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IMPROVED UPPER BOUND OF THIRD ORDER HANKEL DETERMINANT FOR OZAKI CLOSE–TO–CONVEX FUNCTIONS

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Abstract. In this paper we improve the upper bound of the third order Hankel determinant for the class of Ozaki close-to-convex functions. The sharp bound is conjectured.

1. Introduction and preliminaries

Univalent functions are functions which are analytic, one-on-one and onto on a certain domain. Their study for more than a century shows that problems are significantly more difficult to be solved over the general class instead of its subclasses. This is also the case for the upper bound of the Hankel determinant, a problem rediscovered and extensively studied in recent years. Over the class $\mathscr{A}$ of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ analytic on the unit disk, this determinant is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

where $q \geqslant 1$ and $n \geqslant 1$. The second order Hankel determinants is

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

and the third order one is

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

For the general class $\mathscr{S}$ of univalent functions in the class $\mathscr{A}$ tehre are very few results concerning the Hankel determinant. The best known for the second order case is due to Hayman ([5]), saying that $|H_2(n)| \leqslant An^{1/2}$, where A is an absolute constant, and that this rate of growth is the best possible. Another one is [15], where

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it was proven that $|H_2(2)| \leqslant A$, with $1 \leqslant A \leqslant \frac{11}{3} = 3,66\ldots$ and $|H_3(1)| \leqslant B$, with $\frac{4}{9} \leqslant B \leqslant \frac{32+\sqrt{285}}{15} = 3.258796\cdots$.

There are much more results for the subclasses of $\mathscr{S}$. Namely, for starlike functions the upper bounds for the second and the third order Hankel determinant are 1 ([7]) and $= 0.777987\ldots$ ([13]), respectively, while for the same bounds for the convex functions they are $1/8$ ([7]) and $\frac{4}{135} = 0.0296\ldots$ ([8]). The estimates for the second order case are sharp, while of the third order are not, but are best known. For the class $\mathscr{R} \subset \mathscr{A}$ of functions with bounded turning satisfying $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$, we have sharp estimate $|H_2(1)| \leqslant \frac{4}{9} = 0.444\ldots$, ([6]) and probably non-sharp $|H_3(1)| \leqslant \frac{207}{540} = 0.38333\ldots$ ([14]). Other related work is published in [9, 10, 11, 17].

In this paper we study two classes introduced by Ozaki.

The first one is the class of Ozaki close-to-convex functions

$$\mathscr{F} = \{f \in \mathscr{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > -\frac{1}{2}, z \in \mathbb{D}\}$$

introduced by Ozaki in 1941 ([16]) and it is a subclass of the class of close-to-convex functions. For this class the non-sharp estiamtes are known $|H_2(2)| \leqslant \frac{21}{64}$ ([12]) and $|H_3(1)| \leqslant \frac{180+69\sqrt{15}}{32\sqrt{15}} = 3.6086187\ldots$ ([1]). We will significantly improve the second estimate to the value $0.1375\ldots$. More about this class one can find in [21, Sect. 9.5].

The other class that we will be considered is

$$\mathscr{G} = \{f \in \mathscr{A} : \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] < \frac{3}{2}, z \in \mathbb{D}\},$$

Ozaki in [16] introduced this class and proved that it is subclass of $\mathscr{S}$. Later, Sakaguchi in [19] and R. Singh and S. Singh in [20] showed, respectively, that functions in $\mathscr{G}$ are close-to-convex and starlike. Again in [12] it was shown that $|H_2(2)| \leqslant \frac{9}{320} = 0.028125\ldots$. Here we will give estimate of the third Hankel determinant.

In the studies given in this paper we use approach based on the estimates of the coefficients of Shwartz function due to Prokhorov and Szynal (Lemma 1 given below). This approach is essentially different than the commonly used and is the main reason for the improvement in the estimate for the class $\mathscr{F}$ mentioned above. Usually the research is done using a result on coefficients of Carathéodory functions (functions from with positive real part on the unit disk) that involves Toeplitz determinants (see [21, Theorem 3.1.4, p.26] and [4]).

Here is the result of Prokhorov and Szynal that we will need. In more general form it can be found in [18, Lemma 2].

LEMMA 1. *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function, i.e., be analytic in the unit disk and $|\omega(z)| < 1$ when $z \in \mathbb{D}$ and μ and ν be real numbers. If $\frac{1}{2} \leqslant |\mu| \leqslant 2$ and $\frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leqslant \nu \leqslant 1$, then $\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| \leqslant 1$.*

We will also need the following, almost forgotten result of Carleson ([2]).

LEMMA 2. *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_2| \leqslant 1 - |c_1|^2 \quad \textit{and} \quad |c_4| \leqslant 1 - |c_1|^2 - |c_2|^2.$$

2. Main results

We begin with improvement of the upper bound of the third Hankel determinant for the class $\mathscr{F}$ of Ozaki close-to-convex functions.

THEOREM 1. *Let $f \in \mathscr{F}$ be of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then*

$$|H_3(1)| \leqslant \frac{1}{8} = 0.125.$$

Proof. For a function $f \in \mathscr{F}$ there exists a Schwarz function $\omega(z) = c_1z + c_2z^2 + \cdots$ such that

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{1}{2} + \frac{3}{2} \cdot \frac{1+\omega(z)}{1-\omega(z)}, \tag{1}$$

i.e.,

$$[zf'(z)]' \cdot [1-\omega(z)] = [1+2\omega(z)] \cdot f'(z).$$

By equating the coefficients in the above expression we receive

$$\begin{aligned} a_2 &= \frac{3}{2}c_1, \\ a_3 &= \frac{1}{2}(4c_1^2 + c_2), \\ a_4 &= \frac{1}{2}(2c_3 + 13c_1c_2 + 20c_1^3), \\ a_5 &= \frac{3}{40}(2c_4 + 12c_1c_3 + 46c_1^2c_2 + 40c_1^4 + 5c_2^2). \end{aligned} \tag{2}$$

Using (2) we have

$$H_3(1) = \frac{1}{320}\left[4c_1^4c_2 + 8c_1^3c_3 + 4c_1c_2c_3 - 23c_1^2c_2^2 - 12c_1^2c_4 + 20c_2^3 - 20c_3^2 + 24c_2c_4\right]$$

and

$$\begin{aligned} 320H_3(1) = -20&\left[c_3^2 - \frac{1}{5}c_1c_2c_3 + \left(\frac{1}{10}\right)^2 c_1^2c_2^2\right] + \frac{1}{5}c_1^2c_2^2 - 23c_1^2c_2^2 \\ &+ 8c_1^3\left(c_3 + \frac{1}{2}c_1c_2\right) + 20c_2^3 + 12c^4(2c_2 - c_1^2). \end{aligned}$$

From here

$$\begin{aligned} 320|H_3(1)| \leqslant 20&\left|c_3 - \frac{1}{10}c_1c_2\right|^2 + \frac{114}{5}|c_1|^2|c_2|^2 + 8|c_1|^3\left|c_3 + \frac{1}{2}c_1c_2\right| \\ &+ 20|c_2|^3 + 12\left(2|c_2| + |c_1|^2\right)|c_4|. \end{aligned} \tag{3}$$

By applying Lemma 1 (with $(\mu,\nu)=(1/10,0)$ and $(\mu,\nu)=(1/2,0)$) and Lemma 2, we receive

$$\begin{aligned}
320|H_3(1)| &\leqslant 20+\frac{114}{5}|c_1|^2|c_2|^2+8|c_1|^3+20|c_2|^3\\
&\quad +12\left(2|c_2|+|c_1|^2\right)\left(1-|c_1|^2-|c_2|^2\right)\\
&=\frac{54}{5}|c_1|^2|c_2|^2+8|c_1|^3-4|c_2|^3+24|c_2|\\
&\quad -24|c_1|^2|c_2|+12|c_1|^2-12|c_1|^4\\
&=20+h(|c_1|,|c_2|),
\end{aligned}$$

where

$$h(x,y)=\frac{54}{5}x^2y^2+8x^3-4y^3+24y-24x^2y+12x^2-12x^4,$$

$0\leqslant x\leqslant 1$ and $0\leqslant y\leqslant 1-x^2$.

We continue with finding the maximum of the function h on the region $\Omega=\{(x,y):0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1-x^2\}$.

The function h has no critical points in the interior of Ω because $h'_y(x,y)=x^2\left(\frac{108}{5}y-24\right)-12y^2+24=0$ has only one positive solution for x, that is $\sqrt{\frac{5(2-y^2)}{10-9y}}$, an increasing function of y over $(0,\infty)$ with $x(0)=1$.

Therefore, we continue studying h on the edges of Ω.

For $x=0$, $h(0,y)=24y-4y^3\leqslant h(0,1)=20$.

For $x=1$, we have $y=0$, and $h(1,0)=8$.

For $y=0$, $h(x,0)=x^2(-12x^2+8x+12)$ which can be easily shown to increasing function on the segment $[0,1]$, with maximal value $h(1,0)=8$.

Finally, $g(x):=h(x,1-x^2)=\frac{74}{5}x^6-\frac{108}{5}x^4+8x^3-\frac{66}{5}x^2+20$ is a decreasing function on the interval $[0,1]$, since $g'(x)=-\frac{12}{5}x(1-x)(11+x+37x^2+37x^3)$ and $g'(x)=0$ has no solutions on $(0,1)$. Thus, $h(x,1-x)\leqslant g(0)=20$.

The above analysis brings the final conclusion that h has maximal value 20 on Ω, i.e.,

$$|H_3(1)|\leqslant\frac{1}{320}(20+20)=\frac{1}{8}.\quad\square$$

The previous result, although significantly improves the one from [1], still is not sharp, as the following one dealing with the class $\mathcal{G}$.

THEOREM 2. *Let $f\in\mathcal{G}$ and is of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$. Then*

$$|H_3(1)|\leqslant\frac{17}{1080}=0.01574\ldots.$$

Proof. Similarly as in the proof of the previous theorem, for each function f from $\mathcal{G}$, there exists a function $\omega(z)=c_1z+c_2z^2+\cdots$, analytic in $\mathbb{D}$, such that $|\omega(z)|<1$ for all z in $\mathbb{D}$, and

$$1+\frac{zf''(z)}{f'(z)}=\frac{3}{2}-\frac{1}{2}\cdot\frac{1+\omega(z)}{1-\omega(z)},\tag{4}$$

i.e.,

$$[zf'(z)]' \cdot [1-\omega(z)] = [1-2\omega(z)] \cdot f'(z).$$

From here, by equating the coefficients we receive

$$\begin{aligned}
a_2 &= -\frac{1}{2}c_1,\\
a_3 &= -\frac{1}{6}c_2,\\
a_4 &= -\frac{1}{24}(2c_3+c_1c_2),\\
a_5 &= -\frac{1}{120}(6c_4+4c_1c_3+3c_2^2+2c_1^2c_2).
\end{aligned}$$

From here, after some calculations we receive

$$\begin{aligned}
H_3(1) = \frac{1}{8640}\big[&-60c_3^2-132c_1c_2c_3+72c_1^3c_3+36c_4(2c_2+3c_1^2)\\
&+36c_1^4c_2+76c_2^3+3c_1^2c_2^2\big],
\end{aligned}$$

i.e.,

$$\begin{aligned}
8640H_3(1) = -60&\left[c_3^2+\frac{11}{5}c_1c_2c_3+\left(\frac{11}{10}\right)^2c_1^2c_2^2\right]+\left[60\left(\frac{11}{10}\right)^2+3\right]c_1^2c_2^2\\
&+72c_1^3\left(c_3+\frac{1}{2}c_1c_2\right)+76c_2^2+36(2c_2+3c_1^2)c_4
\end{aligned}$$

and further

$$\begin{aligned}
8640|H_3(1)| = 60&\left|c_3+\frac{11}{10}c_1c_2\right|^2+\frac{756}{10}|c_1|^2|c_2|^2+72|c_1|^3\left|c_3+\frac{1}{2}c_1c_2\right|\\
&+76|c_2|^2+36(2|c_2|+3|c_1|^2)|c_4|.
\end{aligned}$$

In a similar way as in the proof of the previous theorem, from Lemma 1 (with $(\mu,\nu)=(11/10,0)$ and $(\mu,\nu)=(1/2,0)$) and Lemma 2, we receive

$$\begin{aligned}
8640|H_3(1)| &= 60+\frac{756}{10}|c_1|^2|c_2|^2+72|c_1|^3+76|c_2|^2\\
&\quad+36(2+|c_1|^2)(1-|c_1|^2-|c_2|^2)\\
&= 60+h(|c_1|,|c_2|),
\end{aligned}$$

where

$$h(x,y)=\frac{756}{10}x^2y^2+72x^3+76y^3+36\left(2+x^2\right)\left(1-x^2-y^2\right),$$

$(x,y)\in\{(x,y): 0\leqslant x\leqslant 1, 0\leqslant y\leqslant 1-x^2\} =: \Omega.$

Since $h'_x(x,y)=\frac{36}{5}x\left(10x(3-2x)+11y^2-10\right)$, we obtain that $h'_x(x,y)=0$ has a positive solution $y_*(x)=\sqrt{\frac{10}{11}(2x^2-3x+1)}$ for $x\in(0,1/2)$. Further, $h'_y(x,y)=\frac{12}{5}y\left(33x^2+95y-60\right)$ and

$$\begin{aligned} g(x) &= h'_y(x,y_*(x)) \\ &= \frac{24}{\sqrt{55}}\sqrt{(x-1)\left(x-\frac{1}{2}\right)}\left[33x^2+95\sqrt{\frac{20}{11}}\sqrt{(x-1)\left(x-\frac{1}{2}\right)}-60\right] \end{aligned}$$

on the interval $(0,1)$, has solutions $x_1=0.5$ and $x_2=0.2311\ldots$, with $y_1=g(x_1)=0.75$ and $y_2=g(x_2)=0.6130\ldots$, respectively. Both, (x_1,y_1) and (x_2,y_2) are in Ω, so are critical points of h in the interior of Ω, such that $h(x_1,y_1)=69.75$ and $h(x_2,y_2)=62.10899\ldots$.

Further, on the edges of Ω we have the following.

For $x=0$, $h(0,y)=76y^3-72y^2+72\leqslant h(0,1)=76$.

For $x=1$, $h(0,1)=72$.

For $y=0$, $h(x,0)=-36x^4+72x^3-36x^2+72\leqslant h(0,0)=h(1,0)=72$.

For $y=1-x^2$, we have $h(x,1-x^2)=-\frac{182x^6}{5}+\frac{204x^4}{5}+72x^3-\frac{402x^2}{5}+76\leqslant 76$ obtained for $x=0$.

All the analysis from above leads to the conclusion that h has maximal value 76 on Ω obtained for $x=0$ and $y=1$, i.e.,

$$|H_3(1)|\leqslant\frac{1}{8640}(60+76)=\frac{17}{1080}=0.01574\ldots.\qquad\square$$

The estimates of the third Hankel determinant given in Theorem 1 and Theorem 2 are probably not sharp. Here is a conjecture of the sharp values.

CONJECTURE 1. *Let $f\in\mathcal{A}$ and is of the form $f(z)=z+a_2z^2+a_3z^3+\cdots$.*

(*i*) *If $f\in\mathcal{F}$, then $|H_3(1)|\leqslant\frac{1}{16}=0.0625$;*

(*ii*) *If $f\in\mathcal{G}$, then $|H_3(1)|\leqslant\frac{19}{2160}=0.00879\ldots$.*

Both estimates are sharp with extremal functions $\frac{1+2z^2}{1-z^2}$ and $\frac{1}{2}\left(z\sqrt{1-z^2}+\arcsin z\right)$, respectively, obtained for $\omega(z)=z^2$ in (1) and (4).

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