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ORIGINAL RESEARCH PAPER

Coefficients of the inverse of functions for the subclass of the class $\mathcal{U}(\lambda)$

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Abstract

Let $\mathcal{A}$ be the class of functions f that are analytic in the unit disk $\mathbb{D}$ and normalized such that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Let $0 < \lambda \leq 1$ and

$$\mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < \lambda,\ z \in \mathbb{D} \right\}.$$

In this paper sharp upper bounds of the first three coefficients of the inverse function f^{-1} are given in the case when

$$\frac{f(z)}{z} \prec \frac{1}{(1 - z)(1 - \lambda z)}.$$

Keywords Univalent · Inverse functions · Coefficients · Sharp bound

Mathematics Subject Classification 30C45

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Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathbb{D}$ and let $\mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, denote the set of all $f \in \mathcal{A}$ satisfying the condition

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \lambda \qquad (z \in \mathbb{D}). \tag{1}$$

For $\lambda = 1$ we put $\mathcal{U}(1) = \mathcal{U}$. More about these classes can be found in [5–8, 10].

In [7] it was claimed that all functions f from $\mathcal{U}(\lambda)$ satisfy

$$\frac{f(z)}{z} \prec \frac{1}{(1+z)(1+\lambda z)}. \tag{2}$$

Here "$\prec$" denotes the usual subordination, i.e., $F(z) \prec G(z)$, for f and G being analytic functions in $\mathbb{D}$, means that there exists a function $\omega(z)$, also analytic in $\|D$, such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Recently, in [3], the authors gave a counterexample that the subordination (2) is not necessarily satisfied by all functions from $\mathcal{U}(\lambda)$.

For the functions f from $\mathcal{U}(\lambda)$ satisfying subordination (2) we have

$$\frac{f(z)}{z} = \frac{1}{(1-\omega(z))(1-\lambda\omega(z))}, \tag{3}$$

where ω is a Schwarz function, i.e., it is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1, z \in \mathbb{D}$. Let's put

$$\omega(z) = c_1 z + c_2 z^2 + \cdots.$$

Later on we will use the fact due to Schur [9] that $|c_2| \leq 1 - |c_1|^2$ (which can be found also in Carlson's work [1]).

Further, the inequality (1) for the function f from $\mathcal{U}(\lambda)$ can be rewritten in the following, equivalent, form

$$\left| \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1 \right| < \lambda \qquad (z \in \mathbb{D})$$

and further

$$\left| \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1 \right| \leq \lambda |z|^2 \qquad (z \in \mathbb{D}).$$

From here, after some calculations we obtain

$$|(1+\lambda)c_2 - \lambda c_1^2 + (2(1+\lambda)c_3 - 4\lambda c_1 c_2)z + \cdots| \leq \lambda$$

for all $z \in \mathbb{D}$, and next,

$$|(1+\lambda)c_2-\lambda c_1^2|\leq\lambda,\quad |2(1+\lambda)c_3-4\lambda c_1c_2|\leq\lambda-\frac{1}{\lambda}|(1+\lambda)c_2-\lambda c_1^2|^2,\tag{4}$$

for all $z\in\mathbb{D}$. The last inequality follows from the result of Carlson for the second coefficient of Schwarz functions cited above.

If $f\in\mathcal{S}$ and

$$f(z)=z+a_2z^2+a_3z^3+\cdots,\tag{5}$$

then the inverse of f has an expansion

$$f^{-1}(w)=w+A_2w^2+A_3w^3+\cdots\tag{6}$$

near the origin (or precisely at least in $|w|<\frac{1}{4}$). By using the identity $f(f^{-1}(w))=w$ and the representations for the functions f and f^{-1}, we can obtain the next relations

$$\begin{cases}A_2=-a_2,\\ A_3=-a_3+2a_2^2,\\ A_4=-a_4+5a_2a_3-5a_2^3.\end{cases}\tag{7}$$

The main results of this paper are the sharp upper bounds for the modulus of these three initial coefficients of f^{-1}.

Theorem 1 *Let $f\in\mathcal{U}(\lambda)$, $0<\lambda\leq 1$, satisfy the subordination (2), and let f and f^{-1} be given by (5) and (6), respectively. Then*

$$|A_2|\leq 1+\lambda,$$
$$|A_3|\leq 1+3\lambda+\lambda^2,$$
$$|A_4|\leq(1+\lambda)(1+5\lambda+\lambda^2).$$

All these results are best possible.

Proof For $f\in\mathcal{U}(\lambda)$, from the relation (3) we have (see [4, 7])

$$\sum_{n=1}^{\infty}a_{n+1}z^n=\sum_{n=1}^{\infty}\frac{1-\lambda^{n+1}}{1-\lambda}\omega^n(z).\tag{8}$$

If we put $\omega(z)=c_1z+c_2z^2+\cdots$, then from (8) by comparing the coefficients we obtain

$$\begin{cases}a_2=(1+\lambda)c_1,\\ a_3=(1+\lambda)c_2+(1+\lambda+\lambda^2)c_1^2,\\ a_4=(1+\lambda)c_3+2(1+\lambda+\lambda^2)c_1c_2+(1+\lambda+\lambda^2+\lambda^3)c_1^3.\end{cases}\tag{9}$$

Using (7) and (9) we also have

$$\begin{cases} A_2 = -(1+\lambda)c_1, \\ A_3 = -(1+\lambda)c_2 + (1+3\lambda+\lambda^2)c_1^2, \\ A_4 = -(1+\lambda)c_3 + (3+8\lambda+3\lambda^2)c_1c_2 - (1+\lambda)(1+5\lambda+\lambda^2)c_1^3. \end{cases} \tag{10}$$

Since $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$, from (10) we receive

$$|A_2| \leq 1 + \lambda$$

and

$$\begin{aligned} |A_3| &\leq (1+\lambda)|c_2| + (1+3\lambda+\lambda^2)|c_1|^2 \\ &\leq (1+\lambda)(1-|c_1|^2) + (1+3\lambda+\lambda^2)|c_1|^2 \\ &\leq (1+\lambda) + (2\lambda+\lambda^2)|c_1|^2 \\ &\leq 1 + 3\lambda + \lambda^2. \end{aligned}$$

Also, from (10) we obtain

$$A_4 = -\frac{1}{2}\left[2(1+\lambda)c_3 - 4\lambda c_1c_2 - 6(1+\lambda)c_1((1+\lambda)c_2 - \lambda c_1^2) + 2(1+\lambda)^3c_1^3\right],$$

and from here, by applying (4),

$$\begin{aligned} |A_4| &\leq \frac{1}{2}\left[|2(1+\lambda)c_3 - 4\lambda c_1c_2| + 6(1+\lambda)|c_1||(1+\lambda)c_2 - \lambda c_1^2| + 2(1+\lambda)^3|c_1|^3\right] \\ &\leq \frac{1}{2}\left[\lambda - \frac{1}{\lambda}|(1+\lambda)c_2 - \lambda c_1^2|^2 + 6(1+\lambda)|c_1||(1+\lambda)c_2 - \lambda c_1^2| + 2(1+\lambda)^3|c_1|^3\right] \\ &= \frac{1}{2}\left[\lambda - \frac{1}{\lambda}t^2 + 6(1+\lambda)|c_1|t + 2(1+\lambda)^3|c_1|^3\right] \\ &=: \frac{1}{2}h(t), \end{aligned}$$

where $t = |(1+\lambda)c_2 - \lambda c_1^2|$ and $0 \leq t \leq \lambda$, since

$$|(1+\lambda)c_2 - \lambda c_1^2| \leq (1+\lambda)|c_2| + \lambda|c_1|^2 \leq (1+\lambda)(1-|c_1|^2) + \lambda|c_1|^2 = \lambda.$$

As for the maximal value of the function h, we consider two cases:

Case 1: When $0 \leq |c_1| \leq \frac{1}{3(1+\lambda)}$ the function h attains its maximum for $t_0 = 3(1+\lambda)\lambda|c_1|$ and we have

$$h(t_0) \leq \lambda + 27\lambda(1+\lambda)^2|c_1|^2 + 2(1+\lambda)^3|c_1|^3 \leq 4\lambda + \frac{2}{27},$$

i.e.,

$$|A_4| \leq 2\lambda + \frac{1}{27}.$$

Case 2: For $\frac{1}{3(1+\lambda)} \leq |c_1| \leq 1$, the function h attains its maximum for $t = \lambda$ and we have

$$h(t) \leq 6(1+\lambda)\lambda|c_1| + 2(1+\lambda)^3|c_1|^3 \leq 2(1+\lambda)(1+5\lambda+\lambda^2),$$

when $0 \leq t \leq \lambda$. So,

$$|A_4| \leq (1+\lambda)(1+5\lambda+\lambda^2).$$

From cases 1 and 2, since $(1+\lambda)(1+5\lambda+\lambda^2) > 2\lambda + \frac{1}{27}$ when $0 < \lambda \leq 1$, we receive the estimate for $|A_4|$.

For the proof of sharpness of the theorem, let us consider the function

$$w = f_\lambda(z) = \frac{z}{(1-z)(1-\lambda z)}.$$

Then

$$z = f_\lambda^{-1}(w) = w - (1+\lambda)w^2 + (1+3\lambda+\lambda^2)w^3 - (1+\lambda)(1+5\lambda+\lambda^2)w^4 - \cdots, \tag{11}$$

which shows that our results are the best possible. □

Note that for $\lambda = 1$ in Theorem 1 we have the estimates for class $\mathcal{U}$ and in that case the inverse of the Koebe function is extremal, as for the class $\mathcal{S}$ (see, for example Goodman's book, Vol II, p. 205, [2]).

In the next theorem we study the Fekete-Szegő functional for the inverse functions of the class $\mathcal{U}(\lambda)$. Namely, we have

Theorem 2 *For the inverse functions of functions from* $\mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, *satisfying subordination* (2), *we have*

$$|A_3 - \mu A_2^2| \leq \lambda + |1-\mu|(1+\lambda)^2,$$

where μ *is a complex number. The result is sharp for* $0 \leq \mu \leq 1$.

Proof From the relations (10) and (4) we obtain

$$\begin{aligned} |A_3 - \mu A_2^2| &= |-(1+\lambda)c_2 + (1+3\lambda+\lambda^2)c_1^2 - \mu(1+\lambda)^2 c_1^2| \\ &= |-((1+\lambda)c_2 - \lambda c_1^2) + (1-\mu)(1+\lambda)^2 c_1^2| \\ &\leq |(1+\lambda)c_2 - \lambda c_1^2| + |1-\mu|(1+\lambda)^2|c_1|^2 \\ &\leq \lambda + |1-\mu|(1+\lambda)^2. \end{aligned}$$

The sharpness of the estimate in the case when $0 \leq \mu \leq 1$ follows from the function f_λ^{-1} defined by (11). □

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Conflict of interest The authors declare that they have no conflict of interest.

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