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Geometric Studies on the Class $\mathcal{U}(\lambda)$

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Abstract The article deals with the family $\mathcal{U}(\lambda)$ of all functions f normalized and analytic in the unit disk such that $\left|\left(z/f(z)\right)^2 f'(z) - 1\right| < \lambda$ for some $0 < \lambda \leq 1$. The family $\mathcal{U}(\lambda)$ has been studied extensively in the recent past and functions in this family are known to be univalent in $\mathbb{D}$. However, the problem of determining sharp bounds for the second coefficients of functions in this family was solved recently by Vasudevarao and Yanagihara but the proof was complicated. In this article, we first present a simpler proof of it. We obtain a number of new subordination results for this family and their consequences. Also, we obtain sharp estimate for the classical Fekete–Szegö inequality for functions in $\mathcal{U}(\lambda)$. In addition, we show that the family $\mathcal{U}(\lambda)$ is preserved under a number of elementary transformations such as rotation, conjugation, dilation, and omitted-value transformations, but surprisingly this family is not preserved under the n-th root transformation for any $n \geq 2$. This is a basic here which helps to generate a number of new theorems and in particular provides a way

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for constructions of functions from the family $\mathcal{U}(\lambda)$. Finally, we deal with a radius problem and the paper ends with a coefficient conjecture.

Keywords Analytic, univalent, and starlike functions · Coefficient estimates · Subordination · Schwarz' lemma · Radius problem · Square-root and n-th root transformation

Mathematics Subject Classification Primary 30C45

1 Introduction and Basic Properties

Let $\mathcal{A}$ be the family of all functions f analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the Taylor series expansion $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. Let $\mathcal{S}$ denote the subset of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$. See [5,8] for the general theory of univalent functions. For $0 < \lambda \leq 1$, consider the class

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A} : \ |U_f(z)| < \lambda \text{ in } \mathbb{D}\},$$

where $U_f(z) = \big(z/f(z)\big)^2 f'(z) - 1$. Set $\mathcal{U} := \mathcal{U}(1)$, $\mathcal{U}_2(\lambda) := \mathcal{U}(\lambda) \cap \{f \in \mathcal{A} : f''(0) = 0\}$ and $\mathcal{U}_2 := \mathcal{U}_2(1)$. Because $f'(z)(z/f(z))^2$ $(f \in \mathcal{U})$ is bounded, it follows that $(z/f(z))^2 f'(z) \neq 0$ in $\mathbb{D}$ and thus, each $f \in \mathcal{U}$ is non-vanishing in $\mathbb{D}\backslash\{0\}$. It is well recognized that the set Σ of meromorphic and univalent functions F on $\{\zeta : 1 < |\zeta| < \infty\}$ of the form $F(\zeta) = \zeta + \sum_{n=1}^{\infty} b_n \zeta^{-n}$ plays an indispensable role in the study of $\mathcal{S}$. For $f(z) = 1/F(1/z)$, $\zeta = 1/z$, we have the formula

$$F'(\zeta) = \left(\frac{z}{f(z)}\right)^2 f'(z)$$

and thus, functions f in $\mathcal{U}$ are associated with functions F in Σ such that $|F'(\zeta) - 1| < 1$ for $|\zeta| > 1$. In [1], it was shown that $\mathcal{U} \subsetneqq \mathcal{S}$ and hence functions in $\mathcal{U}(\lambda)$, that are generalizations of $\mathcal{U}$, are univalent in $\mathbb{D}$ for $0 < \lambda \leq 1$. Moreover, if $f \in \mathcal{S}$ and $1/f$ is a concave schlicht function with the pole at the origin, then $f \in \mathcal{U}$ and this fact is indicated by Aksentév and Avhadiev in [2]. It follows [6,13,18] that neither $\mathcal{U}$ is included in $\mathcal{S}^{\star}$ nor includes $\mathcal{S}^{\star}$. Here $\mathcal{S}^{\star}$ denotes the class of starlike functions, namely, functions $f \in \mathcal{S}$ such that $f(\mathbb{D})$ is starlike with respect to the origin. In 1995, among many results for the class $\mathcal{U}$, Obradović [12] proved that if $f \in \mathcal{U}$ then one has the subordination result

$$\frac{z}{f(z)} \prec (1+z)^2, \quad z \in \mathbb{D}.$$

For the definition of subordination, denoted by the symbol $\prec$, we refer to [5,8].

The class $\mathcal{U}(\lambda)$ has found many interesting properties [13–17,21]. It is a simple exercise to see that each $f \in \mathcal{U}(\lambda)$ has the characterization [15]

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z \int_0^z \omega(t)\,\mathrm{d}t, \tag{1}$$

for some $\omega \in \mathcal{B}$, where $a_2 = f''(0)/2$, and $\mathcal{B}$ denotes the class of functions ω analytic in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ for $z \in \mathbb{D}$. Here is a typical set of functions in $\mathcal{U} \cap \mathcal{S}^*$ given by

$$L = \left\{ z, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{1 \pm z + z^2} \right\},$$

where L is exactly the set of functions in $\mathcal{S}$ having integral coefficients in the power series expansion, [7]. Since $\mathcal{U} \subsetneq \mathcal{S}$ and the Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}$, $|a_2| \leq 2$ is obvious for $f \in \mathcal{U}$. The sharp estimation for the second coefficient of functions in $\mathcal{U}(\lambda)$ was known only recently in [21]. One of our main aims in this article is to give a simpler and different proof of this result. More precisely, in Theorem 1, we present a new proof that if $\mathcal{U}(\lambda)$, then $|a_2| \leq 1 + \lambda$ holds, and, in Theorem 2, we show that if $|a_2| = 1 + \lambda$, then f must be of the form

$$f(z) = \frac{z}{1 - a_2 z + \lambda e^{i\theta} z^2} \tag{2}$$

for some $\theta \in [0, 2\pi]$.

It is well known that the class $\mathcal{S}$ is preserved under a number of elementary transformations, e.g., conjugation, rotation, dilation, disk automorphisms (i.e., the Koebe transformations), range, omitted-value, and square-root transformations to say a few.

Lemma 1 *The class $\mathcal{U}(\lambda)$ is preserved under rotation, conjugation, dilation, and omitted-value transformations.*

Proof Let $f \in \mathcal{U}(\lambda)$ and define $g(z) = e^{-i\theta} f(ze^{i\theta})$, $h(z) = \overline{f(\overline{z})}$, and $\psi(z) = r^{-1} f(rz)$. Then we see that $g'(z) = f'(ze^{i\theta})$, $h'(z) = \overline{f'(\overline{z})}$, $\psi'(z) = f'(rz)$,

$$\begin{aligned}
\left(\frac{z}{g(z)}\right)^2 g'(z) - 1 &= \left(\frac{ze^{i\theta}}{f(ze^{i\theta})}\right)^2 f'(ze^{i\theta}) - 1,\\
\left(\frac{z}{h(z)}\right)^2 h'(z) - 1 &= \overline{\left(\frac{\overline{z}}{f(\overline{z})}\right)^2 f'(\overline{z}) - 1}, \quad \text{and}\\
\left(\frac{z}{\psi(z)}\right)^2 \psi'(z) - 1 &= \left(\frac{rz}{f(rz)}\right)^2 f'(rz) - 1.
\end{aligned}$$

It follows that g, h, and ψ belong to $\mathcal{U}(\lambda)$.

Finally, if $f \in \mathcal{U}(\lambda)$ and $f(z) \neq c$ for some $c \neq 0$, then the function F defined by

$$F(z) = \frac{cf(z)}{c - f(z)}$$

obviously belongs to $\mathcal{S}$. Thus, $z/F(z)$ is non-vanishing in $\mathbb{D}$, and it is a simple exercise to see that

$$U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1, \quad z \in \mathbb{D}. \tag{3}$$

Using (3), it is easy to see that $U_F(z) = U_f(z)$ for $z \in \mathbb{D}$. Consequently, $F \in \mathcal{U}(\lambda)$. The proof is complete. □

Corollary 1 *Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \leq 1$ and $a_2 = f''(0)/2$. If $a_2 + \mu \neq 0$ for some complex number μ with $|\mu| \leq 1 - \lambda$, then*

$$-\frac{1}{a_2 + \mu} \notin f(\mathbb{D}).$$

Proof Let $f \in \mathcal{U}(\lambda)$. Suppose that there exists a point $z_0 \in \mathbb{D}$ such that $f(z_0) = -\frac{1}{a_2+\mu}$. Then

$$\frac{z_0}{f(z_0)} = -(a_2 + \mu)z_0$$

and thus, according to the representation (1), the last relation implies that

$$1 + \mu z_0 + \lambda z_0 \int_0^{z_0} \omega(t)\, \mathrm{d}t = 0$$

for some $\omega \in \mathcal{B}$. But, this is not possible because

$$\begin{aligned}\left|1 + \mu z_0 + \lambda z_0 \int_0^{z_0} \omega(t)\, \mathrm{d}t\right| &\geq 1 - |\mu|\,|z_0| - \lambda|z_0|^2 \\ &\geq 1 - (1-\lambda)|z_0| - \lambda|z_0|^2 \\ &= (1 - |z_0|)(1 + \lambda|z_0|) > 0.\end{aligned}$$

We complete the proof. □

According to Corollary 1, the function F defined by

$$F(z) = \frac{f(z)}{1 + (a_2 + \mu)f(z)}$$

belongs to the class $\mathcal{U}(\lambda)$ whenever $f \in \mathcal{U}(\lambda)$ and $a_2 + \mu \neq 0$ with $|\mu| \leq 1 - \lambda$. In particular,

$$F(z) = \frac{f(z)}{1 + (a_2 + 1 - \lambda)f(z)}$$

belongs to the class $\mathcal{U}(\lambda)$ if $f \in \mathcal{U}(\lambda)$ and $a_2 \neq \lambda - 1$.

On the other hand, the class $\mathcal{U}$ (and hence, $\mathcal{U}(\lambda)$) is not preserved under the square-root transformation. For example, we consider the function

$$f_1(z) = \frac{z}{1 + (1/2)z + (1/3)z^3}.$$

Then we see that $z/f_1(z)$ is non-vanishing in $\mathbb{D}$, and it is a simple exercise to see that $U_{f_1}(z) = -(2/3)z^3$ showing that $f_1 \in \mathcal{U}$. In particular, f_1 is univalent in $\mathbb{D}$. On the other hand if we consider g_1 defined by

$$g_1(z) = \sqrt{f_1(z^2)} = z\sqrt{\frac{f_1(z^2)}{z^2}}$$

then, because $\mathcal{S}$ is preserved under the square-root transformation, it follows that g_1 is univalent in $\mathbb{D}$, whereas

$$\left(\frac{z}{g_1(z)}\right)^2 g_1'(z) - 1 = \left(\frac{z}{f_1(z)}\right)^{3/2} f_1'(z) - 1 = \frac{1 - (2/3)z^6}{\sqrt{1 + (1/2)z^2 + (1/3)z^6}} - 1$$

which approaches the value $\frac{5\sqrt{6}-3}{3} > 1$ as $z \to i$. This means that $U_{g_1}(\mathbb{D})$ cannot be a subset of the unit disk $\mathbb{D}$ and hence, the square-root transformation g_1 of f_1 does not belong to $\mathcal{U}$.

More generally if we consider

$$f(z) = \frac{z}{1 + (1/n)z + (-1)^n(1/(n+1))z^{n+1}}$$

then a computation shows that $f \in \mathcal{U}$, whereas the n-th root transformation g of f, given by

$$g(z) = \sqrt[n]{f(z^n)} = z\sqrt[n]{\frac{f(z^n)}{z^n}},$$

does not belong to the class $\mathcal{U}$ for each $n \geq 2$. Thus, for any $n \geq 2$, $\mathcal{U}$ is not preserved under the n-th root transformation unlike the class $\mathcal{S}$.

The remaining part of the article is organized as follows. In Sect. 2, we present a sharp coefficient bound for the second Taylor coefficient of $f \in \mathcal{U}(\lambda)$ and prove, in particular, several subordination results for $z/f(z)$ implying growth theorems for the family $\mathcal{U}(\lambda)$. In Sect. 3, we derive subordination results for functions in the family $\mathcal{U}(\lambda)$ and in Sect. 4, we present a number of consequences of Lemma 1. Section 5 is dedicated to examples of construction principles for functions in $\mathcal{U}(\lambda)$. The aim of Sect. 6 is the calculation of a radius r_0 such that $f(r_0z)/r_0$ belongs to $\mathcal{U}$ if f is univalent in the unit disk.

2 Second Coefficient for Functions in $\mathcal{U}(\lambda)$

First, we present a direct approach and later we shall obtain the following result as a simple consequence of a subordination result (see Theorems 4 and 5).

Theorem 1 *Let* $f \in \mathcal{U}(\lambda)$ *for some* $0 < \lambda \leq 1$. *Then* $|a_2| \leq 1 + \lambda$.

Proof Recall the fact that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ if and only if

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda z \int_0^z \omega(t)\,\mathrm{d}t \neq 0, \quad z \in \mathbb{D}, \tag{4}$$

where $\omega \in \mathcal{B}$.

It suffices to prove that for $|a_2| > 1 + \lambda$ and for any $\omega \in \mathcal{B}$, there exists a $z_0 \in \mathbb{D}$ such that

$$1 - a_2 z_0 + \lambda z_0 \int_0^{z_0} \omega(t)\,\mathrm{d}t = 0.$$

We may now assume that

$$|a_2| = \frac{1+\lambda}{r}, \quad r \in (0, 1), \tag{5}$$

and prove that the map F defined by

$$a_2 F(z) = 1 + \lambda z \int_0^z \omega(t)\,\mathrm{d}t$$

is a contracting map of $\mathbb{D}_r$ into $\mathbb{D}_r$, where $\mathbb{D}_r = \{z : |z| \leq r\}$.

We see that for $z \in \mathbb{D}_r$,

$$|F(z)| = \frac{r}{1+\lambda}\left|1 + \lambda z \int_0^z \omega(t)\,\mathrm{d}t\right| \leq \frac{r(1+\lambda|z|^2)}{1+\lambda} < r.$$

Now let $z_1, z_2 \in D_r$. This gives that

$$\begin{aligned}
|F(z_1) - F(z_2)| &= \frac{\lambda r}{1+\lambda}\left|z_1 \int_0^{z_1} \omega(t)\,\mathrm{d}t + (-z_1 + z_1 - z_2)\int_0^{z_2} \omega(t)\,\mathrm{d}t\right| \\
&\leq \frac{\lambda r}{1+\lambda}\left(|z_1|\left|\int_{z_2}^{z_1} \omega(t)\,\mathrm{d}t\right| + |z_1 - z_2|\left|\int_0^{z_2} \omega(t)\,\mathrm{d}t\right|\right) \\
&\leq \frac{\lambda r}{1+\lambda}(|z_1| + |z_2|)|z_1 - z_2| \\
&\leq r^2 |z_1 - z_2|.
\end{aligned}$$

Thus, F is a contracting map of $\mathbb{D}_r$ into $\mathbb{D}_r$. This implies, according to Banach's fixed point theorem, that there exists a $z_0 \in \mathbb{D}_r$ such that $F(z_0) = z_0$ which contradicts

(4) at $z_0 \in \mathbb{D}$ (and thus, (5) is not true for any $r \in (0,1)$). Hence, we must have $|a_2| \leq 1+\lambda$ for $f \in \mathcal{U}(\lambda)$. □

Determining the sharp bound for the Taylor coefficients $|a_n|$ ($n \geq 3$), for $f \in \mathcal{U}(\lambda)$, remains an open problem.

Next we deal with the equality case.

Theorem 2 *If $f \in \mathcal{U}(\lambda)$, and $|a_2| = 1+\lambda$, then f must be of the form (2) and especially,*

$$f(z) = \frac{z}{1-(1+\lambda)e^{i\phi}z + \lambda e^{2i\phi}z^2}.$$

Proof Let $f \in \mathcal{U}(\lambda)$. Then f must be of the form (4) for some $\omega \in \mathcal{B}$. If $|a_2| = 1+\lambda$, then we must show that ω in (4) takes the form $\omega(z) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$ and all $z \in \mathbb{D}$.

Assume on the contrary that $\omega(0) = a \in \mathbb{D}$ and f as in (4). Then, according to Schwarz–Pick's Lemma applied to $\omega \in \mathcal{B}$, we get

$$\left|\frac{a-\omega(z)}{1-\overline{a}\omega(z)}\right| \leq |z|, \quad z \in \mathbb{D},$$

from which we can immediately obtain that

$$|\omega(z)| \leq \frac{|a|+|z|}{1+|az|}, \quad z \in \mathbb{D},$$

and thus, we see that

$$\begin{aligned}\left|\int_0^z \omega(t)\,\mathrm{d}t\right| &\leq \int_0^{|z|} \frac{|a|+s}{1+|a|s}\,\mathrm{d}s = \frac{|z|}{|a|} - \frac{1-|a|^2}{|a|^2}\log(1+|az|)\\ &\leq \frac{1}{|a|} - \frac{1-|a|^2}{|a|^2}\log(1+|a|) =: v(|a|) < 1.\end{aligned}$$

Now, we let as in Theorem 1,

$$F(z) = \frac{1+\lambda z\int_0^z \omega(t)\,\mathrm{d}t}{a_2}$$

and define

$$\frac{1+\lambda v(|a|)}{1+\lambda} =: r < 1.$$

For $z \in \mathbb{D}_r$ we have

$$|F(z)| \leq \frac{1+\lambda r v(|a|)}{1+\lambda} < r,$$

and for $z_1, z_2 \in \mathbb{D}_r$ we get as above

$$
\begin{aligned}
|F(z_1) - F(z_2)| &= \frac{\lambda}{1+\lambda} \left| z_1 \int_{z_2}^{z_1} \omega(t)\,\mathrm{d}t + (z_1 - z_2) \int_0^{z_2} \omega(t)\,\mathrm{d}t \right| \\
&\le \frac{1}{2}(|z_1| + |z_2|)|z_1 - z_2| \le r|z_1 - z_2|.
\end{aligned}
$$

Hence F has a fixed point in $\mathbb{D}_r$ which contradicts $f \in \mathcal{U}(\lambda)$.

At last, we consider for fixed $\varphi, \psi \in [0, 2\pi]$ the cases

$$
\frac{z}{f(z)} = 1 - (1+\lambda)e^{i\varphi} z + \lambda e^{i\psi} z^2 =: p(\varphi, \psi, z)
$$

and prove that $p(\varphi, \psi, z)$ is non-vanishing in the unit disk if and only if $\psi = 2\varphi$.

Without restriction of generality we may assume $\varphi = 0$ and prove that among the functions $p(0, \psi, z)$ the only one non-vanishing in $\mathbb{D}$ is the function $p(0, 0, z)$.

To that end we consider the functions

$$
q_\psi(z) := (1+\lambda)z - \lambda e^{i\psi} z^2.
$$

Since for $z = re^{i\tau}$, $r \in [0, 1)$, $\tau \in [0, 2\pi]$, the inequality

$$
\operatorname{Re} q_\psi'(z) = 1 + \lambda - 2\lambda r \cos(\psi + \tau) > 0
$$

is valid, the function q_ψ is univalent in $\mathbb{D}$. In our case $q_\psi(\partial\mathbb{D})$ is a Jordan curve and $q_\psi(\mathbb{D})$ is the simply connected domain bounded by this curve. If we consider the curve $q_\psi(\partial\mathbb{D})$, we see that

$$
\left| q_\psi\left(e^{i\tau}\right) \right| \ge 1 + \lambda - \lambda = 1, \quad \tau \in [0, 2\pi],
$$

and the minimum modulus is attained if and only if $e^{i\tau} = e^{i(\psi + 2\tau)}$, i.e., $\tau = -\psi$. Hence, $1 \notin q_\psi(\mathbb{D})$, if and only if

$$
\operatorname{Re} q_\psi\left(e^{-i\psi}\right) = (1+\lambda)\cos\psi - \lambda\cos\psi = \cos\psi = 1.
$$

This is satisfied if and only if $\psi = 0$. Thus, f must be of the form (2). $\square$

3 Subordination

Theorem 3 *Let $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda \le 1$ and $a_2 = f''(0)/2$. Then*

$$
\frac{z}{f(z)} + a_2 z \prec 1 + 2\lambda z + \lambda z^2.
$$

Proof From (1), we observe that each $f \in \mathcal{U}(\lambda)$ has the form

$$\frac{z}{f(z)} = 1 - a_2 z + \lambda\psi(z), \quad \psi(z) = z\int_0^z \omega(t)\,\mathrm{d}t, \tag{6}$$

where $\omega \in \mathcal{B}$. Since $|\omega(z)| \le 1$ for $z \in \mathbb{D}$ and $\phi(z) = \psi(z)/z$ has the property that $\phi(0) = 0$ and $|\phi(z)| \le 1$, the classical Schwarz' lemma shows that $|\psi(z)| \le |z|^2$ in $\mathbb{D}$. Again, because

$$\frac{z^2}{2} \prec z + \frac{z^2}{2} \text{ and } |\psi(z)| \le |z|^2,$$

it follows that $\psi(z) \prec 2z + z^2$ in $\mathbb{D}$. The desired conclusion follows from (6). □

As remarked earlier, our next result includes a proof of Theorem 1 which will be stated as a corollary below.

Theorem 4 *If* $f \in \mathcal{U}(\lambda)$ *for* $\lambda \in (0, 1]$, *then*

$$\frac{f(z)}{z} \prec \frac{1}{1 + (1+\lambda)z + \lambda z^2}, \quad z \in \mathbb{D}, \tag{7}$$

or equivalently

$$\frac{z}{f(z)} \prec 1 + (1+\lambda)z + \lambda z^2, \quad z \in \mathbb{D};$$

and, for $|z| = r$,

$$\left|\frac{z}{f(z)} - 1\right| \le -1 + (1 + \lambda r)(1 + r).$$

In particular, if $f \in \mathcal{U}$ *then* $\frac{z}{f(z)} \prec (1+z)^2$ *in* $\mathbb{D}$.

Proof It suffices to prove the theorem for $\lambda \in (0, 1)$. Assume that $f \in \mathcal{U}(\lambda)$ and $s(z) = 1 + (1+\lambda)z + \lambda z^2$. First we observe that $s(z)$ is univalent in $\mathbb{D}$ for $\lambda \in (0, 1)$. Indeed for z_1, z_2 in the closed unit disk $\overline{\mathbb{D}}$, we have

$$\left|\frac{s(z_1) - s(z_2)}{z_1 - z_2}\right| = |1 + \lambda + \lambda(z_1 + z_2)| \ge 1 + \lambda - 2\lambda > 0$$

(and also $\mathrm{Re}\, s'(z) \ge 1 + \lambda - 2\lambda > 0$ in $\overline{\mathbb{D}}$) showing that $s(z)$ is univalent in $\mathbb{D}$.

We need to show that $\frac{z}{f(z)} \prec s(z)$. Suppose on the contrary that $\frac{z}{f(z)}$ is not subordinated to $s(z)$. As an application of [11, Lemma 1] (see also [10]), there exist points $z_0 = r_0 e^{i\theta_0} \in \mathbb{D}$ and $\zeta_0 \in \partial\mathbb{D}$ such that

$$\frac{z_0}{f(z_0)} = 1 + (1+\lambda)\zeta_0 + \lambda\zeta_0^2.$$

On the other hand, we know from [21, Theorem 3.2] that $\frac{z_0}{f(z_0)}$ lies in the union of the images of the disks $\{z : |z| \le r_0\}$ under the functions

$$\frac{z}{g(z)} = 1 + (1 + \lambda e^{i\tau})z + \lambda e^{i\varphi} z^2 \tag{8}$$

where one has to consider only those g belonging to $\mathcal{U}(\lambda)$. Hence, for our purposes it is sufficient to prove that the functions of the type (8), where g is restricted as above, are subordinated to the function $s(z)$. We observe that functions of the type g given by (8) belong to $\mathcal{U}(\lambda)$ if and only if

$$0 \ne 1 + (1 + \lambda e^{i\tau})z + \lambda e^{i\varphi} z^2, \quad z \in \mathbb{D}. \tag{9}$$

Using the abbreviation

$$1 + \lambda e^{i\tau} = |1 + \lambda e^{i\tau}| e^{i\gamma}$$

we get

$$(1 + \lambda e^{i\tau})z + \lambda e^{i\varphi} z^2 = e^{i(2\gamma-\varphi)} \left(|1 + \lambda e^{i\tau}| e^{i(\varphi-\gamma)} z + \lambda e^{2i(\varphi-\gamma)} z^2 \right).$$

Hence, (9) is equivalent to

$$-e^{-i(2\gamma-\varphi)} \ne |1 + \lambda e^{i\tau}| u + \lambda u^2, \quad u \in \mathbb{D}.$$

In the following we let $\beta = \varphi - 2\gamma$ and

$$l = |1 + \lambda e^{i\tau}| \in [1 - \lambda, 1 + \lambda].$$

For $u = e^{i\alpha}$ and $x + iy = le^{i\alpha} + \lambda e^{2i\alpha}$, we have

$$x + \lambda = \cos\alpha(l + 2\lambda\cos\alpha) \ \text{ and } \ y = \sin\alpha(l + 2\lambda\cos\alpha). \tag{10}$$

This is the parametrization of a limaçon with center $(-\lambda, 0)$ (see Fig. 1 for the graph of some limaçons parameterized by (10) for various values of λ and l). The implicit equation of this limaçon derived from the above equations is the following

$$\left(x^2 + y^2 - \lambda^2\right)^2 = l^2 \left(x^2 + y^2 + \lambda^2 + 2\lambda x\right).$$

The intersection points (x, y) of the limaçon and the unit circle can be got from this equation and

$$\frac{\left(1 - \lambda^2\right)^2 - l^2 \left(1 + \lambda^2\right)}{2\lambda l^2} = x =: -\cos\beta_1.$$

Hence, for $|\beta| \le \beta_1$ the functions g defined by (8) belong to $\mathcal{U}(\lambda)$.

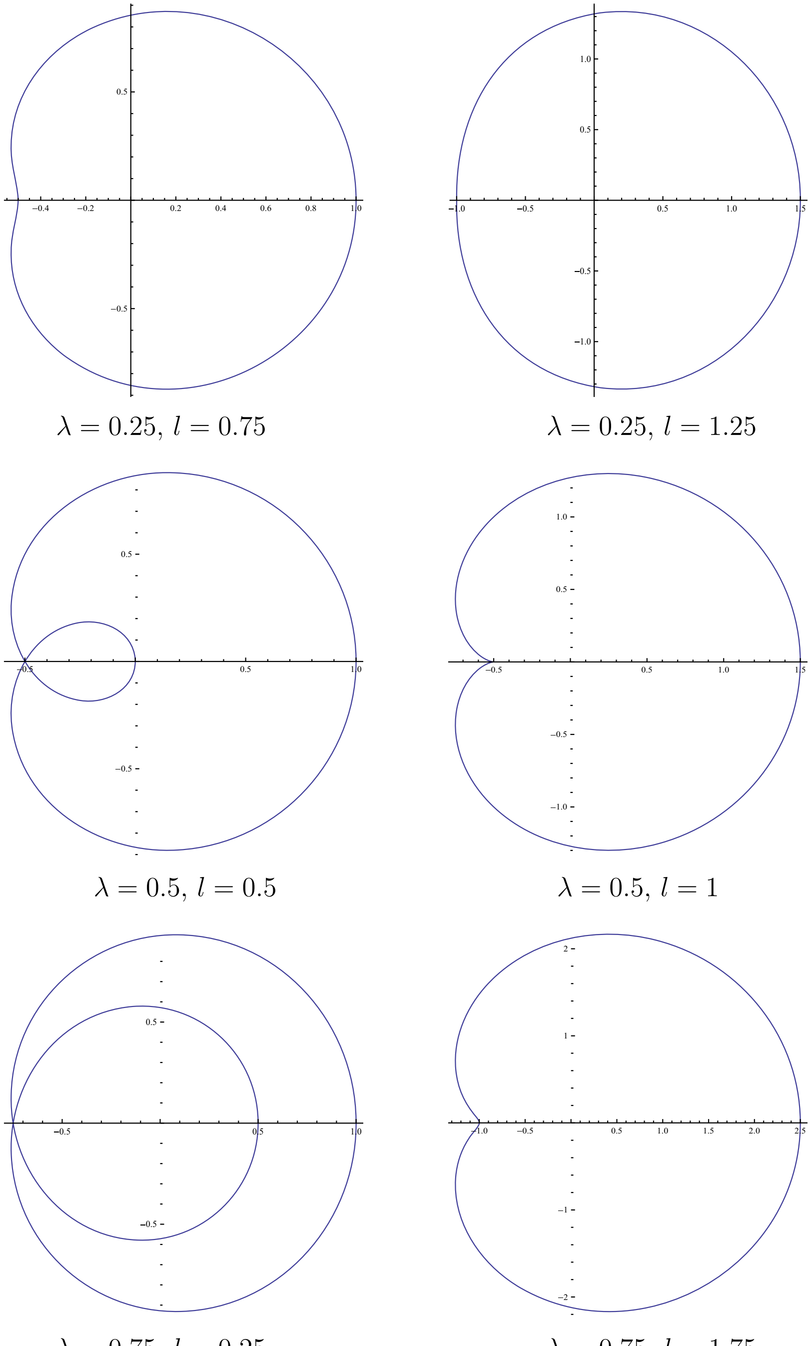

Fig. 1 The graph of some limaçons parameterized by (10) for certain values of λ and l

For $l = 1 + \lambda$, the case $\varphi = 0$ is the only one that produces a member of $\mathcal{U}(\lambda)$ in (8), whereas for $l = 1 - \lambda$ all functions g defined by (8) belong to this family.

Now, we turn to our second duty. Since s is injective in $\mathbb{D}$, we have to show that the image of $\mathbb{D}$ under the functions z/g defined by (8) with $|\beta| \leq \beta_1$ is contained in the domain bounded by the limaçon

$$1 + (1 + \lambda)e^{i\alpha} + \lambda e^{2i\alpha}, \quad \alpha \in [0, 2\pi].$$

By calculations similar to the above ones, we see that this is equivalent to the assertion that for $|\beta| \leq \beta_1$ the points

$$\left\{ lz + \lambda z^2 : z \in \mathbb{D} \right\},$$

are contained in the set

$$\left\{ w : w = e^{i\beta} \left((1 + \lambda)u + \lambda u^2 \right), \quad u \in \mathbb{D} \right\}.$$

This is a simple consequence of the fact that $(-1, 0)$ is the point nearest to the origin of the limaçon (see Fig. 2)

$$(1 + \lambda)e^{i\alpha} + \lambda e^{2i\alpha}, \quad \alpha \in [0, 2\pi],$$

and that the point of intersection of this limaçon turned around with angle β_1, the unit disk and the limaçon

$$le^{i\alpha} + \lambda e^{2i\alpha}, \quad \alpha \in [0, 2\pi],$$

is the point $e^{-i\beta_1}$. This completes the proof of (7).

For the proof of the second part, by the definition of subordination, we simply rewrite (7) as

$$\frac{z}{f(z)} = 1 + (1 + \lambda)\omega(z) + \lambda\omega^2(z),$$

where ω is analytic in $\mathbb{D}$ and $|\omega(z)| \leq |z|$. It follows that from the last equality that

$$\left| \frac{z}{f(z)} - 1 \right| \leq -1 + 1 + (1 + \lambda)|z| + \lambda|z|^2 = -1 + (1 + \lambda|z|)(1 + |z|)$$

and the proof is complete. □

According to Theorem 4, one has the estimate

$$\left| \frac{z}{f(z)} \right| \leq (1 + \lambda r)(1 + r) \text{ for } |z| = r$$

for $f \in \mathcal{U}(\lambda)$, $\lambda \in (0, 1]$.

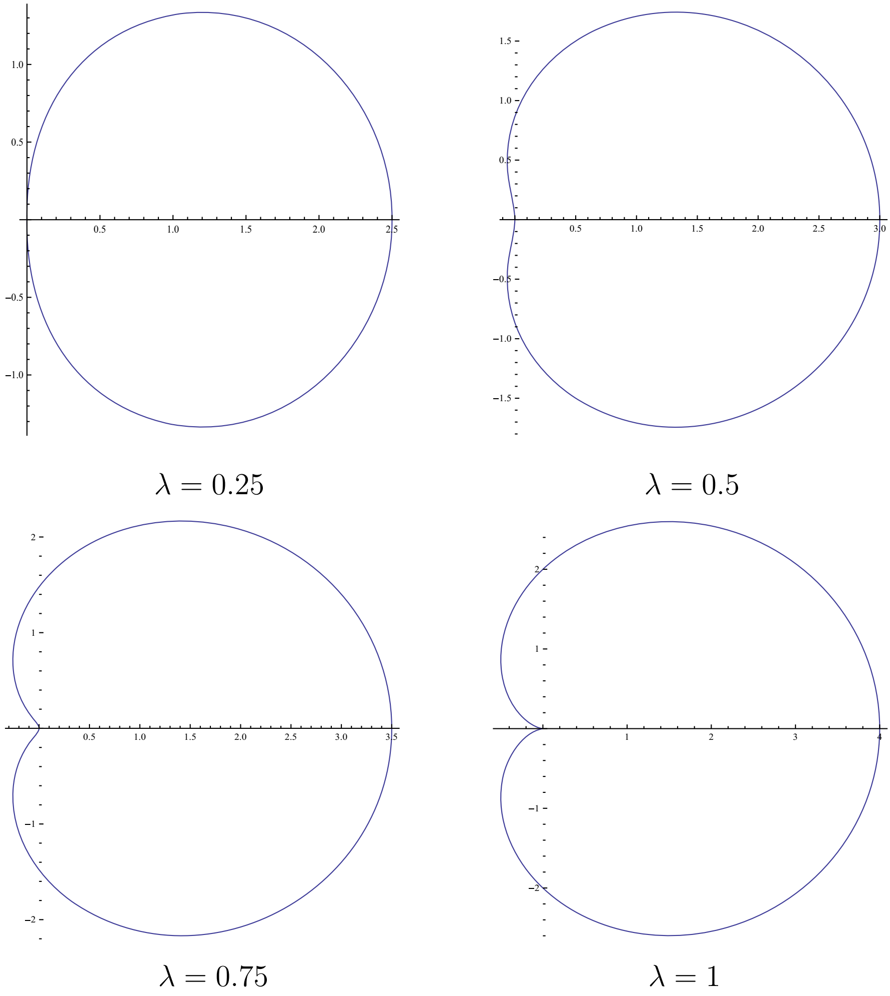

Fig. 2 Graph of $f(\lambda) = 1 + (1+\lambda)e^{i\alpha} + \lambda e^{2i\alpha}$ for certain values of λ, where $0 \le \alpha \le 2\pi$

Remark We remark that Theorem 1 follows from Theorem 4. Indeed, there is nothing to prove if $\lambda = 1$. Thus, if $f \in \mathcal{U}(\lambda)$ for some $0 < \lambda < 1$, then we have

$$\frac{z}{f(z)} \prec 1 + (1+\lambda)z + \lambda z^2.$$

By Rogosinski's theorem [19] (see also [5, Theorem 6.2]), it follows that

$$1 + |a_2|^2 \le 1 + (1+\lambda)^2$$

which implies that $|a_2| \le 1 + \lambda$ for $\lambda \in (0, 1)$.

Under a mild restriction on f, one could improve the bound $|a_2| \leq 1 + \lambda$ by establishing a region of variability of a_2. In the next result we deal with this.

Theorem 5 *Let* $f \in \mathcal{U}(\lambda)$ *for some* $0 < \lambda \leq 1$, *and such that*

$$\frac{z}{f(z)} \neq (1-\lambda)(1+z), \quad z \in \mathbb{D}. \tag{11}$$

Then, we have

$$\frac{z}{f(z)} - (1-\lambda)z \prec 1 + 2\lambda z + \lambda z^2 \tag{12}$$

and the estimate $|a_2 - (1-\lambda)| \leq 2\lambda$ *holds. In particular,* $|a_2| \leq 1+\lambda$ *and the estimate is sharp as the function* $f_\lambda(z) = z/((1+\lambda z)(1+z))$ *shows.*

Proof Notice that there is nothing to prove if we allow $\lambda = 1$. Let $f \in \mathcal{U}(\lambda)$ for some $\lambda \in (0, 1)$. Then, by the assumption (11), the function g is defined by

$$\frac{z}{g(z)} = \frac{z}{f(z)} - (1-\lambda)\,(1+z), \tag{13}$$

has the property that $z/g(z)$ is non-vanishing and $g'(0) = 1/\lambda$ and hence, it is easy to see that $G = \lambda g$ belongs to $\mathcal{U}$. Consequently, by the last subordination relation in Theorem 4, we find that

$$\frac{z}{G(z)} = \frac{1}{\lambda}\left(\frac{z}{f(z)} - (1-\lambda)(1+z)\right) = 1 - \frac{a_2 - (1-\lambda)}{\lambda} z + \cdots \prec (1+z)^2,$$

which is obviously equivalent to (12). The coefficient inequality $|(a_2 - (1-\lambda))/\lambda| \leq 2$ is a consequence of Rogosinski's theorem. Thus, $|a_2 - (1-\lambda)| \leq 2\lambda$ holds. □

It is not clear whether the condition (11) is necessary for a function f to belong to the family $\mathcal{U}(\lambda)$.

Theorem 6 *Suppose that* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *belongs to* $\mathcal{U}(\lambda)$ *for some* $0 < \lambda \leq 1$. *Then, we have the sharp estimate*

$$|a_3 - a_2^2| \leq \lambda.$$

Proof It is a simple exercise to see that

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + \left(a_3 - a_2^2\right) z^2 + \cdots = 1 + \lambda z^2 \omega(z),$$

where $\omega \in \mathcal{B}$, i.e., ω is analytic in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ for $z \in \mathbb{D}$. Hence, we must have $|a_3 - a_2^2| \leq \lambda$. Equality is attained if and only if $\omega(z) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$, i.e., for functions $f \in \mathcal{U}(\lambda)$ of the form

$$f(z) = \frac{z}{1 - a_2 z - \lambda e^{i\theta} z^2}. \tag{14}$$

To get all extremal functions, we consider all those functions, where we may assume $a_2 \geq 0$. The condition

$$1 - a_2 z - \lambda e^{i\theta} z^2 \neq 0$$

is equivalent to this condition. It is clear that this is fulfilled if $a_2 \leq 1 - \lambda$. For $1 - \lambda < a_2 \leq 1 + \lambda$ we get by a reasoning similar to that used in the proof of Theorem 4 that the condition is fulfilled if and only if

$$\cos\theta \leq \frac{\left(1 - \lambda^2\right)^2 - a_2^2\left(1 + \lambda^2\right)}{2\lambda a_2^2}. \tag{15}$$

Hence, the extremal functions are those of the form (14), where in addition (15) is satisfied. □

We observe that for $\lambda = 1$, the above inequality leads to the well-known estimate $|a_3 - a_2^2| \leq 1$ which holds for $f \in \mathcal{S}$ and the Koebe function $k(z) = z/(1-z)^2$ gives the equality.

4 Marx-Type Implication for Functions in $\mathcal{U}$

According to Theorem 4, one has

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}} > \frac{1}{2}, \quad z \in \mathbb{D},$$

if $f \in \mathcal{U}$. This result is known to be true also for functions in the family $\mathcal{S}^\star$ of starlike functions in $\mathbb{D}$ (see Marx [9]) although the class $\mathcal{U}$ neither contains $\mathcal{S}^\star$ nor is contained in $\mathcal{S}^\star$. On the other hand, since the structure of the class $\mathcal{U}$ allows us to determine the lower bound for the functional $\operatorname{Re}\sqrt{f(z)/z}$, as a function of the second Taylor coefficient a_2, it is natural to solve the problem of finding $\alpha = \alpha(|a_2|) \geq 1/2$ such that $f \in \mathcal{U}$ implies that

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}} > \alpha, \quad z \in \mathbb{D}.$$

In the next theorem, we present a solution to this problem. Also, in our result below, we observe that $\alpha(2) = 1/2$ which is indeed the correct bound as the Koebe function $z/(1-z)^2$ shows. However, we could not claim that the bound $\alpha(|a_2|)$ is best possible.

Theorem 7 *Let $f \in \mathcal{U}$ and $a_2 = f''(0)/2$. Then*

$$\operatorname{Re}\sqrt{\frac{f(z)}{z}} > \alpha(|a_2|) \quad \textit{for } z \in \mathbb{D},$$

where

$$\alpha(x) = \frac{20 + x - \sqrt{x^2 + 40x + 16}}{24}, \quad 0 \le x \le 2. \tag{16}$$

Proof We recall from Lemma 1 that the family $\mathcal{U}$ is invariant under rotation and thus, it suffices to prove the theorem for functions $f \in \mathcal{U}$ such that a_2 is real and non-negative and thus, throughout the proof, we may assume that $0 \le a_2 \le 2$. Observe that $\alpha(x)$ is a decreasing function of $x \in [0, 2]$ with $\alpha([0, 2]) = [1/2, 2/3]$. We now let

$$\sqrt{\frac{f(z)}{z}} = p(z) = 1 + \beta z + \cdots, \tag{17}$$

where p is analytic in $\mathbb{D}$, $p(0) = 1$ and a_2 is fixed and $0 \le \beta := (a_2/2) \le 1$.

We wish to prove that

$$p(z) \prec q(z) := \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)z + \cdots,$$

where $\alpha = \alpha(a_2)$ is defined by (16). We prove this by the method of contradiction.

Suppose that $p(z)$ is not subordinate to $q(z)$. Then, according to the result of Miller and Mocanu [10,11] (see also [4]), there exist points $z_0 \in \mathbb{D}$ and $\zeta_0 \in \partial(\mathbb{D})\backslash\{1\}$ such that

$$p(z_0) = q(\zeta_0) \text{ and } z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \tag{18}$$

where

$$m \ge 1 + \frac{q'(0) - \beta}{q'(0) + \beta} = \frac{8(1 - \alpha)}{4(1 - \alpha) + a_2}. \tag{19}$$

We notice that $0 \le \beta = \frac{1}{2}a_2 \le q'(0) = 2(1 - \alpha)$. Also, we see that

$$q(\zeta_0) = \alpha + (1 - \alpha)\frac{1 + \zeta_0}{1 - \zeta_0} =: \alpha + i\rho, \quad \rho \in \mathbb{R}, \tag{20}$$

and a computation gives

$$\zeta_0 q'(\zeta_0) = \frac{2(1 - \alpha)\zeta_0}{(1 - \zeta_0)^2} = -\frac{\left[(1 - \alpha)^2 + \rho^2\right]}{2(1 - \alpha)}. \tag{21}$$

Further, using (17) and (3), it follows easily that

$$U_f(z) = \frac{1}{p^2(z)} + \frac{2zp'(z)}{p^3(z)} - 1$$

and thus, by (18), we obtain that

$$U_f(z_0) = \frac{1}{q^3(\zeta_0)}\left[q(\zeta_0) + 2m\zeta_0 q'(\zeta_0) - q^3(\zeta_0)\right].$$

By (20) and (21), we deduce that

$$\begin{aligned} |U_f(z_0)|^2 &= \frac{1}{|q(\zeta_0)|^6}\left|q(\zeta_0) + 2m\zeta_0 q'(\zeta_0) - q^3(\zeta_0)\right|^2 \\ &= \frac{1}{(\alpha^2+\rho^2)^3}\left|\alpha + i\rho - \frac{m\left[(1-\alpha)^2+\rho^2\right]}{1-\alpha} - (\alpha+i\rho)^3\right|^2 \end{aligned}$$

and a calculation shows that $|U_f(z_0)|^2 = \Phi(\rho^2)$, where

$$\Phi(t) = \frac{(a+bt)^2 + ct(d+t)^2}{(1-\alpha)^2(\alpha^2+t)^3},$$

with

$$\begin{aligned} &t = \rho^2, \quad a = (1-\alpha)^2(m - \alpha(1+\alpha)), \quad b = m - 3\alpha(1-\alpha), \\ &c = (1-\alpha)^2, \quad d = 1 - 3\alpha^2. \end{aligned}$$

Clearly the proof will be completed if we can show that $\Phi(t) \geq 1$ for all $t \geq 0$ under the assumption on $\alpha = \alpha(a_2)$ given by (16). The inequality $\Phi(t) \geq 1$ is equivalent to

$$At^2 + Bt + C \geq 0, \tag{22}$$

where $t \geq 0$,

$$\begin{aligned} &A = b^2 + 2cd - 3\alpha^2(1-\alpha)^2, \quad B = 2ab + cd^2 - 3\alpha^4(1-\alpha)^2, \\ &C = a^2 - \alpha^6(1-\alpha)^2. \end{aligned}$$

In order to prove the inequality (22), it suffices to show that A, B, C are non-negative for $\alpha \in [1/2, 2/3]$. We begin to observe by (19) that

$$\begin{aligned} a - \alpha^3(1-\alpha) &= (1-\alpha)^2(m - \alpha(1+\alpha)) - \alpha^3(1-\alpha) \\ &\geq (1-\alpha)^2\left(\frac{8(1-\alpha)}{4(1-\alpha)+a_2} - \alpha(1+\alpha)\right) - \alpha^3(1-\alpha) = 0, \end{aligned}$$

provided

$$\frac{8(1-\alpha)}{4(1-\alpha)+a_2} - \alpha(1+\alpha) = \frac{\alpha^3}{1-\alpha}, \tag{23}$$

which is the same as $12\alpha^2 - \alpha(20 + a_2) + 8 = 0$. Solving this equation gives the solution $\alpha = \alpha(a_2)$ expressed by (16), and hence, $C \geq 0$. It remains to show that $A \geq 0$, $B \geq 0$ for $\alpha \in [1/2, 2/3]$. The last inequality shows that $a \geq \alpha^3(1-\alpha)$ and

$$\begin{aligned} b &= m - 3\alpha(1-\alpha) \\ &\geq \frac{8(1-\alpha)}{4(1-\alpha)+a_2} - 3\alpha(1-\alpha), \quad \text{by (19)}, \\ &= \frac{\alpha}{1-\alpha} - 3\alpha(1-\alpha), \quad \text{by (23)}, \\ &= \frac{\alpha\left[1-3(1-\alpha)^2\right]}{1-\alpha} > 0 \ \text{ for } \alpha \in [1/2, 2/3]. \end{aligned}$$

Using these facts, we can prove that $A \geq 0$ for $\alpha \in [1/2, 2/3]$. We now find that

$$\begin{aligned} A &= b^2 + 2cd - 3\alpha^2(1-\alpha)^2 \\ &\geq \left(\frac{\alpha}{1-\alpha} - 3\alpha(1-\alpha)\right)^2 + 2(1-\alpha)^2(1-3\alpha^2) - 3\alpha^2(1-\alpha)^2 \\ &= \frac{\alpha^2}{(1-\alpha)^2} - 6\alpha^2 + 2(1-\alpha)^2 \\ &= \frac{(2\alpha-1)^2(2-\alpha^2)}{(1-\alpha)^2} \end{aligned}$$

which is non-negative for $\alpha \in [1/2, 2/3]$. Similarly, we have

$$\begin{aligned} B &= 2ab + cd^2 - 3\alpha^4(1-\alpha)^2 \\ &\geq 2\alpha^3(1-\alpha)\left(\frac{\alpha}{1-\alpha} - 3\alpha(1-\alpha)\right) + (1-\alpha)^2(1-3\alpha^2)^2 - 3\alpha^4(1-\alpha)^2 \\ &= (2\alpha-1)^2(1+2\alpha-\alpha^2) \end{aligned}$$

which is again non-negative for $\alpha \in [1/2, 2/3]$.

Finally, we have shown that $\Phi(t) \geq 1$, i.e., $|U_f(z_0)| \geq 1$, which is a contradiction to $|U_f(z)| < 1$ in $\mathbb{D}$ and hence to the assumption that p is not subordinate to q. Hence, we must have $p(z) \prec q(z)$ in $\mathbb{D}$ which is equivalent to the desired result. $\square$

5 Applications of Elementary Transformations

Because each $f \in \mathcal{U}$ is non-vanishing in $\mathbb{D}\backslash\{0\}$, $z/f(z)$ can be written as

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k, \quad z \in \mathbb{D}. \tag{24}$$

One of the sufficient conditions for functions f of this form to belong to the class $\mathcal{U}$ is that (see [13,15])

$$\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1. \tag{25}$$

Theorem 8 *Let* $f \in \mathcal{A}$ *and*

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty}(-1)^n b_n z^n,$$

where $b_n \geq 0$ *for* $n \geq 2$*. Then* $f \in \mathcal{S}$ *if and only if* $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$.

Proof For $f \in \mathcal{S}$, by Lemma 1, we have that $g(z) = -f(-z) \in \mathcal{S}$. Since

$$\frac{z}{-f(-z)} = 1 - b_1 z + \sum_{n=2}^{\infty} b_n z^n,$$

then by the characterization given in [16] (see also the survey article [17]), $g \in \mathcal{U}$ if and only if $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$ if and only if $g \in \mathcal{S}$. The desired conclusion follows. □

Problem 1 *It will be interesting to find necessary and/or sufficient conditions (as in [16]) for the function* $f \in \mathcal{A}$ *of the following form to be univalent in* $\mathbb{D}$*:*

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty}(-1)^{n-1} b_n z^n \quad \textit{or} \quad \frac{z}{f(z)} = 1 + b_1 z - \sum_{n=2}^{\infty} b_n z^n,$$

where $b_n \geq 0$ *for* $n \geq 2$.

A function f analytic in $\mathbb{D}$ is called n-fold symmetric ($n = 1, 2, \ldots$) if

$$f(e^{i2\pi/n} z) = e^{i2\pi/n} f(z) \quad \text{for } z \in \mathbb{D}.$$

In particular, every $f \in \mathcal{A}$ is onefold symmetric and every odd f is twofold symmetric. Every n-fold symmetric function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ can be written as

$$f(z) = z + a_{n+1} z^{n+1} + a_{2n+1} z^{2n+1} + \cdots .$$

Properties of various geometric subclasses of n-fold symmetric functions from $\mathcal{S}$ have been investigated by many authors [8]. We now investigate certain analogous problems associated with the class $\mathcal{U}$.

Theorem 9 *Let $f \in \mathcal{U}$ be given by (24). Then for each $n \geq 2$, the function $f_n(z)$ defined by*

$$\frac{z}{f_n(z)} = 1 + \sum_{k=1}^{\infty} b_{nk} z^{nk}$$

also belongs to the class $\mathcal{U}$, whenever $z/f_n(z) \neq 0$ in $\mathbb{D}$. More generally, if $f \in \mathcal{U}(\lambda)$ is given by (24), then $f_n \in \mathcal{U}(\lambda)$ whenever it is non-vanishing in $\mathbb{D}$.

Proof Let $f \in \mathcal{U}$ with $\phi(z) = z/f(z)$. Then $\phi(z)$ is non-vanishing and analytic in $\mathbb{D}$ and has the form

$$\frac{z}{f(z)} = \phi(z) = 1 + \sum_{k=1}^{\infty} b_k z^k.$$

Now, we define Φ_n by $\Phi_n(z) = z/f_n(z)$ and $\omega = e^{i2\pi/n}$. Then, $\{\omega^k : k = 1, 2, \ldots, n\}$ is the set of all n n-th roots of unity. It is a simple exercise to see that

$$\Phi_n(z) := \frac{1}{n} \sum_{k=1}^{n} \phi(\omega^k z) = \frac{1}{n} \sum_{k=1}^{n} \frac{z}{\omega^{-k} f(\omega^k z)} = 1 + \sum_{k=1}^{\infty} b_{nk} z^{nk}.$$

Since $f \in \mathcal{U}$, by Lemma 1, for each k, the function $F_k(z)$ defined by $F_k(z) = \omega^{-k} f(\omega^k z)$ clearly belongs to the class $\mathcal{U}$. By calculation and the relation (3), it follows that

$$U_{f_n}(z) = \frac{1}{n} \sum_{k=1}^{n} U_{F_k}(z) = \frac{1}{n} \sum_{k=1}^{n} \left[\left(\frac{\omega^k z}{f(\omega^k z)} \right)^2 f'(\omega^k z) - 1 \right]$$

and thus, $|U_{f_n}(z)| < 1$ in $\mathbb{D}$ for each $n \geq 2$. The proof is complete. □

From the proof of the following corollary, we see that the non-vanishing condition $f_n(z) \neq 0$ in $\mathbb{D}$ in the above theorem can be dropped for the case $n = 2$.

Corollary 2 *If $f \in \mathcal{U}$, then the odd function f_2 defined by*

$$\frac{z}{f_2(z)} = \frac{1}{2} \left(\frac{z}{f(z)} + \frac{z}{-f(-z)} \right)$$

also belongs to the class $\mathcal{U}$. More generally, if $f \in \mathcal{U}(\lambda)$, then $f_2 \in \mathcal{U}(\lambda)$.

Proof Let $f \in \mathcal{U}$. Then, by Lemma 1, F defined by $F(z) = -f(-z)$ belongs to $\mathcal{U}$. Moreover, the condition $f(z) - f(-z) \neq 0$ for $z \in \mathbb{D} \setminus \{0\}$ is satisfied, because if $f(z) = f(-z)$ for some $z \in \mathbb{D} \setminus \{0\}$, then, since f is univalent, we have $z = -z$, i.e., $z = 0$, which is a contradiction. Consequently,

$$\frac{z}{f_2(z)} = \frac{z^2}{f(z)f(-z)} \left(\frac{f(z) - f(-z)}{2} \right)$$

is non-vanishing in $\mathbb{D}$. Moreover, a calculation gives that if $f \in \mathcal{U}$ is given by (24), then f_2 takes the form

$$\frac{z}{f_2(z)} = 1 + \sum_{k=1}^{\infty} b_{2k} z^{2k}$$

and thus, by Theorem 9, $f_2 \in \mathcal{U}$. □

From the proof of Theorem 9, the following general result could be proved easily and so, we omit its details.

Corollary 3 *Let $g_k \in \mathcal{U}(\lambda_k)$ for $k = 1, 2, \ldots, n$ and $\mu_k, \lambda_k \in [0, 1]$ for $k = 1, 2, \ldots, n$ such that $\mu_1\lambda_1 + \cdots + \mu_n\lambda_n = 1$. If Φ defined by*

$$\Phi(z) = \sum_{k=1}^{n} \mu_k \frac{z}{g_k(z)} = \frac{z}{\Psi(z)}$$

is non-vanishing in $\mathbb{D}$, then the function $\Psi(z) = \frac{z}{\Phi(z)}$ belongs to the class $\mathcal{U}$.

Proof It suffices to observe that

$$U_{\Psi}(z) = \sum_{k=1}^{n} \mu_k U_{g_k}(z)$$

and the rest follows by taking the modulus on both sides and use the triangle inequality. □

Corollary 4 *Let $f \in \mathcal{U}$ be given by (24). For $\theta \in [0, 2\pi)$, the functions f_3 and f_4 defined by*

$$\frac{z}{f_3(z)} = 1 + \sum_{n=1}^{\infty} b_n \cos(n\theta) z^n \quad \text{and} \quad \frac{z}{f_4(z)} = 1 + \sum_{n=1}^{\infty} b_n \sin(n\theta) z^n$$

also belong to the class $\mathcal{U}$ (whenever z/f_3 and z/f_4 are non-vanishing in $\mathbb{D}$).

Proof Lemma 1 shows that the functions $g_1(z) = e^{-i\theta} f(ze^{i\theta})$ and $g_2(z) = e^{i\theta} f(ze^{-i\theta})$ belong to the class $\mathcal{U}$ and so does its convex combination (by Corollary 3 with $\mu_1 = \mu_2 = 1/2$ and $\lambda_1 = \lambda_2 = 1$). Moreover, it follows from the power series representation of $z/f(z)$ that

$$\frac{z}{f_3(z)} = \frac{1}{2}\left(\frac{z}{e^{-i\theta} f(ze^{i\theta})} + \frac{z}{e^{i\theta} f(ze^{-i\theta})}\right) = 1 + \sum_{n=1}^{\infty} b_n \cos(n\theta) z^n$$

from which we conclude that $f_3 \in \mathcal{U}$, by Corollary 3.

In order to prove that f_4 belongs to $\mathcal{U}$, we first observe that

$$\frac{z}{f_4(z)} = 1 + \frac{1}{2i}\left(\frac{ze^{i\theta}}{f(ze^{i\theta})} - \frac{ze^{-i\theta}}{f(ze^{-i\theta})}\right) = 1 + \sum_{n=1}^{\infty} b_n \sin(n\theta) z^n,$$

and, by a computation, we have

$$\left|U_{f_4}(z)\right| = \left|\frac{1}{2i}\left(U_f(ze^{i\theta}) - U_f(ze^{-i\theta})\right)\right| \leq \frac{1}{2}\left(|U_f(ze^{i\theta})| + |U_f(ze^{-i\theta})|\right) < 1,$$

showing that $f_4 \in \mathcal{U}$. □

In particular, if we set $\theta = \pi/2$, then $f_3(z)$ and $f_4(z)$ take the forms

$$\frac{z}{f_3(z)} = 1 - b_2 z^2 + b_4 z^4 - \cdots \quad \text{and} \quad \frac{z}{f_4(z)} = 1 + b_1 z - b_3 z^3 + \cdots,$$

respectively, and thus, the above corollary provides us with new functions from $\mathcal{U}$.

Theorem 10 *Let $f \in \mathcal{U}$ be given by (24). Then the function g defined by*

$$\frac{z}{g(z)} = 1 + \sum_{k=1}^{\infty} \mathrm{Re}\,\{b_k\} z^k,$$

with $z/g(z) \neq 0$ in $\mathbb{D}$, also belongs to the class $\mathcal{U}$. More generally, if $f \in \mathcal{U}(\lambda)$, then $g \in \mathcal{U}(\lambda)$.

Proof Let $f \in \mathcal{U}$. Then, by Lemma 1, $h(z) = \overline{f(\overline{z})}$ belongs to $\mathcal{U}$. Now, we observe that

$$\frac{z}{g(z)} = \frac{1}{2}\left[\left(1 + \sum_{k=1}^{\infty} b_k z^k\right) + \overline{\left(1 + \sum_{k=1}^{\infty} b_k \overline{z}^k\right)}\right] = \frac{1}{2}\left(\frac{z}{f(z)} + \frac{z}{h(z)}\right)$$

and thus, we easily have

$$U_g(z) = \frac{z}{g(z)} - z\left(\frac{z}{g(z)}\right)' - 1 = \frac{U_f(z) + U_h(z)}{2}.$$

Clearly, the last relation implies that $g \in \mathcal{U}$. □

Theorem 11 *Let $f \in \mathcal{U}$ be given by (24). Then the function F defined by*

$$\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} b_{2n} z^n \tag{26}$$

belongs to the class $\mathcal{U}$. More generally, if $f \in \mathcal{U}(\lambda)$ is given by (24), then $F \in \mathcal{U}(\lambda)$.

Proof If $f \in \mathcal{U}$, then we have the representation

$$\frac{z}{f(z)} = 1 + b_1 z + z\int_0^z \frac{\omega(t)}{t^2}\,\mathrm{d}t, \quad b_1 = -a_2, \tag{27}$$

where $\omega \in \mathcal{B}_1$. Here $\mathcal{B}_1$ denotes the class of functions ω analytic in $\mathbb{D}$ such that $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. If we put

$$\omega_1(z) = \int_0^z \frac{\omega(t)}{t^2}\,\mathrm{d}t,$$

then ω_1 is analytic in $\mathbb{D}$, $\omega_1(0) = 0$ and $|\omega_1(z)| \le |z|$. Moreover, $|\omega_1'(z)| = |\omega(z)/z^2| \le 1$ for every $z \in \mathbb{D}$. Consequently, for $f \in \mathcal{U}$ one has

$$\frac{z}{f(z)} = 1 + b_1 z + z\omega_1(z). \tag{28}$$

and thus, the function Ψ defined by

$$\Psi(z) = \frac{1}{2}\left(\frac{z}{f(z)} + \frac{-z}{f(-z)}\right) = 1 + \frac{z}{2}\left(\omega_1(z) - \omega_1(-z)\right)$$

is analytic in $\mathbb{D}$ and $|\Psi(z) - 1| < 1$ for $z \in \mathbb{D}$. Consequently, $\Psi(z) \neq 0$ in $\mathbb{D}$,

$$\Psi(z) = 1 + \sum_{n=1}^{\infty} b_{2n} z^{2n}$$

and observe that F defined by

$$\frac{z}{F(z)} = \Psi(\sqrt{z}) = 1 - zW(z) := 1 + \frac{z}{2}\left(\frac{\omega_1(\sqrt{z})}{\sqrt{z}} - \frac{\omega_1(-\sqrt{z})}{\sqrt{z}}\right)$$

is analytic in $\mathbb{D}$, where W is analytic in $\mathbb{D}$. Next, we observe that

$$U_F(z) = \frac{z}{F(z)} - z\left(\frac{z}{F(z)}\right)' - 1 = z^2 W'(z)$$

and, in view of the fact that $|\omega(z)| \le |z|^2$ and $|\omega_1'(z)| = |\omega(z)/z^2| \le 1$, we can easily see that $|z^2 W'(z)| < 1$ in $\mathbb{D}$, which means that $F \in \mathcal{U}$. $\square$

6 Some Radius Problem

When we say that $f \in \mathcal{U}$ in $|z| < r$ it means that the inequality $|U_f(z)| < 1$ holds in the subdisk $|z| < r$ of $\mathbb{D}$, which is indeed same as saying that $r^{-1} f(rz)$ belongs to the class $\mathcal{U}$.

Theorem 12 *Let $f \in \mathcal{S}$ and f be given by* (24). *Then the function F defined by*

$$\frac{z}{F(z)} = 1 + \sum_{n=1}^{\infty} b_{2n} z^n$$

belongs to the class $\mathcal{U}$ at least in the disk $|z| < r_0 = 0.778387$ (implying F is univalent in $|z| < r_0$), where $r_0 \in (0, 1)$ is the root of the equation

$$\frac{r(1-r^2)^2}{2} \log\left(\frac{1+r}{1-r}\right) - (4 + r^4 - 7r^2) = 0. \tag{29}$$

Proof Assume that $f \in \mathcal{S}$ and is given by (24). In order to show that $F \in \mathcal{U}$ in the disk $|z| < r_0$, we need to prove that the function G defined by $G(z) = r^{-1}F(rz)$ belongs to $\mathcal{U}$ in $\mathbb{D}$ for each $0 < r \le r_0$. Thus, we begin to consider the function G defined by

$$\frac{z}{G(z)} = 1 + \sum_{n=1}^{\infty} b_{2n} r^n z^n,$$

where $0 < r \le 1$. To prove $G \in \mathcal{U}$, by (25), it suffices to show that

$$S =: \sum_{n=2}^{\infty} (n-1)|b_{2n}| r^n \le 1$$

for $0 < r \le r_0$. To do this, we need to recall first the following inequality, namely, for $f \in \mathcal{S}$, the necessary coefficient inequality ([8, Theorem 11 on p. 193 of Vol. 2])

$$\sum_{n=2}^{\infty} (n-1)|b_n|^2 \le 1.$$

This in particular gives that $\sum_{n=2}^{\infty}(2n-1)|b_{2n}|^2 \le 1$. Now, we find that

$$\begin{aligned} S &= \sum_{n=2}^{\infty} \sqrt{2n-1}|b_{2n}| \frac{(n-1)}{\sqrt{2n-1}} r^n \\ &\le \left(\sum_{n=2}^{\infty} (2n-1)|b_{2n}|^2\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} \frac{(n-1)^2}{2n-1} r^{2n}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=2}^{\infty} \frac{(n-1)^2}{2n-1} r^{2n}\right)^{\frac{1}{2}}. \end{aligned}$$

By a computation we see that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(n-1)^2}{2n-1}r^{2n} &= \frac{1}{2}\sum_{n=2}^{\infty}\left(n-\frac{3}{2}+\frac{1}{2(2n-1)}\right)r^{2n}\\ &= \frac{1}{2}\left(\frac{r^2}{(1-r^2)^2}-r^2\right)-\frac{3r^4}{4(1-r^2)}-\frac{r^2}{4}+\frac{r}{8}\log\left(\frac{1+r}{1-r}\right)\\ &= \frac{r^2(3r^2-1)}{4(1-r^2)^2}+\frac{r}{8}\log\left(\frac{1+r}{1-r}\right)\end{aligned}$$

and thus, $S\leq 1$ holds provided

$$\frac{r^2(3r^2-1)}{4(1-r^2)^2}+\frac{r}{8}\log\left(\frac{1+r}{1-r}\right)\leq 1,$$

i.e., if $0<r\leq r_0=0.778387$, where r_0 is the root of the Eq. (29). It means that F is in the class $\mathcal{U}$ in the disk $|z|<r_0$. □

In [14], as a corollary to a general result, it has been shown that $|z|<1/\sqrt{2}$ is the largest disk centered at the origin such that every function in $\mathcal{S}$ is included in $\mathcal{U}$. More precisely (see also [20]),

$$\sup\left\{r>0:\ r^{-1}f(rz)\in\mathcal{U}\text{ for every }f\in\mathcal{S}\right\}=1/\sqrt{2}.$$

In this case, $1/\sqrt{2}$ is referred to as the $\mathcal{U}$-radius in $\mathcal{S}$. Recently, Ali and Alarifi [3] investigated $\mathcal{U}$-radius problems for a number of subclasses of analytic functions.

We conclude the paper with the following conjecture.

Conjecture 1 *If $f\in\mathcal{U}(\lambda)$ for some $0<\lambda\leq 1$. Then $|a_n|\leq\sum_{k=0}^{n-1}\lambda^k$ for $n>2$.*

There is nothing to prove if $\lambda=1$. Also, we have verified the truth of the conjecture for $n=3$.

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