5[TH] INTERNATIONAL CONFERENCE
CIVIL ENGINEERING - SCIENCE AND PRACTICE

ŽABLJAK, 17-21 FEBRUARY 2014

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FREE VIBRATIONS OF DELAMINATED COMPOSITE AND SANDWICH PLATES

Summary

The aim of this paper is comparison of fundamental dynamic characteristics of intact and damaged laminated composite and sandwich plates. Generalized Layerwise Plate Theory is used for analysis of different plates with presence of delaminations. Jumps in displacement field are incorporated using Heaviside step functions. Different layerwise FE are derived. Mathematical model is coded in MATLAB® for free vibrations analysis of laminated composite and sandwich plates. Effects of plate aspect ratio, lamination scheme, degree of orthotropy and delamination position on fundamental dynamic characteristics are analyzed. All numerical solutions are compared with existing data from literature. Excellent agreement is obtained. The variety of new results is presented.

Key words

laminated composite plate, sandwich plate, free vibrations, finite element

SLOBODNE VIBRACIJE LAMINATNIH KOMPOZITNIH I SENDVIČ PLOČA SA DELAMINACIJAMA

Rezime

U ovom radu prikazano je poređenje dinamičkih karakteristika laminatnih kompozitnih i sendvič ploča sa ili bez oštećenja. Laminatna teorija je korišćena za analizu različitih ploča sa prisustvom delaminacija. Skokovi u polju pomeranja su uzeti u obzir primenom Heaviside-ovih f-ja. Izvedeni su različiti slojeviti KE. Za analizu slobodnih vibracija laminatnih kompozitnih i sendvič ploča korišćen je MATLAB® program. Analiziran je uticaj geometrije ploče, šeme laminacije, stepena ortotropije i položaja delaminacije na dinamičke karakteristike ploča. Sva rešenja su upoređena sa postojećim rezultatima iz literature i dobijeno je odlično poklapanje. Prikazano je mnoštvo novih rezultata.

Ključne reči

laminatna kompozitna ploča, sendvič ploča, slobodne vibracije, konačni element

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1. INTRODUCTION

Laminar composites are applicable in many engineering disciplines. In order to predict the behavior of laminated composite and sandwich plates under different types of loading, we have to describe local and global structural response. Global behavior is accurately predicted by the use of ESL plate theories. However, these models are primarily intended for thin plates. If the scope of the analysis is the thick structural component, we must take into account the transverse shear deformation. Mindlin-Reissner ESL model accounts for the transverse shear effects by use of shear correction factors. ESL theories are not adequate for thick plates for two reasons: (1) neglection of transverse shear deformation and (2) inability to account for jumps in transverse shear strains at layer interfaces.

Kirchhoff plate theory overpredicts natural frequencies [1]. The overview of ESL plate theories is given in [2]. The great effort has been made in the analysis of laminar composites with the presence of different forms of damage. Generalized Laminated Plate Theory [3] is used here to analyze the overall plate response. Barbero and Reddy further extended this theory to account for jump discontinuities in displacement field [4].

Piece-wise linear variation of in-plane displacements is imposed, as well as constant transverse displacement. It is assumed that C_0 continuity through the plate thickness is satisfied, so the nodal variables in the FE model are translations in three orthogonal directions. Cross-sectional warping is accounted by using of the layerwise expansion of in-plane displacements. Consistent mass matrix is employed [5]. Owen and Li [6] used the ESL theory to analyze the natural frequencies of intact laminar composites. Vuksanović [1] derived the FE model based on the HSDT theory. Refined theory is used in work of Noor [7]. Ćetković and Vuksanović [8] have used the LW theory for the derivation of the analytical and numerical solutions for intact plates. LW model is also used in Ref. [9] and it was implemented in ABAQUS® to study the influence of delamination on mode shapes.

Comparative study between fundamental dynamic characteristics of intact and damaged laminar composites, as well as of the soft-core sandwich plates, is the scope of the analysis in this paper. Mode shapes of damaged composite and soft-core sandwich plates are plotted using the originally coded MATLAB® program, and local mode shapes are presented. Influence of embedded delamination on higher bending modes is investigated.

2. GENERALIZED LAMINATED PLATE THEORY

We will analyze the laminated plate made of n orthotropic laminas. Global CS is shown in Figure 1. Material (local) CS of each lamina coincides with fiber direction. The number of numerical layers is denoted as N. This number can be equal or higher than the number of layers. The number of delaminated interfaces is ND. Plate thickness is denoted as h. Linearly elastic orthotropic material is used; inextensibility of transverse normal is imposed (plane stress model), and kinematic relations are linear and follow Hooke's law.

2.1. DISPLACEMENT FIELD

Displacement components (u_1,u_2,u_3) at the point (x,y,z) of the laminate are written as:

$$\begin{aligned}
u_1(x,y,z) &= u(x,y) + \sum_{I=1}^{N} u^I(x,y)\Phi^I(z) + \sum_{I=1}^{ND} U^I(x,y)H^I(z) \\
u_2(x,y,z) &= v(x,y) + \sum_{I=1}^{N} v^I(x,y)\Phi^I(z) + \sum_{I=1}^{ND} V^I(x,y)H^I(z) \\
u_3(x,y,z) &= w(x,y) + \sum_{I=1}^{ND} W^I(x,y)H^I(z)
\end{aligned} \tag{1}$$

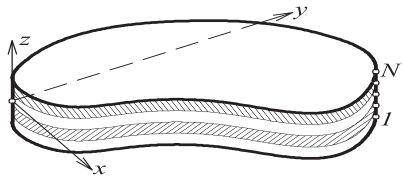

Figure 1. Typical laminated composite plate in global coordinate system (4 layers)

In Eq. (1), (u,v,w) are mid-plane displacement components, (u^I,v^I) are undetermined coefficients, and (U^I,V^I,W^I) are jump discontinuities in the displacement field in the I^{th} delaminated interface. $\Phi^J(z)$ are layerwise continuous functions of z-coordinate. $H^I(z)$ are Heaviside step functions defined as follows:

$$H^I(z \ge 0) = \begin{cases} +1, & z^I \le z \le h/2 \\ 0, & -h/2 \le z < z^I \end{cases} \qquad H^I(z < 0) = \begin{cases} 0, & z^I \le z \le h/2 \\ -1, & -h/2 \le z < z^I \end{cases} \tag{2}$$

In the FE model, (u,v,w) are mid-plane values of (u_1,u_2,u_3), (u^I,v^I) are additions of (u_1,u_2) to the mid-plane displacement components in the I^{th} numerical layer, and (U^I,V^I,W^I) are displacement jumps in the I^{th} delaminated interface. $\Phi^J(z)$ are the one-dimensional Lagrange interpolations of z-coordinate, and $H^I(z)$ are Heaviside step functions.

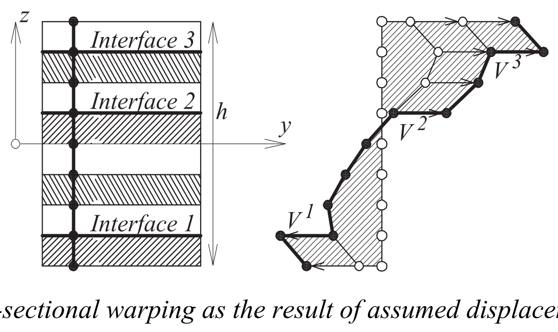

Figure 2. Cross-sectional warping as the result of assumed displacement field, for u_2

Linear Lagrange interpolations of in-plane displacements are assumed. They are piecewise continuous through the laminate thickness in intact region, and discontinuous in delaminated planes. Distribution of in-plane displacements u_2 through the plate thickness is shown in Figure 2. It is obvious that every delamination incorporates jump discontinuity in displacement field in all nodes above/below the delaminated interface.

2.2. KINEMATIC AND CONSTITUTIVE RELATIONS

Linear strain field based on the previously derived displacement field (1) is:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \sum_{I=1}^{N} \frac{\partial u^I}{\partial x}\Phi^I + \sum_{I=1}^{ND} \frac{\partial U^I}{\partial x} H^I, \qquad \varepsilon_y = \frac{\partial v}{\partial y} + \sum_{I=1}^{N} \frac{\partial v^I}{\partial y}\Phi^I + \sum_{I=1}^{ND} \frac{\partial V^I}{\partial y} H^I$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \sum_{I=1}^{N} u^I \frac{d\Phi^I}{dz} + \sum_{I=1}^{ND} \frac{\partial W^I}{\partial x} H^I, \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \sum_{I=1}^{N} v^I \frac{d\Phi^I}{dz} + \sum_{I=1}^{ND} \frac{\partial W^I}{\partial y} H^I \tag{3}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \sum_{I=1}^{N} \left(\frac{\partial u^I}{\partial y} + \frac{\partial v^I}{\partial x} \right)\Phi^I + \sum_{I=1}^{ND} \left(\frac{\partial U^I}{\partial x} + \frac{\partial V^I}{\partial y} \right) H^I$$

Constitutive relations for the k^{th} orthotropic layer in local coordinate system are given in Eq. (4). Constitutive relations in global coordinate system are written in the transformed form in Eq. (5). Q_{ij} are reduced stiffness components for the plane stress case:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & & & \\ Q_{12} & Q_{22} & & & \\ & & Q_{66} & & \\ & & & Q_{44} & \\ & & & & Q_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}^{(k)} \tag{4}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & & \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & & \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & & \\ & & & \bar{Q}_{55} & \bar{Q}_{45} \\ & & & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} \tag{5}$$

2.3. PRINCIPLE OF VIRTUAL WORK

The virtual work statement is derived using Hamilton's principle (see Eq. (6)). Stress resultants and constitutive matrices from the extended form of Eq. (6) can be found in detail in Ref. [4]. In Eqs. (7), only the matrix form of stress resultants is shown:

$$\int_V \left(\sigma_{ij} \delta\varepsilon_j \right) dV - \int_V \rho \left(\ddot{u}_i \delta u_i \right) dV = 0 \tag{6}$$

$$\begin{gathered} \{N\} = [A]\{\varepsilon\} + \sum_{I=1}^{N} \left[B^I \right] \left\{ \varepsilon^I \right\} + \sum_{I=1}^{ND} \left[E^I \right] \left\{ \bar{\varepsilon}^I \right\} \\ \left\{ N^I \right\} = \left[B^I \right] \left\{ \varepsilon^I \right\} + \sum_{J=1}^{N} \left[D^{IJ} \right] \left\{ \varepsilon^J \right\} + \sum_{J=1}^{ND} \left[L^{IJ} \right] \left\{ \bar{\varepsilon}^J \right\} \\ \left\{ \bar{N}^I \right\} = \left[E^I \right] \left\{ \varepsilon^I \right\} + \sum_{J=1}^{N} \left[L^{JI} \right] \left\{ \varepsilon^J \right\} + \sum_{J=1}^{ND} \left[F^{IJ} \right] \left\{ \bar{\varepsilon}^J \right\} \end{gathered} \tag{7}$$

3. LAYERWISE FINITE ELEMENT MODEL

In this paper, layerwise FE model is presented. It consists of the mid-plane numerical layer, *n* numerical layers through the plate thickness and *ND* delaminated interfaces. Natural coordinate system of the single FE is located in the middle of the FE. In each node of the FE mesh, we have the following variables: (u,v,w) as mid-plane absolute displacements, (u^I,v^I) as relative displacements in I^{th} layer (say node) through the thickness, and (U^I,V^I,W^I) as displacement jumps in I^{th} delaminated interface. It is obvious that the proposed model allows arbitrary number of delaminations to be modeled. Note that in delaminated interfaces, nodes on line segment (transverse normal) are "virtually duplicated". In-plane displacement field is interpolated using the *2D* classical Lagrangian interpolations. After that, strain field is interpolated in the same manner, using the derivatives of the Lagrangian interpolations (matrices ***B*** in the FE model). For deriving of the LW FE model based on the displacement field and kinematics given in previous chapters, we substitute the assumed interpolation of the displacement field into the virtual work principle for a representative finite element of the delaminated plate, and derive the element stiffness and mass matrices. Assembly procedure of the element matrices is done in a usual manner. From the previous considerations we derive system equation of the FE assembly for free vibrations problem:

$$\left[K_{nn}\right]-\omega^2\left[M_{nn}\right]\left\{d_n\right\}=0 \tag{8}$$

In Eq. (8), $[\boldsymbol{K_{nn}}]$ and $[\boldsymbol{M_{nn}}]$ are stiffness and mass submatrices in the FE assembly, respectively, $\{\boldsymbol{d_n}\}$ is the displacement subvector, and ω is the angular frequency. By solving the eigenvalue problem, natural frequencies and corresponding mode shapes are derived.

4. RESULTS AND DISCUSSION

For the purpose of this work, 4-node and 9-node quadrilateral FE are derived. All procedures are coded in MATLAB®, for the free vibrations analysis of laminated composite and sandwich plates. Element stiffness and mass matrices were evaluated using Full and Reduced Integration. Reduced Integration is used when side-to-thickness ratio of the FE exceeded the thin plate limit. Consistent mass matrix is implemented. Accuracy of the proposed model is verified by comparison with existing results from literature.

Problem 4.1. Simply supported square laminated composite plate is considered. Symmetric cross-ply lamination scheme and layers of equal thickness are adopted. **Five layers** through overall plate height ***h*** are considered, as well as different levels of orthotropy. Each layer is made of material with following mechanical characteristics:

$E_2=2.10\times10^6$ N/cm^2, $G_{12}=1.26\times10^6$ N/cm^2, $G_{13}=G_{23}=1.05\times10^6$N/cm^2,

$\upsilon_{12}=0.25$, $\rho=8\times10^{-6}$Ns^2/cm^4

The effects of orthotropy and lamination scheme of simply supported cross-ply laminated composite plate with ***a/h*=5** are presented in Table 1. The fundamental frequencies are presented in non-dimenzionalized form: $\bar{\omega}=\omega\sqrt{\rho h^2/E_2}$.

Problem 4.2. We will consider an 8-layer free square plate with a side length of 178mm. The total thickness of the plate is 1.58mm, and all plies are of 0° orientation. Each layer is of equal thickness and it has the following mechanical properties:

$E_1 = 172.7$ GPa, $E_2 = E_3 = 7.2$ GPa, $G_{12} = G_{13} = 3.76$ GPa, $G_{23} = 2.71$ GPa,

$\upsilon_{12} = \upsilon_{13} = 0.25$, $\upsilon_{23} = 0.33$, $\rho = 1566$ kg/m^3

Table 1. Non-dimenzionalized natural frequencies of intact SS cross-ply square laminates

Numerical model	$E_1 / E_2 = 3$	$E_1 / E_2 = 10$	$E_1 / E_2 = 20$
Owen and Li - Refined [6]	0.2699	0.3453	0.4030
Vuksanović - HSDT [1]	0.2684	0.3442	0.3939
Noor – 3D Elasticity [7]	0.2659	0.3409	0.3979
Present (9-node; 25 FE)	0.2618	0.3330	0.3858

Table 2. Comparison of natural frequencies (Hz) of intact (0/0/0/0)s free square laminate

Numerical model	FE	Mode 1	Mode 2	Mode 3	Mode 4
Alnefaie [9] (3D layerwise model; ABAQUS®/CAE v6.3)	25	78.51	100.17	277.31	316.42
	100	81.23	107.20	207.72	294.01
Layerwise model (9-node-model; Reduced Integration)	25	81.67	110.07	199.37	306.17
	100	81.50	109.90	199.52	303.02
Layerwise model (4-node-model; Reduced Integration)	25	80.23	116.22	206.38	372.85
	100	81.86	111.56	201.59	320.33

Table 2 shows natural frequencies of the first 4 mode shapes for two different mesh densities, using 25 or 100 FE. The proposed model uses significantly lower number of nodal variables and saves the computational time in comparison with the full 3D model.

Problem 4.3. An anti-symmetric simply supported square soft-core sandwich plate is analyzed. Plate is composed from the cross-ply rigid face sheets, each of thickness t_f and the core of thickness t_c. The following lamination scheme is used: (0/90/core/90/0). Face sheets are made of Graphite-Epoxy T300/934 with following mechanical characteristics:

$E_1 = 131$ GPa, $E_2 = E_3 = 10.34$ GPa, $G_{12} = G_{23} = 6.895$ GPa, $G_{13} = 6.205$GPa,

$\upsilon_{12} = \upsilon_{13} = 0.22$, $\upsilon_{23} = 0.49$, $\rho = 1627$ kg/m^3

Isotropic soft-core has the following mechanical characteristics:

$E = 6.89$ MPa, $G = 3.45$ MPa, $\upsilon = 0$, $\rho = 97$ kg/m^3

Table 3. Non-dimenzionalized natural frequencies of SS square soft-core sandwich plate with $t_c/t_f = 10$ and $a/h = 10$, for different delamination positions

Mode	Intact [8]	Intact	Interface 1	Interface 2
1,1	1.87	1.85	1.78	1.83
1,2	3.24	3.25	2.89	3.18
1,3	5.32	5.36	5.01	5.53
2,2	4.40	4.32	4.10	4.29
2,3	6.26	6.21	5.96	6.18
3,3	7.97	7.82	7.59	7.73
Local	-	-	4.86	3.95

Natural freqs. are presented in the non-dimenzionalized form: $\bar{\omega} = \omega a^2 \sqrt{\left(\rho / E_2\right)_f}$.
Different positions of centrally located square delamination of side length $\boldsymbol{a_{del} = a/2}$ were studied using the 4-node-model, with 256 finite elements. Two interface planes were considered. Interface 1 is between the soft-core and the upper face sheet and Interface 2 is between the 0^o and 90^o laminas in the upper face sheet. It is obvious that if the delamination is closer to the mid-plane, natural frequency is more reduced, so the influence of delamination is higher. If the plate is delaminated, local mode shapes occur beside the global mode shapes. Global and local mode shapes for different delamination positions are shown in Figures 3-5. From the performed modal analysis it is obvious that the proposed model is capable to calculate local and global natural frequencies of delaminated composite and sandwich plates. If the delamination is located closer to the mid-plane, the local natural frequency rises (Table 3). The reason of this is the increasing of the bending stiffness of the delaminated segment, which starts to vibrate independently from the intact rest of the plate. Natural frequency of the delaminated segment is denoted as local natural frequency.

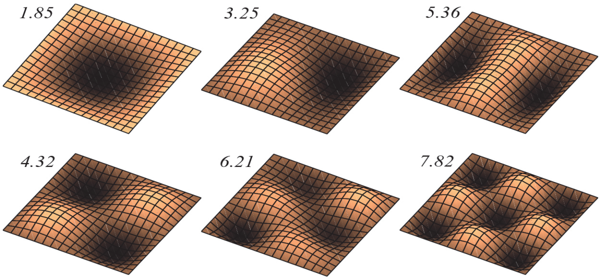

Figure 3. Non-dimenzionalized natural freqs. and mode shapes of intact sandwich plate

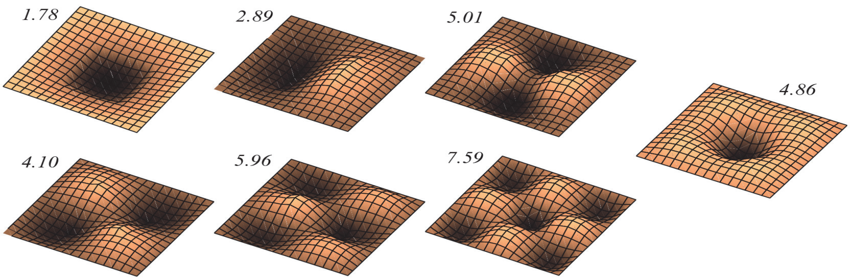

Figure 4. Non-dimenzionalized natural frequencies and mode shapes of sandwich plate with delamination in Interface 1

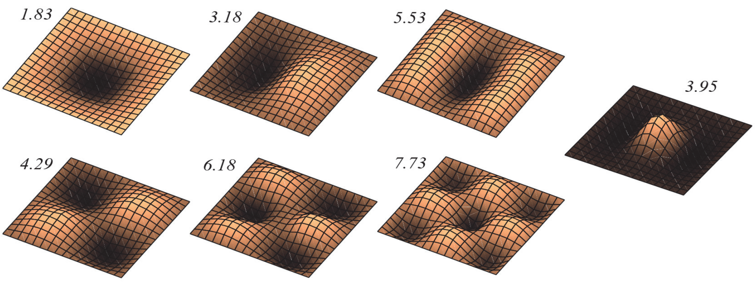

Figure 5. Non-dimenzionalized natural frequencies and mode shapes of sandwich plate with delamination in Interface 2

5. CONCLUSIONS

Layerwise FE based on the extended GLPT, are derived in this paper. Original program is coded in MATLAB® for FE analysis of free vibrations of laminated composite and soft-core sandwich plates. The proposed model is capable to accurately predict the natural frequencies both for the intact and delaminated plates. From the presented results we may conclude that incorporation of the transverse shear deformation reduces the fundamental frequencies of laminar composites. This reduction is more pronounced for the plates with higher E_1/E_2 ratios. The proposed model implies only simple 2D-FE mesh, in contrast to the conventional 3D-FE models. Numerical capacity is still kept on the very high level. Using of Reduced Integration was necessary for avoiding of shear locking, in the case of coarse FE mesh. When delamination occurs, the delaminated segment starts to oscillate independently from the rigid rest of the plate. The proposed model is able to capture the local mode shape and frequency of the delaminated segment. This important feature serves as the basis for defining the new set of boundary conditions which can be incorporated on crack boundaries. Global and local mode shapes of delaminated soft-core sandwich plate are plotted using MATLAB program and may serve as benchmark for future investigations.

ACKNOWLEDGEMENTS

The authors are thankful for the financial support received by Ministry of Education and Science of the Republic of Serbia, through the Project TR 36048.

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