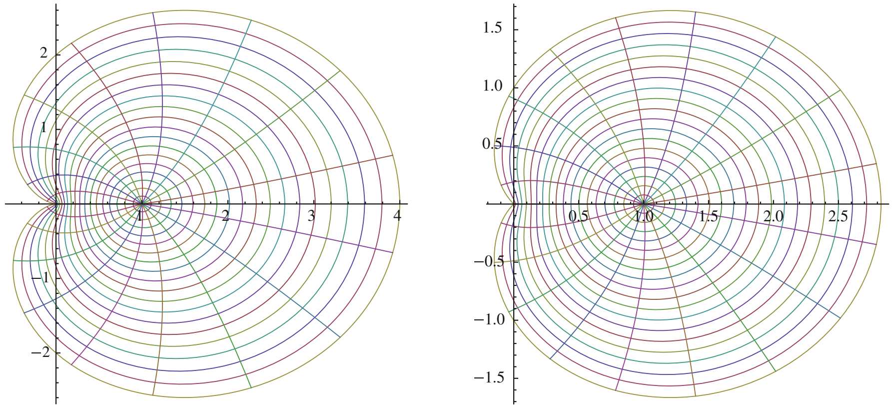

Fig. 1 Images of the unit disk under g_0 and $g_{1/4}$

it follows that

$$A_\beta(r) \le A_\beta(1) = 4\pi(1-\beta)^2 F(2\beta-1, 2\beta-1; 2, 1) = \frac{\Gamma(4-4\beta)}{\Gamma^2(3-2\beta)}.$$

In order to illustrate another property of the functional in (5), we now include here Table 1 for the values of $A_\beta(1)$ for different values of β, and present typical shapes of the images of the unit disk under the extremal mappings $g_\beta(z) = z/f_\beta(z) = (1-z)^{2(1-\beta)}$ for the corresponding values of β from Table 1 (see Figs. 1, 2, 3, 4).

Remark 1 Note that $\mathcal{S}^\star(\beta) \subseteq \mathcal{S}^\star(\beta')$ for $0 \le \beta' \le \beta < 1$ and therefore, the functional $\max_{f \in \mathcal{S}^\star(\beta)} \Delta\left(r, \frac{z}{f(z)}\right)$ is monotone trivially and is also continuous as a function of the parameter β, since the maximizing hypergeometric function for the functional is continuous and convex. This observation shows that it suffices to present a proof of Theorem 3 for $\beta \ne 1/2$ since the proof for the case $\beta = 1/2$ follows as a limiting case. Because of its independent interest, a direct proof of Theorem 3 for the case $\beta = 1/2$ has been given in Theorem 2.

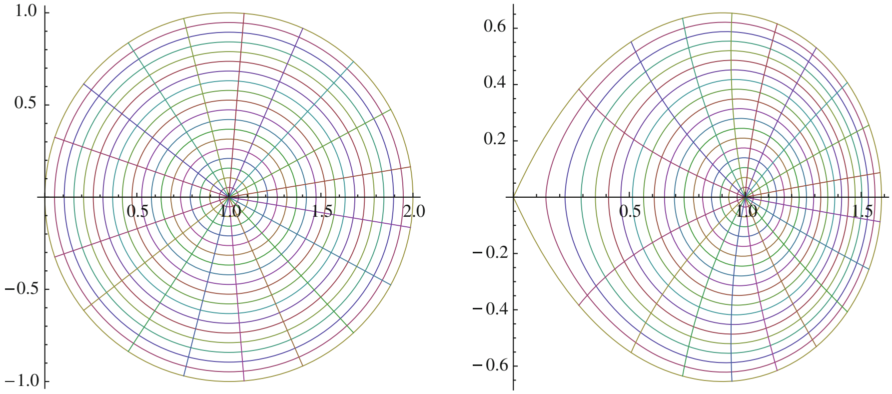

Fig. 2 Images of the unit disk under $g_{1/2}$ and $g_{2/3}$

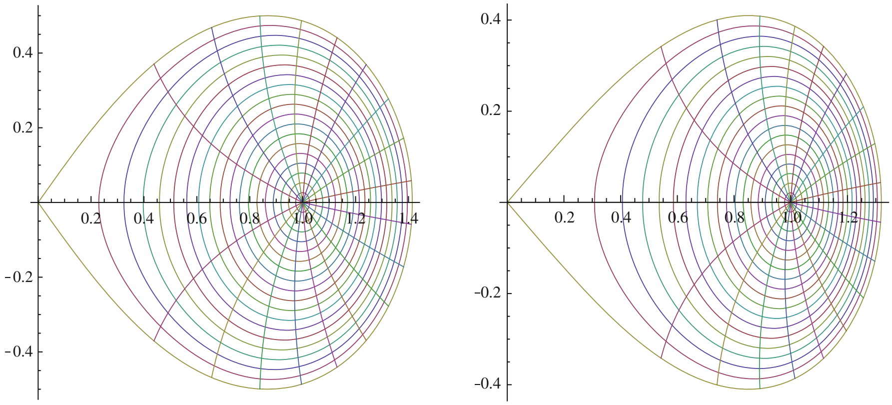

Fig. 3 Images of the unit disk under $g_{3/4}$ and $g_{4/5}$

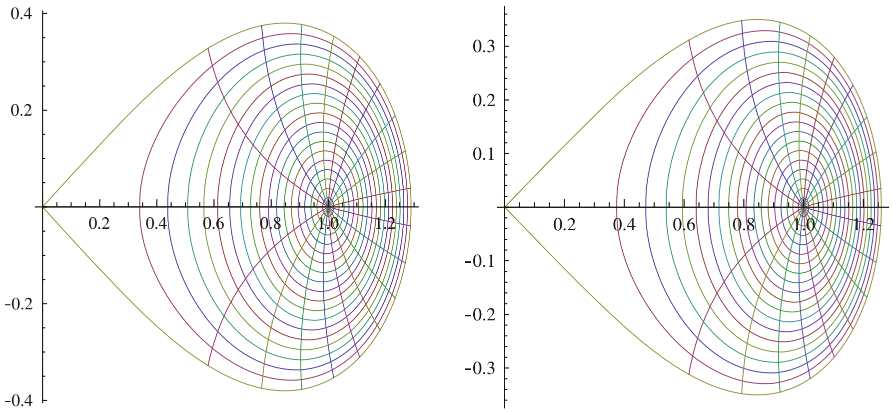

Fig. 4 Images of the unit disk under $g_{49/60}$ and $g_{5/6}$

Finally, we would like to point out that it is possible to refine Theorem A for a special situation when the second coefficient of $f \in \mathcal{S}$ is fixed, for example. Let $\mathcal{S}_+$ denote the subclass of $\mathcal{S}$ so that $f \in \mathcal{S}_+$ has the form (4) with $b_n \geq 0$ for $n \geq 2$.

Theorem 4 *Let $f \in \mathcal{S}_+$ and have the form (4), where $b_n \geq 0$ for $n \geq 2$. Then*

$$\max_{f \in \mathcal{S}_+} \Delta\left(r, \frac{z}{f(z)}\right) = \pi r^2\left(|b_1|^2 + 2r^2\right).$$

The result is sharp.

2 Proofs of Theorems 3 and 4

As remarked above, it suffices to prove the theorem for $\beta \neq 1/2$. Moreover, our proof below covers the case $\beta = 1/2$ (see also Theorem 2). It is a simple exercise to see that $f \in \mathcal{S}^\star(\beta)$ $(\beta \in [0, 1))$ if and only if F defined by

$$F(z) = z\left(\frac{f(z)}{z}\right)^{1/(1-\beta)}$$

belongs to $\mathcal{S}^\star$. Further, $F \in \mathcal{S}^\star$ implies that

$$\frac{z}{F(z)} = \left(\frac{z}{f(z)}\right)^{1/(1-\beta)} \prec (1-z)^2, \text{ i.e. } \frac{z}{f(z)} \prec \frac{z}{f_\beta(z)} = (1-z)^{2(1-\beta)},$$

where $\prec$ denotes the usual subordination. Extension of Rogosinki's result [13] observed by Goluzin [3, Thm. 6.3, p. 193] shows that if

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \text{and} \quad \frac{z}{f_\beta(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

then by the assumption and the above observations one has

$$\sum_{n=1}^{\infty} n|b_n|^2 r^{2n} \leq \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}$$

whenever the sequence $\{nr^{2n}\}$ is non-increasing. This gives the condition that $0 < r \leq 1/\sqrt{2}$. Thus, the theorem is obviously true for $0 < r \leq 1/\sqrt{2}$. On the other hand, in order to present a proof to include the case $r > 1/\sqrt{2}$, it suffices to prove

$$\sum_{n=1}^{N} n|b_n|^2 r^{2n} \leq \sum_{n=1}^{N} n|c_n|^2 r^{2n}$$

for any $N \in \mathbb{N}$ and for each $r \in (0, 1)$. This follows from the next lemma and the proof of Theorem 3 is complete. □

Since the contents of the following lemma are implied by Yamashita's theorem in the case $\beta = 0$ and by Theorem 2 in the case $\beta = \frac{1}{2}$ and these cases would be the reason for some complications, we exclude these cases in the considerations below.

Lemma 2 *Let* $\beta \in (0, 1) \setminus \left\{\frac{1}{2}\right\}$ *and* $f \in \mathcal{S}^\star(\beta)$. *Let further*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \ \ and \ \ (1+z)^{2-2\beta} = 1 + \sum_{k=1}^{\infty} (-1)^k c_k z^k,$$

and $r \in (0, 1)$. *Then for any* $N \in \mathbb{N}$ *the inequality*

$$\sum_{k=1}^{N} k|b_k|^2 r^{2k} \leq \sum_{k=1}^{N} k|c_k|^2 r^{2k} \tag{6}$$

is valid.

Proof We will divide the proof into several steps. But first we describe the plan of the proof. In Step 1, following the proof of Lemma 1, we derive inequalities for certain weighted sums of the moduli $|b_k|^2$ by a method due to Clunie [1], Robertson [12] and Rogosinski [13] which can be found in Pommerenke [11, Thm. 2.2]. In Step 2 we multiply these inequalities by factors such that the addition of the left side of these modified inequalities will result in the left side of (6). The multipliers will be evaluated by Cramer's rule. In Step 3 we shall show that these multipliers are all positive.

Step 1: Following the proof of Lemma 1, we let $g(z) = z/f(z)$ and obtain

$$\sum_{k=1}^{n-1} |b_k|^2 r^{2k-2}(k^2 - (k - 2(1-\beta))^2 r^2) + |b_n|^2 r^{2n-2} n^2 \leq 4(1-\beta)^2.$$

If we multiply these inequalities by r^2 and set as an abbreviation $\gamma = 2(1-\beta) \in (0, 2) \setminus \{1\}$, we get the inequalities

$$\sum_{k=1}^{n-1} |b_k|^2 r^{2k} \left(k^2 - (k-\gamma)^2 r^2\right) + |b_n|^2 r^{2n} n^2 \leq r^2 \gamma^2. \tag{7}$$

As the function $b(z) = (1+z)^{2-2\beta}$ satisfies the differential equation

$$b'(z) = \gamma b(z) - z b'(z), \quad z \in \mathbb{D},$$

it is clear that in the inequalities (7) equality is attained for $b_k = (-1)^k c_k$.

Step 2: Cramer's rule.

We consider the inequalities (7) for $n = 1, \ldots, N$ and multiply the n-th inequality by a factor $\lambda_{n,N}$. These factors are chosen such that the addition of the left sides of the modified inequalities results in the left side of (6). For the calculation of the factors $\lambda_{n,N}$ we get the following system of linear equations

$$k = k^2 \lambda_{k,N} + \sum_{n=k+1}^{N} \lambda_{n,N} \left(k^2 - (k-\gamma)^2 r^2\right), \quad k = 1, \ldots, N. \tag{8}$$

Since the matrix of this system is an upper triangular matrix with positive integers as diagonal elements, the solution of this system is uniquely determined.

Cramer's rule allows us to write the solution of the system (8) in the form

$$\lambda_{n,N} = \frac{((n-1)!)^2}{(N!)^2} \operatorname{Det} A_{n,N},$$

where $A_{n,N}$ is the $(N-n+1) \times (N-n+1)$ matrix constructed as follows. In the first column are the positive integers $n, n+1, \ldots, N$. In the rest of the first row stands $(N-n)$ times the constant $n^2 - (n-\gamma)^2 r^2$. The rest of the matrix is an upper triangular matrix, the main diagonal consists of the squares $(n+1)^2, \ldots, N^2$. In the rest of the second row stands $(N-n-1)$ times the constant $(n+1)^2 - (n+1-\gamma)^2 r^2$, the next row ends with the constant $(n+2)^2 - (n+2-\gamma)^2 r^2$, and so on, i.e.,

$$A_{n,N} = \begin{pmatrix} n & n^2-(n-\gamma)^2r^2 & \ldots & n^2-(n-\gamma)^2r^2 \\ n+1 & (n+1)^2 & \ldots & (n+1)^2-(n+1-\gamma)^2r^2 \\ \vdots & \vdots & \vdots & \vdots \\ N & 0 & \ldots & N^2 \end{pmatrix}.$$

The evaluation of the determinants of these matrices can be done by expanding according to Laplace's rule with respect to the last row, wherein the first coefficient is N and the last one is N^2. The rest of the entries are zeros. This expansion and a little mathematical induction results in the following formula. If $k \le N-1$, then

$$\lambda_{k,N} = \lambda_{k,N-1} - \frac{1}{N}\left(1 - \left(1-\frac{\gamma}{k}\right)^2 r^2\right) \prod_{m=k+1}^{N-1} \left(\left(1-\frac{\gamma}{m}\right)^2 r^2\right).$$

We see that for fixed $k \in \mathbb{N}$, $N \ge k$, the sequence $\{\lambda_{k,N}\}$ is a strictly decreasing sequence with

$$\lambda_k := \lim_{N\to\infty} \lambda_{k,N} = \frac{1}{k} - \left(1 - \left(1-\frac{\gamma}{k}\right)^2 r^2\right) \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\left(1-\frac{\gamma}{m}\right)^2 r^2\right).$$

To prove that $\lambda_{k,N} > 0$ for all $N \in \mathbb{N}$, $1 \le k \le N$, it is sufficient to prove $\lambda_k \ge 0$ for $k \in \mathbb{N}$. This will be done in Step 3. But before that we want to remark that the proof of this inequality is sufficient for the proof of the theorem, since, as we remarked in Step 1, equality is attained for $b_k = (-1)^k c_k$.

Step 3: Positivity of the multipliers.
Let us use the abbreviation

$$S_k = \sum_{n=k+1}^{\infty} \frac{1}{n} \prod_{m=k+1}^{n-1} \left(\left(1 - \frac{\gamma}{m}\right)^2 r^2 \right)$$

for $k \in \mathbb{N} \cup \{0\}$. We want to prove that for $k \in \mathbb{N}$

$$S_k \leq \frac{1}{k\left(1 - \left(1 - \frac{\gamma}{k}\right)^2 r^2\right)}. \tag{9}$$

The identity

$$S_{k-1} = \frac{1}{k} + \left(1 - \frac{\gamma}{k}\right)^2 r^2 S_k$$

implies that (9) is equivalent with

$$S_{k-1} \leq \frac{1}{k\left(1 - \left(1 - \frac{\gamma}{k}\right)^2 r^2\right)}. \tag{10}$$

To prove (10) we use the inequality

$$\frac{1}{n\left(1 - \left(1 - \frac{\gamma}{n}\right)^2 r^2\right)} > \frac{1}{(n+1)\left(1 - \left(1 - \frac{\gamma}{n+1}\right)^2 r^2\right)} \tag{11}$$

and the identity

$$\frac{1}{n\left(1 - \left(1 - \frac{\gamma}{n}\right)^2 r^2\right)} = \frac{1}{n} + \frac{\left(1 - \frac{\gamma}{n}\right)^2 r^2}{n\left(1 - \left(1 - \frac{\gamma}{n}\right)^2 r^2\right)}, \tag{12}$$

which are valid for each $n \in \mathbb{N}$. Repeated application of (11) and (12) for $n = k, k+1, \ldots, K$ results in the inequality

$$\frac{1}{k\left(1 - \left(1 - \frac{\gamma}{k}\right)^2 r^2\right)} > \sum_{n=k}^{K} \frac{1}{n} \prod_{m=k}^{n-1} \left(\left(1 - \frac{\gamma}{m}\right)^2 r^2 \right)$$
$$+ \frac{\prod_{m=k}^{K} \left(\left(1 - \frac{\gamma}{m}\right)^2 r^2 \right)}{K\left(1 - \left(1 - \frac{\gamma}{K}\right)^2 r^2\right)} =: s_{k,K} + R_{k,K}.$$

Since $0 < R_{k,K} < 1/(K(1-r^2))$ and $\lim_{K\to\infty} s_{k,K} = S_{k-1}$, these inequalities for $k \leq K$ imply the inequality (10). According to the above considerations in Step 1 and Step 2, the proof of the lemma is complete. □

Proof of Theorem 4. It is known that [8]

$$f \in \mathcal{S}_+ \Longleftrightarrow \sum_{n=2}^{\infty}(n-1)b_n \leq 1.$$

Now, we have

$$\begin{aligned}
\pi^{-1}\Delta\left(r, \frac{z}{f(z)}\right) &= \sum_{n=1}^{\infty} n|b_n|^2 r^{2n} = |b_1|^2 r^2 + \sum_{n=2}^{\infty} n b_n^2 r^{2n} \\
&\leq |b_1|^2 r^2 + r^4 \sum_{n=2}^{\infty} n b_n^2 \\
&\leq |b_1|^2 r^2 + 2r^4 \sum_{n=2}^{\infty} (n-1) b_n \\
&\leq |b_1|^2 r^2 + 2r^4,
\end{aligned}$$

since $0 \leq b_n \leq 1$ and $nb_n^2 \leq 2(n-1)b_n$ for $n = 2, 3, \ldots$. For the function f_0 defined by $\frac{z}{f_0(z)} = 1 + bz + z^2$, where $-2 \leq b \leq 2$, we have the equality:

$$\pi^{-1}\Delta\left(r, \frac{z}{f_0(z)}\right) = r^2\left(|b|^2 + 2r^2\right).$$

□

It would be interesting for the reader to recall that a variety of probabilistic processes can be interpreted in terms of the analytic fixed point function of the form $z/f(z)$ in the unit disk. Moreover, many useful probabilities can be expressed in terms of the Taylor coefficients of $z/f(z)$, its derivatives, or their combinations and as a consequence of it, certain known inequalities for such combinations allow us to find explicit estimates for probabilities. For example, the authors in [5–7, 14, 15] recently have found many interesting applications in the theory of the analytic fixed point function and even in questions in probability.

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