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Coefficient Criteria for Univalent and Close-to-Convex Functions

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In this paper, we provide sufficient conditions for constructing univalent analytic functions in the unit disk $|z| < 1$. We motivate our results through several examples and compare with the previously known coefficient conditions. Finally as an application, we present an interesting theorem involving Gaussian hypergeometric function.

KEYWORDS: univalent, starlike and close-to-convex functions, Gaussian hypergeometric functions

1. Introduction and Main Results

Let $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the family of all functions analytic in $\mathbb{D}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$, and set

$$\mathcal{S} = \{f \in \mathcal{A}\colon f \text{ is univalent in } \mathbb{D}\}.$$

The class $\mathcal{S}$ has been central in the development of geometric function theory since Bieberbach stated the conjecture $|a_n| \le n$ for $n \ge 2$, for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, with equality precisely when $f(z)$ equals the Koebe function $k(z) = z/(1-z)^2$ or its rotation $e^{-i\theta}k(ze^{i\theta})$. This conjecture was finally solved in 1985 by de Branges [1]. Deriving sufficient coefficient conditions for f to belong to $\mathcal{S}$ or some of its natural geometric subclasses (such as convex, starlike and close-to-convex) has been some of the important issues in the theory of univalent functions. For example if $\sum_{n=2}^{\infty} n|a_n| \le 1$ then $f(z)$ defined by the above (normalized) power series satisfies the condition $|f'(z) - 1| < 1$ in $\mathbb{D}$ and moreover, $f(\mathbb{D})$ is starlike. This sufficient condition is also necessary for the range $f(\mathbb{D})$ to be a starlike domain whenever $a_n < 0$ for all $n \ge 2$ (cf. [7]). Recall that $f \in \mathcal{S}$ is called starlike if the range $f(\mathbb{D})$ is starlike (with respect to the origin), and is analytically characterized by the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ in } \mathbb{D}.$$

For instance the Koebe function $k(z) = z/(1-z)^2$ is starlike in $\mathbb{D}$. A univalent function g (need not be normalized) is said to be convex if the range $g(\mathbb{D})$ is a convex domain. A function $f \in \mathcal{A}$ is called close-to-convex if $\mathrm{Re}(e^{i\theta}zf'(z)/g(z)) > 0$ on $|z| < 1$ for some $\theta \in \mathbb{R}$ and for some starlike function $g \in \mathcal{S}$. It is known that close-to-convex functions are univalent in $\mathbb{D}$, but not necessarily the converse. Moreover, it is convenient (or rather enough) to show that f is close-to-convex in order to check the univalency of f.

Theorem 1. *Let $f \in \mathcal{A}$ and g be a convex univalent function in $\mathbb{D}$ such that $m = \inf_{z\in\mathbb{D}} |g'(z)| > 0$. If*

$$|f'(z) - g'(z)| < m \text{ for } z \in \mathbb{D},$$

then f is close-to-convex (with respect to g) in $\mathbb{D}$.

Proof. From the hypothesis we get,

$$|f'(z) - g'(z)| < |g'(z)| \text{ for } z \in \mathbb{D}.$$

Since $g'(z) \neq 0$ in $\mathbb{D}$, the above calculation gives $\left|\frac{f'(z)}{g'(z)} - 1\right| < 1$ which implies that f is close-to-convex in $\mathbb{D}$. □

In [5], Nicolae N. Pascu obtained the following result:

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Theorem 1.1. *Let f and g be analytic in $\mathbb{D}$. If there exists an $m > 0$ such that*

$$|f'(z) - g'(z)| \leq m \leq |\mathrm{Re}\{g'(z)\}|$$

for every $z \in \mathbb{D}$, then f is univalent in $\mathbb{D}$.

Proof. Since the theorem is not well-known, it may be appropriate to outline a proof of this result. By hypothesis, we have $|\mathrm{Re}\{g'(z)\}| \geq m > 0$ and hence, without loss of generality, we may assume that $\mathrm{Re}\, g'(z) \geq m$. Since $\mathrm{Re}\, g'(z) \geq m > 0$, it follows from the Noshiro–Warschawski Theorem that g is univalent in $\mathbb{D}$. Then for each $z_0, z_1 \in \mathbb{D}$ with $z_1 \neq z_0$, one has

$$\begin{aligned}
|f(z_1) - f(z_0) - (g(z_1) - g(z_0))| &= \left|(z_1 - z_0)\int_0^1 \{f'((1-t)z_0 + tz_1) - g'((1-t)z_0 + tz_1)\}\, dt\right| \\
&\leq |z_1 - z_0| \int_0^1 |f'((1-t)z_0 + tz_1) - g'((1-t)z_0 + tz_1)|\, dt \\
&\leq |z_1 - z_0| \int_0^1 \mathrm{Re}\{g'((1-t)z_0 + tz_1)\}\, dt \\
&= \mathrm{Re}\left\{e^{-i\arg(z_1 - z_0)}(z_1 - z_0)\int_0^1 g'((1-t)z_0 + tz_1)\, dt\right\} \\
&= \mathrm{Re}\{e^{-i\arg(z_1 - z_0)}(g(z_1) - g(z_0))\} \\
&\leq |g(z_1) - g(z_0)|.
\end{aligned}$$

In the above estimates if the first quantity is equal to the last quantity, then this case can happen only when f and g are linear functions. Thus in this case f and g are univalent. Otherwise, we obtain the following strict inequality

$$|f(z_1) - f(z_0) - (g(z_1) - g(z_0))|| < |g(z_1) - g(z_0)|.$$

Since g is univalent in $\mathbb{D}$, it follows that $g(z_1) - g(z_0) \neq 0$ and so, we have

$$\left|\frac{f(z_1) - f(z_0)}{g(z_1) - g(z_0)} - 1\right| < 1.$$

Thus we have $f(z_1) - f(z_0) \neq 0$ and so f is univalent in $\mathbb{D}$. □

According to Theorem 1.1, univalency of f can be achieved without g being convex.

Theorem 2. *Let $f \in \mathcal{A}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Suppose that g is analytic and convex (univalent) in $\mathbb{D}$, such that $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $m = \inf_{z\in\mathbb{D}} |g'(z)| > 0$. If*

$$\sum_{n=2}^{\infty} n|a_n - b_n| < m - |1 - b_1|, \tag{1.1}$$

then f is close-to-convex in $\mathbb{D}$.

Proof. Using the power series expansion for $f(z)$ and $g(z)$, we see that

$$\begin{aligned}
|f'(z) - g'(z)| &= \left|1 - b_1 + \sum_{n=2}^{\infty}(na_n - nb_n)z^{n-1}\right| \\
&\leq |1 - b_1| + \sum_{n=2}^{\infty} |na_n - nb_n| < m.
\end{aligned}$$

Hence by the hypothesis and Theorem 1, f is close-to-convex in $\mathbb{D}$. □

We remark that the univalency of f follows under the condition of (1.1) with g being convex. Moreover, by using the Theorem 1.1, we can easily obtain the following result.

Theorem 3. *Let $f \in \mathcal{A}$, and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If*

$$\sum_{n=2}^{\infty} |na_n - \beta| \leq 1 - \frac{\beta}{2}, \tag{1.2}$$

for some β with $0 \leq \beta < 2$, then f is univalent in $\mathbb{D}$.

Proof. Let $g(z) = z - \beta \log(1 - z)$. Then $\mathrm{Re}\, g'(z) > 1 + \beta/2$ for $z \in \mathbb{D}$. Now,

$$|f'(z) - g'(z)| = \left|\beta + \sum_{n=2}^{\infty}(na_n - \beta)z^{n-1}\right|$$
$$< \beta + \sum_{n=2}^{\infty}|na_n - \beta| < 1 + \beta/2.$$

Thus, by Theorem 1.1, we see that f is univalent in $\mathbb{D}$. □

The case $\beta = 1$ clearly gives the following.

Corollary 1. *Let $f \in \mathcal{A}$ and let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If*

$$\sum_{n=2}^{\infty} |na_n - 1| \leq \frac{1}{2}, \tag{1.3}$$

then f is univalent in $\mathbb{D}$.

Remark 1. *If we take $\beta = 0$ in Theorem 3, then we get that the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in $\mathbb{D}$ whenever*

$$\sum_{n=2}^{\infty} n|a_n| \leq 1. \tag{1.4}$$

But we know that if the coefficients of f satisfies the inequality (1.4)*, then f is also close-to-convex and starlike in $\mathbb{D}$. Therefore, it is natural to ask whether the condition on g in Theorem 1, or the coefficient condition* (1.1) *in Theorem 2 ensures the starlikeness of f.*

Next, we present an application of our coefficient inequality. For complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function defined by the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^n$$

is analytic in $|z| < 1$, where $(a, 0) = 1$ and $(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$. Also, for $\operatorname{Re} c > \operatorname{Re}(a + b)$, we have

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c - a - b)}{\Gamma(c - a)\,\Gamma(c - b)}.$$

We have

$$F'(a, b; c; z) = \left(\frac{ab}{c}\right) F(a + 1, b + 1; c + 1; z). \tag{1.5}$$

Theorem 4. *Let a, b, and c be either positive real numbers satisfying $c > a + b + 1$ or $a, b \in \mathbb{C}$, $c \in \mathbb{R}$ with $b = \overline{a}$ and $c > 2\operatorname{Re} a + 1$. Then the analytic function*

$$f(z) = zF(a, b; c; z) - \log(1 - z) - z \tag{1.6}$$

is univalent in $\mathbb{D}$, whenever

$$\frac{\Gamma(c)\Gamma(c - a - b - 1)}{\Gamma(c - a)\Gamma(c - b)} [ab + c - a - b - 1] \leq \frac{3}{2}. \tag{1.7}$$

Proof. Let $f(z) = \phi(z) - \log(1 - z) - z = z + \sum_{n=2}^{\infty} a_n z^n$, where

$$\phi(z) = zF(a, b; c; z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{1.8}$$

Then

$$f'(z) = \frac{1}{1 - z} + \sum_{n=2}^{\infty} nb_n z^{n-1}$$

so that $a_1 = 1$ and $na_n = nb_n + 1$ for all $n \geq 2$. Thus, by Corollary 1, it suffices to show that $\sum_{n=2}^{\infty} nb_n \leq 1/2$. Now from (1.8) we have,

$$\sum_{n=2}^{\infty} nb_n z^{n-1} = zF'(a, b; c; z) + F(a, b; c; z) - 1.$$

Now letting $z \to 1^-$, we get

$$\begin{aligned}\sum_{n=2}^{\infty} nb_n &= F'(a,b;c;1) + F(a,b;c;1) - 1 \\ &= \frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \text{ (Using (1.5))} \\ &= \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}[ab+c-a-b-1] - 1\end{aligned}$$

and therefore, by (1.7) it follows that $\sum_{n=2}^{\infty} |na_n - 1| \le 1/2$. The conclusion follows. □

Remark 2. *If* $a, b > 0$, $T_1(a,b) = \max\{a+b, a+b+(ab-1)/2, 2ab\}$ *and* c *satisfies either*

$$c \ge T_1(a,b), \tag{1.9}$$

or $c = a + b$ *with*

$$ab \ge 1,\ a+b \le 2ab \text{ and } \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \le 1, \tag{1.10}$$

then the analytic function f *defined by* (1.6) *is close-to-convex with respect to* $-\log(1-z)$. *Indeed, from the proof of Theorem 4, we have* $na_n = nb_n + 1$ *and so*

$$na_n - (n+1)a_{n+1} = nb_n - (n+1)b_{n+1}$$

and hence the first part of the remark follows from Theorem 2.1 [6], so we omit the details. Similarly the second part of the last remark, follows easily from the proof of the second part of Theorem 2.1 [6]. However, it seems that (1.10) *does not hold for values other than* $a = 1$ *and* $b = 1$.

In order to demonstrate the usefulness of our coefficient condition (1.3), we present an example.

Example 1.2. Let f be an analytic function in the unit disk $\mathbb{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n = \left(\frac{1}{\lambda^{n-1}} + \frac{a}{\mu^{n-1}}\right)\frac{1}{n}, \tag{1.11}$$

where λ, μ are real numbers greater than or equal to 1 and a is a positive real number. If $\lambda(\mu+1) \ge a(\lambda+1)$, then it can be easily seen that f is close-to-convex in $\mathbb{D}$ by Noshiro–Warchawski–Wolff univalence criterion [2]. Indeed, a computation reveals

$$f'(z) = 1 + \frac{z}{\lambda - z} + \frac{az}{\mu - z}$$

and so

$$\operatorname{Re} f'(z) > f'(-1) = 1 - \frac{1}{\lambda+1} - \frac{a}{\mu+1} = \frac{\lambda(\mu+1) - a(\lambda+1)}{(\lambda+1)(\mu+1)},\ z \in \mathbb{D},$$

which shows that $\operatorname{Re} f'(z) > 0$ if and only if $\lambda(\mu+1) \ge a(\lambda+1)$.

Consider the special choice $\lambda = 1$. Then we have $\operatorname{Re} f'(z) > 0$ if and only if $a \le \frac{\mu+1}{2}$. Also we have,

$$\sum_{n=2}^{\infty} |na_n - 1| = \sum_{n=2}^{\infty} \frac{a}{\mu^{n-1}} = \frac{a}{\mu - 1},$$

where $\mu \neq 1$. Therefore, by Corollary 1, f is univalent in $\mathbb{D}$ if $a \le \frac{\mu-1}{2}$.

Moreover, for the a_n's given by (1.11), a simple calculation gives,

$$\sum_{n=2}^{\infty} |na_n - (n+1)a_{n+1}| = \begin{cases} \dfrac{1}{\lambda} + \dfrac{a}{\mu}, & \text{if } \lambda > 1,\ \mu > 1 \\ \dfrac{1}{\lambda}, & \text{if } \lambda > 1,\ \mu = 1 \\ \dfrac{a}{\mu}, & \text{if } \lambda = 1,\ \mu > 1. \end{cases}$$

If either $a\lambda \le \mu(\lambda - 1)$ with $\lambda > 1$ and $\mu > 1$, or $a \le \mu$ with $\lambda = 1$ and $\mu > 1$, holds then by a well-known result [4] (see also [6]) the function f defined by (1.11) is close-to-convex with respect to $-\log(1-z)$ in $\mathbb{D}$.

In particular, for $\lambda = 1$, $a = 1/2$, and $\mu = 2$, the function f defined by (1.11) takes the form

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{n}\left(\frac{1}{2^n} + 1\right) z^n$$

so that

$$f'(z) = 1 + \sum_{n=2}^{\infty}\left(\frac{1}{2^n} + 1\right) z^{n-1} = 1 + \frac{z}{1-z} + \frac{z}{2(2-z)}$$

and therefore,

$$\sum_{n=2}^{\infty} |na_n - 1| = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}.$$

Moreover, for this function we see that $\mathrm{Re}\, f'(z) > 1/3$ in $\mathbb{D}$ and $|f'(z) - 1| \to +\infty$ when z approaches 1^- through real values of z. Also in this case,

$$\sum_{n=2}^{\infty} n|a_n| = \sum_{n=2}^{\infty}\left(\frac{1}{2^n} + 1\right) = +\infty,$$

showing that the well-known criterion $\sum_{n=2}^{\infty} n|a_n| \leq 1$ (implying univalency and starlikeness of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$) is not satisfied.

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REFERENCES

[1] de Branges, L., “A proof of the Bieberbach conjecture,” *Acta Math.*, **154(1–2)**: 137–152 (1985).
[2] Duren, P. L., Univalent Functions (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag (1983).
[3] Goodman, A. W., Univalent Functions, Vols. 1–2, Mariner, Tampa, Florida (1983).
[4] Ozaki, S., “On the theory of multivalent functions,” *Sci. Rep. Tokyo Bunrika Daigaku*, **2**: 167–188 (1935).
[5] Pascu, N. N., Criteria for Univalence (Romanian language), University of Transilvania, Brasov (2003).
[6] Ponnusamy, S., and Vuorinen, M., “Univalence and convexity properties for Gaussian hypergeometric functions,” *Rocky Mountain J. Math.*, **31(1)**: 327–353 (2001).
[7] Silverman, H., “Univalent functions with negative coefficients,” *Proc. Amer. Math. Soc.*, **51**: 109–116 (1975).