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The third logarithmic coefficient for the class $\mathcal{S}$

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Abstract: In this paper we give an upper bound of the third logarithmic coefficient for the class $\mathcal{S}$ of univalent functions in the unit disc.

Key words: Univalent, third logarithmic coefficient

1. Introduction

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1.1}$$

and let $\mathcal{S}$ be its subclass consisting of functions that are univalent in the unit disc $\mathbb{D}$.

The logarithmic coefficients of the function f given by (1.1) are defined in $\mathbb{D}$ by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1.2}$$

By using (1.1), after differentiation and comparing the coefficients, we can obtain that $\gamma_1 = \frac{1}{2}a_2$, $\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right)$ and

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right). \tag{1.3}$$

Very little is known about the estimates of the modulus of the logarithmic coefficients for the whole class $\mathcal{S}$ of normalized of univalent functions. The Koebe function $k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n$ with $\gamma_n = \frac{1}{n}$ being extremal in majority estimates over the class $\mathcal{S}$ inspires a conjecture that $|\gamma_n| \le \frac{1}{n}$ for $n = 1, 2, \dots$ and $f \in \mathcal{S}$. Apparently, this is true only for the class of starlike functions ([8]), but not for the class $\mathcal{S}$ in general ([5, Theorem 8.4, p.242]). Sharp estimates for the class $\mathcal{S}$ are known only for the first two coefficients, $|\gamma_1| \le 1$ and $|\gamma_2| \le \frac{1}{2} + \frac{1}{e}$.

In this paper we give an upper bound of $|\gamma_3|$ for the class $\mathcal{S}$.

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It is worth mentioning that the problem of estimating the modulus of the first three logarithmic coefficients is widely studied for the subclasses of $\mathcal{S}$ and in some cases sharp bounds are obtained. Namely, sharp estimates for the class of strongly starlike functions of certain order and γ-starlike functions are given in [8] and [3], respectively, while nonsharp estimates for the class of Bazilevic, close-to-convex and different subclasses of close-to-convex functions are given in [4], [1] and [7], respectively.

2. Main result

As announced before, here is an estimate of the modulus of the third logarithmic coefficient for the whole class of univalent functions.

Theorem 2.1 *For the class $\mathcal{S}$ we have*

$$|\gamma_3| \le \frac{\sqrt{133}}{15} = 0.7688\ldots.$$

Proof In the proof of this theorem we will use mainly the notations and results given in the book of N. A. Lebedev ([6]).

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t)-f(z)}{t-z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are called Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([5, 6]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{2.1}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if f given by (1.1) belongs to $\mathcal{S}$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots \tag{2.2}$$

belongs to the class $\mathcal{S}$. Then for the function f_2 we have the appropriate Grunsky's coefficients of the form $\omega^{(2)}_{2p-1,2q-1}$ and the inequality (2.1) has the form

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega^{(2)}_{2p-1,2q-1} x_{2p-1} \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{2.3}$$

As it has been shown in [6, p.57], if f is given by (1.1) then the coefficients a_2, a_3, a_4 are expressed by Grunsky's coefficients $\omega^{(2)}_{2p-1,2q-1}$ of the function f_2 given by (2.2) in the following way (in the next text we omit upper index 2 in $\omega^{(2)}_{2p-1,2q-1}$):

$$\begin{aligned} a_2 &= 2\omega_{11}, \\ a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\ a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3. \end{aligned} \tag{2.4}$$

Now, from (1.3) and (2.3) we have

$$\gamma_3 = \omega_{33} + 2\omega_{11}\omega_{13}$$

On the other hand, from (2.4) for $x_{2p-1} = 0$, $p = 3, 4, \dots$ we have

$$|\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 \le |x_1|^2 + \frac{|x_3|^2}{3}. \tag{2.5}$$

From (2.5) for $x_1 = 2\omega_{11}$, $x_3 = 1$ and since $\omega_{31} = \omega_{13}$, we have

$$|2\omega_{11}^2 + \omega_{13}|^2 + 3|\gamma_3|^2 \le 4|\omega_{11}|^2 + \frac{1}{3},$$

and from here

$$\begin{aligned}
|\gamma_3|^2 &\le \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{1}{3}|2\omega_{11}^2 + \omega_{13}|^2 \\
&= \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{1}{3}\left(4|\omega_{11}|^4 + |\omega_{13}|^2 + 4\mathrm{Re}\,\left\{\omega_{13}\overline{\omega_{11}}^2\right\}\right) \\
&= \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{4}{3}|\omega_{11}|^4 - \frac{1}{3}|\omega_{13}|^2 - \frac{4}{3}\mathrm{Re}\,\left\{\omega_{13}\overline{\omega_{11}}^2\right\}.
\end{aligned}$$

Using the fact that

$$-|\omega_{13}|^2 \le -\left|\mathrm{Re}\,\{\omega_{13}\}\right|^2 = -\left(\mathrm{Re}\,\{\omega_{13}\}\right)^2,$$

we obtain

$$|\gamma_3|^2 \le \frac{1}{9} + \frac{4}{3}|\omega_{11}|^2 - \frac{4}{3}|\omega_{11}|^4 - \frac{1}{3}\left(\mathrm{Re}\,\{\omega_{13}\}\right)^2 - \frac{4}{3}\mathrm{Re}\,\left\{\omega_{13}\overline{\omega_{11}}^2\right\}.$$

Next, without loss of generality using suitable rotation of f we can assume that $0 \le a_2 \le 2$ and $a_2 = 2\omega_{11}$ receive that $0 \le \omega_{11} \le 1$. So, let put $\omega_{11} = a$, $0 \le a \le 1$, and continue analysing

$$|\gamma_3|^2 \le \frac{1}{9} + \frac{4}{3}a^2 - \frac{4}{3}a^4 - \frac{1}{3}\left(\mathrm{Re}\,\{\omega_{13}\}\right)^2 - \frac{4}{3}a^2\mathrm{Re}\,\{\omega_{13}\}. \tag{2.6}$$

It is a classical result that for the class $\mathcal{S}$ we have $|a_3 - a_2^2| \le 1$ (see [9, p.5]), which is by (2.4) equivalent with

$$|2\omega_{13} - \omega_{11}^2| \le 1.$$

From here,

$$-1 \le \mathrm{Re}\,\{2\omega_{13} - \omega_{11}^2\} \le 1,$$

i.e.

$$-\frac{1}{2}(1 - a^2) \le \mathrm{Re}\,\{\omega_{13}\} \le \frac{1}{2}(1 + a^2). \tag{2.7}$$

If we put $x_1 = 1$ and $x_3 = 0$ in (2.5), then we get

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 \le 1,$$

which implies

$$|\omega_{13}| \le \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2} = \frac{1}{\sqrt{3}}\sqrt{1 - a^2}.$$

Combining this with (2.7), we receive

$$-\frac{1}{2}(1-a^2) \leq \mathrm{Re}\{\omega_{13}\} \leq \frac{1}{\sqrt{3}}\sqrt{1-a^2}$$

(because $-\frac{1}{2}(1-a^2) \geq -\frac{1}{\sqrt{3}}\sqrt{1-a^2}$).

By using (2.6), (2.7) and the notation $t = \mathrm{Re}\{\omega_{13}\}$ we obtain

$$|\gamma_3|^2 \leq \frac{1}{9} + \frac{4}{3}a^2 - \frac{4}{3}a^4 - \frac{1}{3}t^2 - \frac{4}{3}a^2 t :\equiv \psi(a,t) = \frac{1}{9} + \frac{1}{3}\varphi(a,t),$$

where $0 \leq a \leq 1$, $-\frac{1}{2}(1-a^2) \leq t \leq \frac{1}{\sqrt{3}}\sqrt{1-a^2}$ and $\varphi(a,t) = 4a^2 - 4a^4 - t^2 - 4a^2 t$.

It remains to show that the maximal value of the function $\psi(a,t)$ over the region $\Omega = [0,1] \times [-\frac{1}{2}(1-a^2), \frac{1}{\sqrt{3}}\sqrt{1-a^2}]$ equals $\left(\frac{\sqrt{133}}{15}\right)^2 = \frac{133}{225}$, or equivalently that $\varphi(a,t)$ has maximal value $\frac{36}{25}$ on the same region.

Indeed, the system of equations

$$\begin{cases} \varphi'_a(a,t) = 8a - 16a^3 - 8at = 0 \\ \varphi'_t(a,t) = -4a^2 - 2t = 0 \end{cases}$$

has unique real solution $a = t = 0$ with $\varphi(0,0) = 0$, while on the edges of the region Ω we have the following:

- for $a = 0$ we have that the function $\varphi(0,t) = -t^2$ on the interval $-\frac{1}{2} \leq t \leq \frac{1}{\sqrt{3}}$ attains maximal value $\varphi(0,0) = 0$;
- when $a = 1$, t can take single value, $t = 0$, and in that case $\varphi(1,0) = 0$;
- for $t = -\frac{1}{2}(1-a^2)$, the function $\varphi\left(a, -\frac{1}{2}(1-a^2)\right) = -\frac{1}{4}(a^2-1)\left(a^2 - \frac{1}{25}\right)$ is with maximal value $\frac{36}{25}$ on the interval $0 \leq a \leq 1$ attained for $a = \frac{\sqrt{13}}{5}$;
- for $t = \frac{1}{\sqrt{3}}\sqrt{1-a^2}$, the values of the function

$$\begin{aligned} \varphi\left(a, \frac{1}{\sqrt{3}}\sqrt{1-a^2}\right) &= \frac{1}{3}(-12a^4 + 13a^2 - 1) - \frac{4a^2}{\sqrt{3}}\sqrt{1-a^2} \\ &\leq \frac{1}{3}(-12a^4 + 13a^2 - 1) < \frac{36}{25}. \end{aligned}$$

  on the interval $0 \leq a \leq 1$ are smaller than $\frac{36}{25}$.

This completes the proof. □

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