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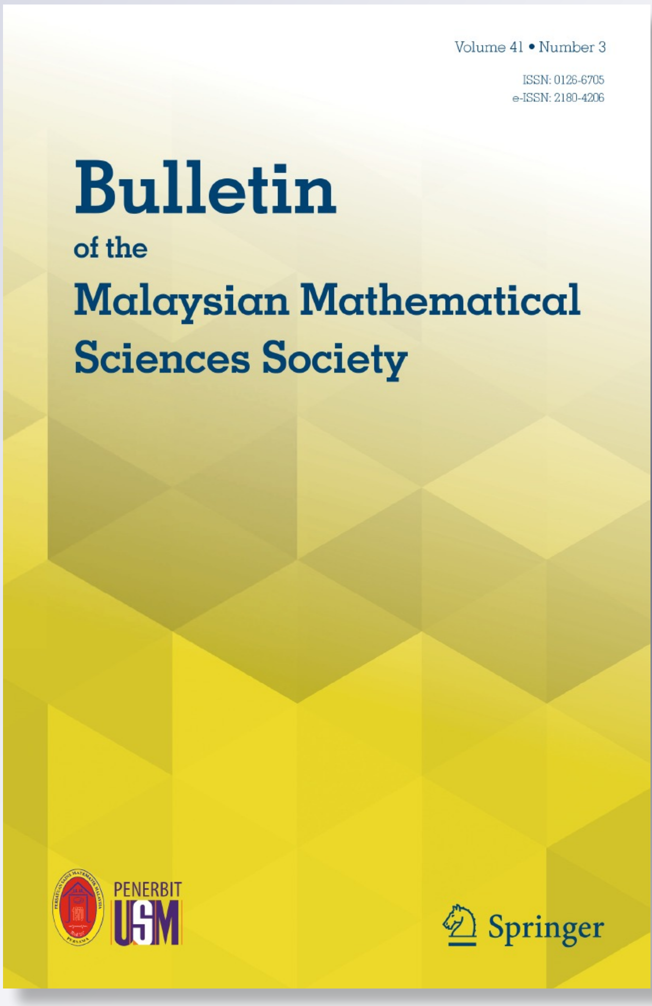

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Some New Results for Certain Classes of Univalent Functions

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Abstract Let $\mathcal{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let $\mathcal{U}$ denote the subset of functions $f \in \mathcal{A}$ which satisfy

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, \quad z \in \mathbb{D},$$

and let $\mathcal{P}(2)$ be the subclass of all functions $f \in \mathcal{A}$ such that $f(z) \neq 0$ for $0 < |z| < 1$ and

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2, \quad z \in \mathbb{D}.$$

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In this paper, a conjecture on the class $\mathcal{U}$ and $\mathcal{P}(2)$ has been resolved. Furthermore, two sufficient conditions for functions to be univalent are presented.

Keywords Analytic · Univalent · Starlike

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathcal{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let $\mathcal{B}$ denote the set of functions ω that are analytic in $\mathbb{D}$ and satisfy $|\omega(z)| \leq 1 (|z| < 1)$. Let S be the set of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{D}$. Let S^* denote the subset of S consisting of all starlike functions. Let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1, \quad z \in \mathbb{D}, \tag{1}$$

and let $\mathcal{P}(2)$ be the subclass of all functions $f \in \mathcal{A}$ such that $f(z) \neq 0$ for $0 < |z| < 1$ and

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2, \quad z \in \mathbb{D}. \tag{2}$$

It is known that $\mathcal{U} \subset S$(see [1]). In recent years, the class $\mathcal{U}$ were studied in detail (see [2–6]). Obradović and Ponnusamy[3] proved that

$$\mathcal{P}(2) \subset \mathcal{U}.$$

For the function f defined by $\frac{z}{f(z)} = 1 + \frac{1}{2} z^3$, which belongs to the class $\mathcal{U}$, we have that

$$\left| \left(\frac{z}{f(z)} \right)'' \right| = |3z| \leq 2 \text{ for } \quad |z| \leq \frac{2}{3},$$

i.e., $\mathcal{P}(2)$-radius for the above function f is equal to $\frac{2}{3}$. The authors considered a subclass of the class $\mathcal{U}$ and showed that $\mathcal{P}(2)$-radius for that subclass is equal to $\frac{2}{3}$. They conjectured that the same is valid for the class $\mathcal{U}$ [7]. In the second part of this paper, we shall prove that the conjecture is not true by giving the correct $\mathcal{P}(2)$-radius for the class $\mathcal{U}$.

Let Ω be the subset of $\mathcal{A}$ which consists of all functions f satisfying

$$|zf'(z) - f(z)| < \frac{1}{2}, \quad (|z| < 1).$$

It is known that $\Omega \subset S^*$[8]. In the third part of this paper, we shall give two conditions for functions to be in the class Ω.

2 $\mathcal{P}(2)$-Radius for the Class $\mathcal{U}$

Theorem 1 *If $f \in \mathcal{U}$, then*

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le 2$$

for $|z| \le r_0 = \frac{\sqrt{5}-1}{2} = 0.618...$ *and the result is the best possible.*

For the proof of Theorem 1, we need the next lemma given by Shaffer[9].

Lemma 1 *Let* $g(z) = \sum_{n=p}^{\infty} a_n z^n$ $(p \ge 1)$ *be analytic in* $\mathbb{D}$ *and satisfy* $|g(z)| \le 1$ *for* $z \in \mathbb{D}$ *, then*

(a) $|g'(z)| \le p|z|^{p-1}$ *for* $|z| \le \frac{\sqrt{1+p^2}-1}{p}$,

(b) $|g'(z)| \le |z|^{p-2} \frac{4|z|^2+p^2(1-|z|^2)^2}{4(1-|z|^2)}$ *for* $|z| > \frac{\sqrt{1+p^2}-1}{p}$.

These estimates are the best possible.

Proof of Theorem 1 For $f \in \mathcal{U}$ let's put

$$\mathcal{U}_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1. \tag{3}$$

Then,

$$\mathcal{U}_f(z) = \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1.$$

If $f(z) = z + a_2 z^2 + a_3 z^3 + ...$, then

$$\mathcal{U}_f(z) = (a_3 - a_2^2)z^2 + ...$$

and

$$\mathcal{U}'_f(z) = -z\left(\frac{z}{f(z)}\right).'' \tag{4}$$

By using (1), previous notation and other conclusions, we can apply Lemma 1 with $g(z) = \mathcal{U}_f(z)$ and $p = 2$. By Lemma 1(a), we obtain

$$\left|\mathcal{U}'_f(z)\right| \le 2|z| \text{ for } |z| \le r_0 = \frac{\sqrt{5}-1}{2},$$

which by (4) implies

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le 2,\ |z| \le r_0 = \frac{\sqrt{5}-1}{2},$$

i.e., f has $\mathcal{P}(2)$-property in the disk $|z| \le r_0 = \frac{\sqrt{5}-1}{2}$, which was to be proved. □

Similarly, by Lemma 1(b) we have

$$\left|\left(\frac{z}{f(z)}\right)''\right| \le \frac{1-|z|^2+|z|^4}{|z|(1-|z|^2)} =: \varphi(|z|),\ |z| > r_0 = \frac{\sqrt{5}-1}{2},$$

where

$$\varphi(t) = \frac{1-t^2+t^4}{t(1-t^2)},\ r_0 < t < 1.$$

It is easy to check that φ is an increasing function and $\varphi(r_0) = 2 < \varphi(t)$ for $r_0 < t < 1$. For sharpness of the theorem , let us consider the function f_b defined by the condition

$$\frac{z}{f_b(z)} = 1 - z\int_0^z \frac{z+b}{1+bz}\mathrm{d}z, \tag{5}$$

where b is real and $|b| < 1$. Since $\omega(z) = \frac{z+b}{1+bz} : \mathbb{D} \to \mathbb{D}$, then

$$\left|z\int_0^z \frac{z+b}{1+bz}\mathrm{d}z\right| \le |z|^2 < 1,\ z \in \mathbb{D},$$

which by (5) implies $\frac{z}{f_b(z)} \ne 0,\ z \in \mathbb{D}$, i.e., f_b is well defined. Also

$$\left|\mathcal{U}_{f_b}(z)\right| = \left|z^2\frac{z+b}{1+bz}\right| < |z|^2 < 1,\ z \in \mathbb{D},$$

which gives that $f_b \in \mathcal{U}$.

Let r_1 be a fixed real number such that $r_0 < r_1 < 1$ and $b_1 = \frac{1-2r_1^2}{r_1^3}$. We claim that $|b_1| < 1$. In fact,

$$\begin{aligned} -1 < b_1 < 1 \Leftrightarrow -1 < \frac{1-2r_1^2}{r_1^3} < 1 \\ \Leftrightarrow -r_1^3 < 1-2r_1^2 < r_1^3 \\ \Leftrightarrow r_1^2(1-r_1) < 1-r_1^2 < r_1^2(1+r_1). \end{aligned}$$

The left inequality is equivalent to $r_1^2 < 1+r_1$, which is true, and the right is equivalent to $1-r_1-r_1^2 < 0$, which is also true since $r_0 < r_1 < 1$.

After simple calculations, for the function f_{b_1} we have

$$\left|\left(\frac{z}{f_{b_1}(z)}\right)''\right|_{z=r_1} = \frac{1-r_1^2+r_1^4}{r_1(1-r_1^2)} =: \varphi(r_1) > 2,$$

because of the property of the function φ and since $r_0 < r_1 < 1$. It means that the function f_{b_1} is an extremal function for our problem, since it has $\mathcal{P}(2)$-property in the disk $|z| \le r_0 = \frac{\sqrt{5}-1}{2}$ (because $f_{b_1} \in \mathcal{U}$), but not in a disk with longer radius.

3 Sufficient Conditions for Function to be in Ω

Theorem 2 *Let $f \in \mathcal{A}$. If $|f''(z)| \le 1$ then $f \in \Omega$. The number 1 is the best possible.*

Proof Let $g(z) = zf'(z) - f(z)$. Then, $g'(z) = zf''(z)$. Since $f(0) = f'(0) - 1 = 0$ and $|f''(z)| \le 1$ for $z \in \mathbb{D}$, we have

$$g'(z) = z\omega(z) \tag{6}$$

where $\omega(z) \in \mathcal{B}$. It follows from (6) that

$$g(z) = \int_0^z \zeta\omega(\zeta)\mathrm{d}\zeta = z^2\int_0^1 t\omega(zt)\mathrm{d}t.$$

Therefore,

$$|g(z)| = |z^2\int_0^1 t\omega(zt)\mathrm{d}t| < \int_0^1 t\mathrm{d}t = \frac{1}{2}, \quad (z \in \mathbb{D}).$$

That is, $|zf'(z) - f(z)| < \frac{1}{2}$ for $z \in \mathbb{D}$. This implies that $f \in \Omega \subset S^*$.

If $|f''(z)| \le \lambda$ and $\lambda > 1$, then f may be not univalent. For example, $f(z) = z + \frac{1}{2}\lambda z^2$ satisfy $|f''(z)| \le \lambda$, but $f'(z) = 1 + \lambda z$ vanish at $-\frac{1}{\lambda}$, which implies that $f \notin S^*$. □

Theorem 3 *Let $f \in \mathcal{A}$. If*

$$|z^2f''(z) + zf'(z) - f(z)| \le \frac{3}{2}$$

then $f \in \Omega \subset S^*$. The number $\frac{3}{2}$ is the best possible.

Proof Since $f(0) = f'(0) - 1 = 0$ and

$$|z^2f''(z) + zf'(z) - f(z)| \le \frac{3}{2},$$

it follows that

$$[z^2 f'(z) - zf(z)]' = \frac{3}{2} z^2 \omega(z),$$

where $\omega(z) \in \mathcal{B}$. Thus,

$$z^2 f'(z) - zf(z) = \frac{3}{2} \int_0^z \omega(\zeta)\zeta^2 \mathrm{d}\zeta = \frac{3}{2} z^3 \int_0^1 \omega(zt)t^2 \mathrm{d}t,$$

and consequently,

$$|zf'(z) - f(z)| = |\frac{3}{2} z^2 \int_0^1 \omega(zt)t^2 \mathrm{d}t| < \frac{3}{2} \int_0^1 t^2 \mathrm{d}t = \frac{1}{2}$$

for $z \in \mathbb{D}$. This implies that $f \in \Omega \subset S^*$.

If $|z^2 f''(z) + zf'(z) - f(z)| \leq \lambda$ and $\lambda > \frac{3}{2}$, then f may be not univalent. One can see that by investigating the function $f(z) = z + \frac{1}{2}\lambda z^2$, $\lambda > 1$. □

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