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Where is $f(z)/f'(z)$ univalent?

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Dedicated to the bright memory of Late Professor Vikramaditya Singh

Abstract. Let $\mathcal{S}$ denote the family of all univalent functions f in the unit disk $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. There is an intimate relationship between the operator $P_f(z) = f(z)/f'(z)$ and the Danikas-Ruscheweyh operator $T_f := \int_0^z (tf'(t)/f(t))\,dt$. In this paper we mainly consider the univalence problem of $F = P_f$, where f belongs to some subclasses of $\mathcal{S}$. Among several sharp results and non-sharp results, we also show that if $f \in \mathcal{S}$, then $F \in \mathcal{U}$ in the disk $|z| < r$ with $r \leq r_6 \approx 0.360794$ and conjecture that the upper bound for such r is $\sqrt{2} - 1$.

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1. Introduction and Main Results

Let $\mathcal{B}$ denote the class of analytic functions $\omega(z)$ in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. If f, g are two analytic functions in $\mathbb{D}$, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an $\omega \in \mathcal{B}$ such that $f(z) = g(\omega(z))$. We also note that if g is univalent, then it is easy to show that $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

We consider the family $\mathcal{A}$ of all functions f analytic in $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. By $\mathcal{S}$, $\mathcal{S} \subset \mathcal{A}$, we denote the class of univalent functions in $\mathbb{D}$. Certain special subclasses of $\mathcal{S}$ possess various remarkable features due to their geometrical properties. By $\mathcal{C}$, $\mathcal{K}$, and $\mathcal{S}^\star$ we denote the subclasses of $\mathcal{S}$ which consist of convex, close-to-convex, and starlike functions, respectively. For $\beta \in [0, 1)$, let $\mathcal{S}^\star(\beta)$ denote the usual normalized class of all (univalent) starlike

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functions of order β. Analytically, $f \in \mathcal{S}^\star(\beta)$ if $f \in \mathcal{A}$ and satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in \mathbb{D}.$$

It is well-known that $\mathcal{C} \subsetneq \mathcal{S}^\star(1/2)$, and $\mathcal{S}^\star := \mathcal{S}^\star(0)$. At this point it is interesting to note that a function belonging to $\mathcal{S}^\star(1/2)$ may not be convex in $|z| < R$ for any $R > \sqrt{2\sqrt{3}-3} = 0.68\ldots$, see [8, Theorem 1]. We say that $f \in \mathcal{A}$ is starlike in $|z| < r$ (i.e. to say $f \in \mathcal{S}^\star$ in $|z| < r$) for some $0 < r \leq 1$, if $f(|z| < r)$ is starlike with respect to the origin. This means that the last subordination condition is satisfied for $|z| < r$ instead of the full disk $|z| < 1$. Similar convention will be followed for other classes. We refer to [3, 4, 11] for a detailed discussion on these classes. Also let us introduce some notations and definitions as follows:

$$\begin{aligned}
\mathcal{U} &= \{f \in \mathcal{A}:\, |U_f(z)| < 1 \ \text{ for } z \in \mathbb{D}\}\,,\ U_f(z) = f'(z)\left(\frac{z}{f(z)}\right)^2 - 1,\\
\mathcal{C}(-1/2) &= \left\{f \in \mathcal{A}:\, \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \ \text{ for } z \in \mathbb{D}\right\}, \text{ and}\\
\mathcal{G} &= \left\{f \in \mathcal{A}:\, \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \ \text{ for } z \in \mathbb{D}\right\}.
\end{aligned}$$

According to Aksentév's theorem [1] (see also [10]), the strict inclusion $\mathcal{U} \subsetneq \mathcal{S}$ holds. In a recent paper the authors in [14] discussed the class $\mathcal{U}(\lambda)$ in geometric perspectives.

Moreover, $\mathcal{C}(-1/2) \subset \mathcal{K}$, and functions in $\mathcal{G}$ are proved to be starlike in $\mathbb{D}$, see for eg. [12, Example 1, Equation (16)]. See also [7] for further details and investigation on the class $\mathcal{G}$.

This article concerns with the operator

$$F(z) := P_f(z) = \frac{f(z)}{f'(z)} \tag{1.1}$$

for locally univalent functions $f \in \mathcal{A}$. The main problem is to consider the univalency and starlikeness of P_f when f belongs to some of the subclasses of $\mathcal{S}$ defined above.

Among others our interest in the operator P_f arose from the fact that there exists an intimate relation between this one and the Danikas-Ruscheweyh ([2]) operator

$$T_f(z) := \int_0^z \frac{tf'(t)}{f(t)}\,dt = z + \sum_{n=1}^{\infty} \frac{n}{n+1} c_n(f) z^{n+1} \quad (f \in \mathcal{S}), \tag{1.2}$$

where $c_n(f)$ $(n \geq 1)$ denote the logarithmic coefficients of $f \in \mathcal{S}$ defined by

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n.$$

The conjecture that $T_f \in \mathcal{S}$ for each $f \in \mathcal{S}$ remains open.

The relation between (1.1) and (1.2) becomes obvious, when one considers the equivalent operators in the w-plane where $w = f(z)$. Let $g(w) = f^{-1}(w)$ be the function inverse to f. If we transform the operator P_f to the w-plane, we get the operator

$$Q(g)(w) = wg'(w) = q(w).$$

A similar consideration concerning the Danikas-Ruscheweyh operator results in

$$S(g)(w) = \int_0^w \frac{g(u)}{u}\, du = s(w).$$

Now it is immediately seen that

$$Q^{-1}(q)(w) = \int_0^w \frac{q(u)}{u}\, du = S(q)(w) \quad \text{and} \quad S^{-1}(s)(w) = ws'(w) = Q(s)(w).$$

2. Preliminaries and two examples

We remark that if $f \in \mathcal{S}$ then $(z/f(z)) \neq 0$ in $\mathbb{D}$ and hence, f can be represented as Taylor's series of the form

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}. \tag{2.1}$$

According to the well-known Area Theorem [4, Theorem 11 on p.193 of Vol. 2], for $f \in \mathcal{S}$ of the form (2.1), one has

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1 \tag{2.2}$$

but this condition is not sufficient for the univalence of f. On the other hand, if $f \in \mathcal{A}$ of the form (2.1) satisfies the condition

$$\sum_{n=2}^{\infty} (n-1)|b_n| \leq 1, \tag{2.3}$$

then $f \in \mathcal{U}$. The condition (2.3) is also necessary if $b_n \geq 0$ for $n \geq 1$. The constant 1 is the best possible in the sense that if

$$\sum_{n=2}^{\infty} (n-1)|b_n| = 1 + \varepsilon,$$

for some $\varepsilon > 0$, then there exists an f which is not univalent in $\mathbb{D}$.

Let us continue the discussion with two examples. Consider

$$f_1(z) = \frac{z(1-\frac{z}{2})}{(1-z)^2}, \quad \text{and} \quad f_2(z) = z - \frac{z^2}{2}.$$

Then $f_1 \in \mathcal{C}(-1/2)$ and $f_2 \in \mathcal{G}$. Define

$$F_j(z) = P_{f_j}(z) = \frac{f_j(z)}{f_j'(z)}, \quad \text{for } j = 1, 2,$$

so that

$$F_1(z) = z - \frac{3}{2}z^2 + \frac{1}{2}z^3 \quad \text{and} \quad F_2(z) = \frac{z(1-\frac{z}{2})}{1-z}.$$

1. We have that

$$F_1'(z) = \frac{3}{2}z^2 - 3z + 1 = \frac{3}{2}(z - r_+)(z - r_-), \; r_\pm = 1 \pm \frac{\sqrt{3}}{3}$$

and therefore $F_1'(r_-) = 0$, where $r_- = 1 - \frac{\sqrt{3}}{3} = 0.4226497\ldots$. We claim that $\mathrm{Re}\,(F_1'(z)) > 0$ for $|z| < r_-$. To do this, we observe that

$$\mathrm{Re}\,(F_1'(re^{i\theta})) = 3r^2\cos^2\theta - 3r\cos\theta + 1 - \frac{3}{2}r^2,$$

then it is easy to show that $\mathrm{Re}\,(F_1'(re^{i\theta})) > 0$ for $-1 \le \cos\theta \le 1$ and $0 \le r < r_-$. It means that F_1 is univalent in the disc $|z| < r_-$.

2. It is a simple exercise to see that $F_2 \in \mathcal{U}$. In fact,

$$\frac{z}{F_2(z)} = \frac{1-z}{1-\frac{z}{2}} = 1 - \frac{\frac{z}{2}}{1-\frac{z}{2}} = 1 - \frac{z}{2} - \sum_{n=2}^{\infty} b_n z^n, \quad b_n = \frac{1}{2^n},$$

so that $z/F_2(z)$ is non-vanishing in $\mathbb{D}$ and thus,

$$-z\left(\frac{z}{F_2(z)}\right)' + \frac{z}{F_2(z)} - 1 = \left(\frac{z}{F_2(z)}\right)^2 F_2'(z) - 1 = \left(\frac{\frac{z}{2}}{1-\frac{z}{2}}\right)^2$$

from which we easily see that $|U_{F_2}(z)| < 1$ for $z \in \mathbb{D}$. Indeed, by a direct computation, we see that the function $w = (z/2)/(1-(z/2))$ maps $\mathbb{D}$ onto the disk $|w - (1/3)| < 2/3$ so that $w \in \mathbb{D}$ and thus, $w^2 \in \mathbb{D}$. This observation gives that $|U_{F_2}(z)| < 1$ in $\mathbb{D}$ and hence, $F_2 \in \mathcal{U}$. Alternately, using the series expansion for F_2, we find that

$$\sum_{n=2}^{\infty}(n-1)|b_n| = \sum_{n=2}^{\infty}(n-1)\frac{1}{2^n} = 1$$

and, by the sufficient condition (2.3), it follows that $F_2 \in \mathcal{U}$.

3. Main results

Let $\omega \in \mathcal{B}$. Then by the Schwarz lemma it follows that $|\omega(z)| \leq |z|$ for $z \in \mathbb{D}$ and by the Schwarz-Pick lemma we have

$$|\omega'(z)| \leq \frac{1 - |\omega(z)|^2}{1 - |z|^2} \quad \text{for } z \in \mathbb{D}. \tag{3.1}$$

Clearly, $\frac{\omega(z)}{z}$ is analytic in $\mathbb{D}$ and $|\omega(z)/z| \leq 1$ in $\mathbb{D}$. The Schwarz-Pick lemma, namely, (3.1), applied to $\omega(z)/z$ shows that

$$|z\omega'(z) - \omega(z)| \leq \frac{|z|^2 - |\omega(z)|^2}{1 - |z|^2}. \tag{3.2}$$

These three inequalities will be used frequently in the proof of our main results.

Theorem 3.3. *If $f \in \mathcal{S}^\star(\beta)$, then $P_f \in \mathcal{U}$ in the disk $|z| < 1/(1 + \sqrt{2(1-\beta)})$. The result is sharp (as for univalence) as the function $z/(1-z)^{2(1-\beta)}$ shows.*

Proof. Each $f \in \mathcal{S}^\star(\beta)$ and $F = P_f$ defined by (1.1) can be written as

$$\frac{zf'(z)}{f(z)} = \frac{1 + (1-2\beta)\omega(z)}{1 - \omega(z)} \quad \text{and} \quad F(z) = \frac{z(1-\omega(z))}{1 + (1-2\beta)\omega(z)},$$

where $\omega \in \mathcal{B}$. Clearly, $\frac{\omega(z)}{z}$ is analytic in $\mathbb{D}$ and $|\omega(z)/z| \leq 1$ in $\mathbb{D}$. Using the last two relations, we observe that

$$U_F(z) = -z\left(\frac{z}{F(z)}\right)' + \frac{z}{F(z)} - 1 = \frac{zf'(z)}{f(z)} - z\left(\frac{zf'(z)}{f(z)}\right)' - 1 \tag{3.4}$$

and thus,

$$\begin{aligned} U_F(z) &= 2(1-\beta)\left(\frac{\omega(z)}{1-\omega(z)} - \frac{z\omega'(z)}{(1-\omega(z))^2}\right) \\ &= 2(1-\beta)\left(\frac{(\omega(z) - z\omega'(z)) - \omega^2(z)}{(1-\omega(z))^2}\right) \end{aligned}$$

from which and (3.2), we obtain that

$$\begin{aligned}
|U_F(z)| &\leq 2(1-\beta)\left(\frac{|\omega(z)-z\omega'(z)|}{(1-|\omega(z)|)^2}+\frac{|\omega(z)|^2}{(1-|\omega(z)|)^2}\right)\\
&\leq 2(1-\beta)\left(\frac{\frac{|z|^2-|\omega(z)|^2}{1-|z|^2}}{(1-|\omega(z)|)^2}+\frac{|\omega(z)|^2}{(1-|\omega(z)|)^2}\right)\\
&= \frac{2(1-\beta)|z|^2}{1-|z|^2}\left(\frac{1+|\omega(z)|}{1-|\omega(z)|}\right)\\
&\leq \frac{2(1-\beta)|z|^2}{1-|z|^2}\left(\frac{1+|z|}{1-|z|}\right)=\frac{2(1-\beta)|z|^2}{(1-|z|)^2}
\end{aligned}$$

which can easily seen to be less than 1 if $|z|<1/(1+\sqrt{2(1-\beta)})$. Thus, F belongs to $\mathcal{U}$ in the disk $|z|<1/(1+\sqrt{2(1-\beta)})$.

To prove the sharpness part, we consider $k_\beta(z)=z/(1-z)^{2(1-\beta)}$ and define

$$F_\beta(z)=P_{k_\beta}(z)=\frac{k_\beta(z)}{k'_\beta(z)}.$$

Then we see that $k_\beta\in\mathcal{S}^*(\beta)$ and

$$F_\beta(z)=\frac{z(1-z)}{1+(1-2\beta)z}\quad\text{and}\quad\frac{z}{F_\beta(z)}=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)\sum_{n=1}^{\infty}z^n.$$

Define $G_\beta(z)=\frac{1}{r}F_\beta(rz)$ and observe that

$$\frac{z}{G_\beta(z)}=1+2(1-\beta)\sum_{n=1}^{\infty}r^nz^n.$$

According to (2.3), the function G_β is in $\mathcal{U}$ (and hence is univalent in $\mathbb{D}$) if and only if

$$2(1-\beta)\sum_{n=2}^{\infty}(n-1)r^n\leq 1,\quad\text{i.e.}\quad\frac{2(1-\beta)r^2}{(1-r)^2}\leq 1.$$

The gives the condition $0<r\leq r_1=1/(1+\sqrt{2(1-\beta)})$. Thus, the function F_β is univalent in the disk $|z|<r_1$ and not in any larger disk with center at the origin. Note also that

$$F'_\beta(z)=\frac{1-2z-(1-2\beta)z^2}{(1+(1-2\beta)z)^2}$$

and thus, $F'_\beta(r_1)=0$. Moreover,

$$U_{F_\beta}(z)=\frac{1-2z-(1-2\beta)z^2}{(1-z)^2}-1$$

showing that $U_{F_\beta}(r_1) = -1$. Thus, the number r_1 is best both for univalence and also for $\mathcal{U}$. The proof is complete. ■

Corollary 3.5. *If $f \in \mathcal{S}^\star$, then $P_f \in \mathcal{U} \cap \mathcal{S}^\star$ in the disk $|z| < \sqrt{2}-1$. The result is sharp (as for univalence) as the Koebe function $z/(1-z)^2$ shows.*

Proof. It suffices to prove the starlikeness part since $P_f \in \mathcal{U}$ follows from Theorem 3.3 by taking $\beta = 0$. Thus, for the proof of the second part, it suffices to observe by (3.1) that

$$\left|\frac{zF'(z)}{F(z)} - 1\right| = \left|-\frac{2z\omega'(z)}{1-\omega^2(z)}\right| \leq \frac{2|z|\,|\omega'(z)|}{1-|\omega(z)|^2} \leq \frac{2|z|}{1-|z|^2}$$

which is again less than 1 provided $|z| < \sqrt{2}-1$. In particular, F is starlike in the disk $|z| < \sqrt{2}-1$. Sharpness part follows from the discussion in Theorem 3.3 with $\beta = 0$. ■

Corollary 3.6. *If $f \in \mathcal{S}^\star(1/2)$, then $P_f \in \mathcal{U} \cap \mathcal{S}^\star$ in the disk $|z| < 1/2$. The result is sharp as the function $z/(1-z)$ shows.*

Proof. Choose $\beta = 1/2$ in Theorem 3.3 and observe that it suffices to prove the starlikeness part. As in the proof of Theorem 3.3, for each $f \in \mathcal{S}^\star(1/2)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1}{1-\omega(z)} \quad \text{and} \quad F(z) = z(1-\omega(z))$$

for some $\omega \in \mathcal{B}$. By (3.1) and the fact that $|\omega(z)| \leq |z|$, we obtain

$$\begin{aligned}\left|\frac{zF'(z)}{F(z)} - 1\right| = \left|\frac{-z\omega'(z)}{1-\omega(z)}\right| \leq \frac{|z|\,|\omega'(z)|}{1-|\omega(z)|} &\leq \frac{|z|(1+|\omega(z)|)}{1-|z|^2} \\ &\leq \frac{|z|(1+|z|)}{1-|z|^2} = \frac{|z|}{1-|z|}\end{aligned}$$

which is less than 1 if $|z| < 1/2$. Note that for $f(z) = z/(1-z)$, one has $F(z) = z - z^2$ and thus, $|F'(z) - 1| = 2|z| < 1$ for $|z| < 1/2$ and $F'(1/2) = 0$. Thus, F is univalent in the disk $|z| < 1/2$ and not in any larger disk with center at the origin. Also, it is easy to see that $F(z)$ is starlike for $|z| < 1/2$. The desired conclusion follows. ■

Corollary 3.7. *If $f \in \mathcal{S}^\star(1/2)$ such that $f''(0) = 0$, then P_f is starlike in the disk $|z| < r_2$, where $r_2 \approx 0.543689$ is the root of the equation $\phi_2(r) = 0$, where*

$$\phi_2(r) = r^3 + r^2 + r - 1.$$

Proof. Clearly, we just need to apply Corollary 3.6 with $|\omega(z)| \leq |z|^2$. This will lead to the inequality

$$\left|\frac{zF'(z)}{F(z)} - 1\right| \leq \frac{|z|(1+|z|^2)}{1-|z|^2}$$

which is clearly less than 1 if $|z|^3 + |z|^2 + |z| - 1 < 0$. The result follows. ■

Corollary 3.8. *Let f belong to either $\mathcal{S}^\star(1/2)$ or $\mathcal{C}(-1/2)$, such that $f''(0) = 0$. Then $F \in \mathcal{U}$ in the disk $|z| < 1/\sqrt{3}$.*

Proof. It known that [9, p. 68] if $\mathcal{C}(-1/2)$ with $f''(0) = 0$, then $f \in \mathcal{S}^\star(1/2)$. In view of this result, it suffices to prove the corollary when f belongs to $\mathcal{S}^\star(1/2)$ with $f''(0) = 0$. However, using the proof of Theorem 3.3 with $\beta = 1/2$ and $|\omega(z)| \leq |z|^2$, we easily obtain that

$$|U_F(z)| \leq \frac{|z|^2}{1-|z|^2}\left(\frac{1+|\omega(z)|}{1-|\omega(z)|}\right) \leq \frac{|z|^2}{(1-|z|^2)}\left(\frac{1+|z|^2}{1-|z|^2}\right)$$

which is less than 1 provided $1 - 3|z|^2 > 0$ and this gives the disk $|z| < 1/\sqrt{3}$. The proof is complete. ■

A locally univalent function $f \in \mathcal{A}$ is said to belong to $\mathcal{G}(\alpha)$, for some $\alpha \in (0,1]$, if it satisfies the condition

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}, \quad z \in \mathbb{D}. \tag{3.9}$$

Thus, we have $\mathcal{G} := \mathcal{G}(1)$.

Theorem 3.10. *If $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1]$, then P_f is starlike in the disk $|z| < 1 + \alpha - \sqrt{\alpha(1+\alpha)}$.*

Proof. Let $f \in \mathcal{G}(\alpha)$ and F be given by (1.1). Then we have (see eg. [5, Theorem 1])

$$\frac{zf'(z)}{f(z)} \prec \frac{(1+\alpha)(1-z)}{1+\alpha-z}, \quad z \in \mathbb{D},$$

and thus, we may write

$$\frac{zf'(z)}{f(z)} = \frac{(1+\alpha)(1-\omega(z))}{1+\alpha-\omega(z)} \quad \text{and} \quad F(z) = P_f = \frac{z(1+\alpha-\omega(z))}{(1+\alpha)(1-\omega(z))}$$

for some $\omega \in \mathcal{B}$. By a computation, we obtain that

$$\frac{zF'(z)}{F(z)} - 1 = \frac{\alpha z\omega'(z)}{(1-\omega(z))(1+\alpha-\omega(z))}$$

and, as before, it follows from the Schwarz-Pick lemma that

$$\left|\frac{zF'(z)}{F(z)} - 1\right| \leq \frac{\alpha|z|\,|\omega'(z)|}{(1+\alpha-|\omega(z)|)(1-|\omega(z)|)} \leq \frac{\alpha|z|}{(1+\alpha-|z|)(1-|z|)}$$

which is less than 1 provided $\phi_3(|z|) > 0$, where $\phi_3(r) = r^2 - 2(1+\alpha)r + 1 + \alpha$. Thus, we conclude that P_f is starlike in the disk $|z| < r_3(\alpha) = 1+\alpha-\sqrt{\alpha(1+\alpha)}$, where $r_3(\alpha)$ is the root of the equation $\phi_3(r) = 0$ in the interval $(0,1]$. The theorem follows. ∎

Taking $\alpha = 1$ gives

Corollary 3.11. *If $f \in \mathcal{G}$, then P_f is starlike in the disk $|z| < 2-\sqrt{2} \approx 0.585786$.*

The same reasoning gives as in Corollary 3.7 the following.

Corollary 3.12. *If $f \in \mathcal{G}(\alpha)$ such that $f''(0) = 0$ and for some $\alpha \in (0,1]$, then P_f is starlike in $|z| < r_4(\alpha)$, where $r_4(\alpha)$ is the root in the interval $(0,1]$ of the equation $\phi_4(r) = 0$,*

$$\phi_4(r) = r^4 - \alpha r^3 - (2+\alpha)r^2 - \alpha r + 1 + \alpha.$$

Proof. In this case, the corresponding inequality for $f \in \mathcal{G}(\alpha)$ in Theorem 3.10 becomes

$$\left|\frac{zF'(z)}{F(z)} - 1\right| \leq \frac{\alpha|z|}{1-|z|^2}\left(\frac{1+|\omega(z)|}{1+\alpha-|\omega(z)|}\right) \leq \frac{\alpha|z|}{1-|z|^2}\left(\frac{1+|z|^2}{1+\alpha-|z|^2}\right)$$

which is less than 1 if $\phi_4(|z|) > 0$. The result follows. ∎

Setting $\alpha = 1$ gives

Corollary 3.13. *If $f \in \mathcal{G}$ such that $f''(0) = 0$, then P_f is starlike in $|z| < r_4$, where $r_4 \approx 0.64731$ is the root in the interval $(0,1]$ of the equation $r^4 - r^3 - 3r^2 - r + 2 = 0$.*

Theorem 3.14. *If $f \in \mathcal{G}(\alpha)$ for some $\alpha \in (0,1]$, then $F \in \mathcal{U}$ in the disk $|z| < r_5(\alpha)$, where $r_5(\alpha) = \sqrt{\frac{-\alpha+\sqrt{(1+\alpha)^2+1}}{2}}$.*

Proof. Let $f \in \mathcal{G}(\alpha)$ and $F = P_f$ be given by (1.1). Then, following the proof of Theorem 3.10, one has

$$\frac{z}{F(z)} - 1 = -\frac{\alpha\omega(z)}{1+\alpha-\omega(z)}$$

and, using this relation, we find that

$$\begin{aligned} U_F(z) &= -\frac{\alpha\omega(z)}{1+\alpha-\omega(z)} + \frac{\alpha(1+\alpha)z\omega'(z)}{(1+\alpha-\omega(z))^2} \\ &= \frac{\alpha[(1+\alpha)(z\omega'(z)-\omega(z))+\omega^2(z)]}{(1+\alpha-\omega(z))^2} \end{aligned}$$

so that, by (3.2), we easily have as before that

$$\begin{aligned} |U_F(z)| &\leq \frac{\alpha}{(1+\alpha-|\omega(z)|)^2}\left((1+\alpha)\left(\frac{|z|^2-|\omega(z)|^2}{1-|z|^2}\right)+|\omega(z)|^2\right) \\ &= \frac{\alpha}{1-|z|^2}\left(\frac{-(\alpha+|z|^2)|\omega(z)|^2+(1+\alpha)|z|^2}{(1+\alpha-|\omega(z)|)^2}\right) = \frac{\alpha\phi(t)}{1-r^2}, \end{aligned}$$

where we put $|z| = r$, $|\omega(z)| = t$ and

$$\phi(t) = \frac{-(\alpha+r^2)t^2+(1+\alpha)r^2}{(1+\alpha-t)^2},\ \ 0 \leq t \leq r.$$

We compute that

$$\phi'(t) = \frac{2(1+\alpha)}{(1+\alpha-t)^3}\left[-(\alpha+r^2)t+r^2\right],$$

and it is easy to see that ϕ attains its maximum value $\phi(t_0)$, where $t_0 = \frac{r^2}{\alpha+r^2}$ and $\phi''(t_0) < 0$. A calculation gives

$$\phi(t_0) = \frac{r^2(\alpha+r^2)}{\alpha(1+\alpha+r^2)}$$

and thus, we have

$$|U_F(z)| \leq \frac{\alpha\phi(t_0)}{1-r^2} = \frac{r^2(\alpha+r^2)}{(1-r^2)(1+\alpha+r^2)}$$

which is less than 1 if $2r^4+2\alpha r^2-(1+\alpha) < 0$. This gives that $|U_F(z)| < 1$ for $0 < r \leq r_5(\alpha)$, where $r_5(\alpha)$ is the root of the equation $2r^4+2\alpha r^2-(1+\alpha) = 0$, that lies in the interval $(0,1)$. The conclusion follows. ∎

The choice $\alpha = 1$ yields the following.

Corollary 3.15. *If $f \in \mathcal{G}$, then F belongs to the class $\mathcal{U}$ in the disk* $|z| < \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$.

Theorem 3.16. *Let $f \in \mathcal{S}$ with $a_2 = f''(0)/2!$. Then F belongs to $\mathcal{U}$ in the disk $|z| < r_6(|a_2|)$, where $r_6(|a_2|)$ is the root of the equation $\phi_5(r) = 0$ that lies in the interval $(0,1)$, where*

$$\phi_5(r) = (a + 1 - \frac{1}{4}b^2)r^{10} - (5a + 5 - \frac{5}{4}b^2)r^8 + (19a + 10 - \frac{19}{4}b^2)r^6 + (9a - 10 - \frac{9}{4}b^2)r^4 + 5r^2 - 1$$

with $b = |a_2|$ and $a = \frac{2\pi^2 - 12}{3} \approx 2.57974$.

Proof. Let $f \in \mathcal{S}$ and following the idea of [6, Theorem 4], we consider

$$\log \frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n, \tag{3.17}$$

where $c_n(f)$ $(n \geq 1)$ denote the logarithmic coefficients of f with $c_1(f) = a_2$. Further, for $f \in \mathcal{S}$ the following sharp inequality is known from the work of Roth [13, Theorem 1.1]

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^2 |c_n(f)|^2 \leq \frac{2\pi^2 - 12}{3} = a.$$

By (3.17), we obtain

$$\frac{zf'(z)}{f(z)} - 1 = \sum_{n=1}^{\infty} n c_n(f) z^n$$

which by the relation (3.4) gives that

$$U_F(z) = -\sum_{n=1}^{\infty} n(n-1) c_n(f) z^n$$

and thus, by the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned} |U_F(z)| &= \left| \sum_{n=2}^{\infty} n(n-1) c_n(f) z^n \right| \\ &\leq \left(\sum_{n=2}^{\infty} \left(\frac{n}{n+1} \right)^2 |c_n(f)|^2 \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n^2 - 1)^2 |z|^{2n} \right)^{\frac{1}{2}} \\ &\leq \left(a - \frac{1}{4}|c_1(f)|^2 \right)^{\frac{1}{2}} \left(\frac{|z|^4(|z|^6 - 5|z|^4 + 19|z|^2 + 9)}{(1 - |z|^2)^5} \right)^{\frac{1}{2}} \end{aligned}$$

which is less than 1 whenever,

$$\left(a - \frac{1}{4}|c_1(f)|^2 \right) |z|^4(|z|^6 - 5|z|^4 + 19|z|^2 + 9) < (1 - |z|^2)^5.$$

Values of $\lvert a_2\rvert$	**values of** $r_6(\lvert a_2\rvert)$	**Values of** $\lvert a_2\rvert$	**values of** $r_6(\lvert a_2\rvert)$
0.25	0.361166	1.25	0.370874
0.5	0.362294	1.5	0.375923
0.75	0.364226	1.75	0.382504
1	0.367042	2	0.391124

Table 1. Values of $r_6(|a_2|)$ for different values of $|a_2|$

If we put $r = |z|$, then the last inequality is equivalent to $\phi_5(r) := \phi_5(r, |a_2|) < 0$, where $\phi_5(r)$ is as in the statement. The desired result follows. ∎

Corollary 3.18. *Let $f \in \mathcal{S}$ with $f''(0) = 0$, and $a = \frac{2\pi^2-12}{3}$. Then F belongs to $\mathcal{U}$ in the disk $|z| < r_6$, where $r_6 \approx 0.360794$ is the root of the equation*

$$(a+1)r^{10} - 5(a+1)r^8 + (19a+10)r^6 + (9a-10)r^4 + 5r^2 - 1 = 0,$$

that lies in the interval $(0,1)$.

Proof. Set $a_2 = 0$ in Theorem 3.16. ∎

It is a simple exercise to see that the values $r_6(|a_2|)$, as the roots of the equation $\phi_5(r) = 0$, increase with increasing values of $|a_2| \in [0,2]$. For a ready reference, we included in Table 1 a list of values of $r_6(|a_2|)$ for certain choices of $|a_2|$. This observation shows that if $f \in \mathcal{S}$, then $F \in \mathcal{U}$ in the disk $|z| < r$ and the lower bound for r by Corollary 3.18 is $r_6 \approx 0.360794$. We end the discussion with a conjecture that the upper bound for the value of r is $\sqrt{2}-1$ which is attained by the Koebe function.

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