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DEVELOPMENT OF DYNAMIC STIFFNESS METHOD FOR FREE VIBRATION ANALYSIS OF PLATE STRUCTURES

In the paper, an overview of the development of the dynamic-stiffness-method-based computational model for the free vibration analysis of plates has been presented. Starting from several formulations of the so-called dynamic stiffness elements, formulated at the Institute for Numerical Analysis and Design of Structures (INP) at the Faculty of Civil Engineering, University of Belgrade in the last decade, a novel software framework FREEVIB has been developed and validated. FREEVIB is object-oriented software in Python environment, designed to predict free vibration characteristics in a wide range of possible structural problems (stepped, stiffened and folded plate structures, implying isotropic or orthotropic material formulations). The presented methodology still serves as a strong basis for further improvements through the extensive research efforts of authors, their collaborators and students.

Keywords: free vibration, dynamic stiffness method, software development, Python

1. INTRODUCTION

Dynamic stiffness method (DSM), also known in the literature as spectral element method (SEM), is nowadays used as an alternative to the finite element method (FEM) in the free vibration analysis of different engineering structures [1, 2]. It is highly competitive against the FEM in terms of computational time and cost in mid and high frequency ranges where very fine mesh of finite elements is required for the computation of the free vibration response.

Main component of the DSM is the strong form solution of the governing equations of motion of the corresponding elastodynamic problem formulated in the frequency domain, based on which frequency dependent dynamic stiffness matrix is formulated. As a consequence, structural discretization is frequency independent and affected only by the geometrical and/or material discontinuities of the structure, implying that only one dynamic stiffness element can exactly represent structural behavior at any frequency.

First dynamic stiffness matrices have been formulated for one-dimensional elements (beams and bars) for which closed-form solution of the governing equations of motion can be found. Kolousek [3] was the first who developed dynamic stiffness matrix for beam element, based on Bernoulli-Euler theory. Later on, dynamic stiffness matrices for wide range of one-dimensional dynamic stiffness elements have been developed [4-7].

In a series of contributions [8-11], dynamic stiffness matrices of two-dimensional elements have been derived and applied in the free vibration and buckling analysis of both isotropic and anisotropic long plate assemblies based on classical or first order shear deformation plate theory. In the works of Boscolo and Banerjee [12-17], dynamic stiffness elements of Levy-type plates have been developed enabling free vibration study of isotropic and composite plates and stiffened plate assemblies.

All afore mentioned studies were limited to plates and plate assemblies having special boundary conditions (i.e. two opposite edges simply supported), for which closed-form solutions of the governing equations can be found. This issue has been overcome by Casimir et al. [18], who derived dynamic stiffness matrix for a completely free isotropic rectangular plate element for transverse vibration based on classical plate theory (CPT), using the projection and superposition methods.

The research in the field of dynamic stiffness method at the Faculty of Civil Engineering, University of Belgrade, started in a frame of the research project TR-36046: "*Towards development of sustainable cities: Influence of traffic induced vibrations on buildings and humans".* Within the framework of the project, numerical model for dynamic analysis of soil-structure system has been formulated using the substructure approach, where the structure has been modeled using dynamic stiffness elements, while for soil modeling, integral transform method has been applied [19]. In a series of further studies, authors formulated dynamic stiffness matrices for a completely free rectangular isotropic plate element based on the first order (FSDT) [20] and higher order shear deformation theory (HSDT) [21], as well as for the plate element ongoing in-plane vibration [22]. Recently, the above formulations have been extended to the free vibration analysis of sandwich [23], symmetric cross-ply laminated composite plates [24, 25], and composite stiffened plate assemblies [26]. Moreover, starting from the developed dynamic stiffness elements, object-oriented software in Python environment – FREEVIB [27] has been developed, enabling free vibration analysis of a wide range of possible structural problems. Finally, recent efforts in this field are related to the development of the dynamic stiffness element of an open cylindrical shell [28, 29] and its implementation in FREEVIB.

Main objective of the paper is to give an overview of the dynamic-stiffness-method-based computational model for vibration analysis of plates incorporated in the FREEVIB software, developed by the authors.

2. FUNDAMENTALS OF THE DYNAMIC STIFFNESS FORMULATION

The dynamic stiffness matrix of a corresponding plate element is obtained through several steps. The first one is derivation of Euler-Lagrange equations of motion for the considered plate theory, by using the Hamilton's principle in terms of the displacements. The next steps are explained as follows.

Transformation of the governing equations of motion to frequency domain. The equations of motion are transformed into the frequency domain by assuming a harmonic representation of the displacement/rotation field:

$$u(x,y,t)=\hat{u}(x,y,\omega)e^{i\omega t} \quad (1)$$

In. Eq. (1), $\hat{u}(x,y,\omega)$ are the amplitudes of the displacement/rotation $\hat{u}(x,y,t)$ in the frequency domain. Having in mind that Eq. (1) is valid for all angular frequencies ω in the considered frequency range, the argument ω will be omitted in further representations. After the substitution of the above transformation into the governing equations of motion, the equations of motion are transformed into the following set of partial differential equations:

$$\mathbf{L}\hat{u}(x,y)=0 \quad (2)$$

where **L** is the matrix of the differential operators [20-24] defined in terms of the plate stiffness coefficients, the mass moments of inertia and the angular frequency ω.

Superposition of symmetry contributions. Displacement or rotation amplitudes of a rectangular plate element $\hat{u}(x,y)$ can be presented as a superposition of four symmetry contributions: both symmetric (SS), symmetric - anti-symmetric (SA), anti-symmetric - symmetric (AS) and both anti-symmetric (AA), [20-

24]. By the superposition of 4 symmetry contributions, it is possible to analyze only one quarter of a rectangular plate, which significantly reduces the size of the corresponding dynamic stiffness matrices. By using the method of separation of variables, the general solution for each symmetry contribution can be represented in the Fourier series form as:

$$\tilde{u}^{ij}(x,y) = \sum_{m}^{\infty} {}^{1}U_m^{ij}(x)\,{}^{1}f_m^{ij}(y) + \sum_{m}^{\infty} {}^{2}U_m^{ij}(y)\,{}^{2}f_m^{ij}(x) \tag{3}$$

In Eq. (3), $\tilde{u}^{ij}(x,y)$ is the corresponding displacement/rotation function, ${}^{1}U_m^{ij}(x)$ and ${}^{2}U_m^{ij}(y)$ (*ij* = SS, SA, AS or AA) are the unknown displacement/ rotation functions, while $f_m^{ij}(y)$ and $f_m^{ij}(x)$ are the base trigonometric functions, depending on the symmetry case. In practical calculations, the infinite Fourier series must be truncated. Thus, the accuracy of solution depends on the number of terms in the general solution.

The solutions for all symmetry contributions are given in [20-24].

Projection method. The vector of displacement components $\hat{u}^{ij}$ along plate boundaries is denoted as displacement vector $\hat{\mathbf{q}}^{ij}$. The corresponding force vector $\hat{\mathbf{Q}}^{ij}$ consists of force components along plate boundaries. Both vectors are functions of spatial variables *x* and *y*, so they cannot be related explicitly, as in the case of one – dimensional elements. The issue can be overcome with the aid of the projection method [18]. Therefore, instead of using the vectors $\hat{\mathbf{q}}^{ij}$ and $\hat{\mathbf{Q}}^{ij}$, new projection vectors $\tilde{\mathbf{q}}^{ij}$ and $\tilde{\mathbf{Q}}^{ij}$ are introduced, which components are the Fourier coefficients in the series expansion (see [20-26] for details).

The relation between the projection vectors $\tilde{\mathbf{q}}^{ij}$ and $\tilde{\mathbf{Q}}^{ij}$ and the vector of integration constants $\mathbf{C}^{ij}$ is obtained as [20-22]:

$$\tilde{\mathbf{q}}^{ij} = \tilde{\mathbf{D}}_D^{ij}\mathbf{C}^{ij}, \quad \tilde{\mathbf{Q}}^{ij} = \tilde{\mathbf{F}}_D^{ij}\mathbf{C}^{ij} \tag{4}$$

Finally, by eliminating the vector $\mathbf{C}^{ij}$ from Eq. (4) the following relation between the projection vectors $\tilde{\mathbf{Q}}^{ij}$ and $\tilde{\mathbf{q}}^{ij}$ is obtained:

$$\tilde{\mathbf{Q}}^{ij} = \tilde{\mathbf{F}}_D^{ij}\left(\tilde{\mathbf{D}}_D^{ij}\right)^{-1}\tilde{\mathbf{q}}^{ij} = \tilde{\mathbf{K}}_D^{ij}\tilde{\mathbf{q}}^{ij} \tag{5}$$

where $\tilde{\mathbf{K}}_{Dt}^{ij}$ is the dynamic stiffness matrix for the *ij* symmetry contribution.

The dynamic stiffness matrix for a completely free dynamic stiffness element, which relates the projections of the forces and displacements along the four plate boundaries, is obtained by using the transformation matrix **T** (for details see [20-26]). The size of the dynamic stiffness matrix $\tilde{\mathbf{K}}_D$ depends on the number of terms in the general solution *M*, the type of the vibration problem (in plane or transverse) and applied plate theory (CPT, FSDT, HSDT).

Finally, considering that transverse and in-plane vibrations of a single plate represent two independent states, the dynamic stiffness matrix of the single plate can be written as:

$$\tilde{\mathbf{K}}_D = \begin{bmatrix} \tilde{\mathbf{K}}_{Dt} & 0 \\ 0 & \tilde{\mathbf{K}}_{Di} \end{bmatrix} \tag{7}$$

In Eq. (7), $\tilde{\mathbf{K}}_{Dt}$ denotes the dynamic stiffness matrix of plate element for transverse vibrations, while $\tilde{\mathbf{K}}_{Di}$ is the dynamic stiffness matrix of plate element undergoing in-plane vibrations which can be determined in the same way as $\tilde{\mathbf{K}}_{Dt}$.

Rotation of dynamic stiffness matrices to global coordinates. For two stiffened plate assemblies where plate 1 and plate 2 are connected to each other with an arbitrary angle between them, vibrations of plate 1 causes vibrations of the corresponding plate 2, and vice versa. Consequently, it is necessary to transform the displacement and force projection vectors $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{Q}}$ defined in the local coordinate system of the single plate to the corresponding projection vectors $\tilde{\mathbf{q}}^{*}$ and $\tilde{\mathbf{Q}}^{*}$ in the global coordinate system of the plate assembly. This is accomplished by using the rotation matrix $\mathbf{T_R}$, which depends on the number of terms in the general solution, angle between the local and global coordinate system and the selected dynamic stiffness element. After that, the dynamic stiffness matrix of the single plate in global coordinate system is derived in the same way as in the conventional FEM. The assembly procedure is then performed as in the conventional FEM.

Computation of natural frequencies and mode shapes. In the analysis, arbitrary boundary conditions can be applied by removing the rows and columns of the global dynamic stiffness matrix that correspond to the components of the constrained displacement projections. After that, the natural frequencies can be computed from the following equation:

$$\det\left|\tilde{\mathbf{K}}_{D,nn}^{G}(\omega)\right| = 0 \tag{8}$$

where $\tilde{\mathbf{K}}_{D,nn}^{G}$ is the global dynamic stiffness sub-matrix of the plate assembly related to the unknown generalized displacement projections $\tilde{\mathbf{q}}_{n}^{G}$ of the plate assembly. Since the elements of the dynamic stiffness matrix $\tilde{\mathbf{K}}_{D,nn}^{G}$ contain transcendental functions, the solutions can be obtained using some of the search methods. To avoid numerical difficulties when calculating the zeroes of Eq. (8), the natural frequencies can be determined as maxima of the following expression [1]:

$$g(\omega) = \log \frac{1}{\det\left|\tilde{\mathbf{K}}_{D,nn}^{G}(\omega)\right|} \tag{9}$$

The expression (9) is computed for all frequencies in the frequency range of interest with a frequency increment $\Delta\omega$. Consequently, the accuracy of the computed natural frequencies is affected only by the frequency increment. After the natural frequencies have been computed, the i^{th} mode shape corresponding to the natural frequency ω_i is obtained in the usual manner.

3. DEVELOPMENT OF COMPUTER CODE FREEVIB

Formulated dynamic stiffness elements served as basis for the development of computational framework for free vibration analysis of plate-like structures. The object-oriented software FREEVIB has been developed at the Institute for Numerical Analysis and Design of Structures (INP) at the Faculty of Civil Engineering, University of Belgrade.

FREEVIB is written in Python [30], which comes with a large standard library that covers areas such as string processing, software engineering and operating system interfaces. Object-oriented design is introduced to enable code encapsulation, class inheritance and further code reusability.

For input parameters related to the geometry, material and number of terms in trigonometric series (M), simple text file may be created in the prescribed format, or generated using the existing graphical pre-processors. Using the procedure described in the previous sections, a variety of plate-like structural problems, illustrated in Fig. 1, can be analyzed: (a) individual plates, (b) plate assembly of arbitrary shape, (c) stepped plates, (d) stiffened plates, (e) cracked plates. In addition, different material properties of single- and multi-layer plates can be considered: (f) single layer isotropic or orthotropic plates, (g) sandwich panels, or (h) laminated composite plates.

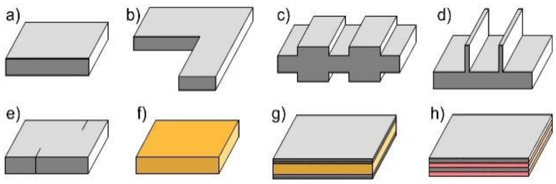

Figure 1. Structural problems which can be analyzed using the proposed computational framework

In Fig. 2, general FREEVIB class structure is illustrated. Note that `DynStiffElement` class is the super class which implements almost all methods related to the creation of the dynamic stiffness matrix. Detailed class structure can be found in [27].

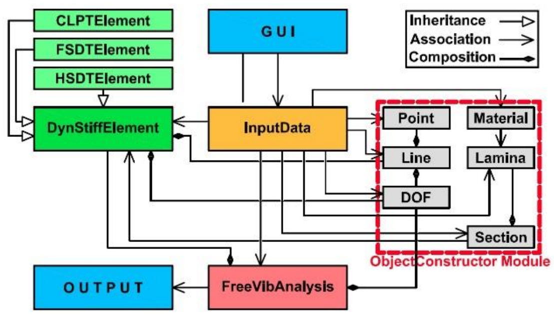

Figure 2. FREEVIB class structure

The so-far developed code could be extended by adding new dynamic stiffness element formulations. The example is the open cylindrical shell element [28, 29]. The family of implemented algorithms could be extended by adding new analysis types, such as response or buckling analysis. With the rapid development of multicore CPU technique, using the multiprocessing module in Python's Standard Library would enable the parallel execution of the code and speed up program execution. Namely, the solving of the equation (9) is time consuming, since it should be performed for every frequency in the considered range. This process is mutually independent for each frequency, thus the parallelization of the code by using the parallel FOR-loops would drastically increase the speed of the sequential execution.

4. ILUSTRATIVE EXAMPLE

To illustrate the reliability of the presented methodology, free vibration properties of the 5-layer rectangular cross-laminated timber (CLT) panels have been derived using FREEVIB. FSDT-based dynamic stiffness elements (M=5) have been used that give accurate results for considered panels [31]. The panels are 2.5m wide (*b*), with different spans (*a*): 2.5m, 5.0m, 10m and 15m. Considered panel thicknesses are 16cm and 20cm. The panels are simply supported along all edges (S-S-S-S) and composed from timber of class C24 (E_1 = 11000 MPa, E_2 = 370 MPa, ν_{12} = 0.44, G_{12} = G_{13} = 690 MPa, G_{23} = 50 MPa and ρ = 420 kg/m^3).

The results from FREEVIB are compared against the predictions from commonly used handbooks for CLT design [32, 33] and illustrated in Figure 3.

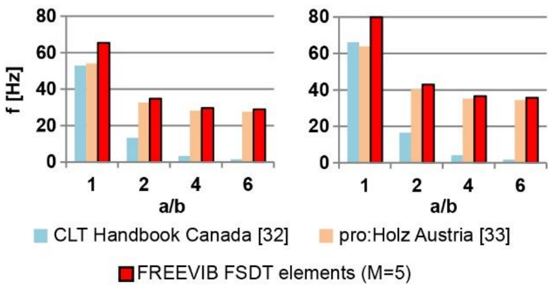

Figure 3. Fundamental frequencies of 5-layer C24 SSSS CLT panel considering different computational models: h=0.16m (left) and h=0.20m (right)

Fundamental frequencies calculated using [32-33] obviously are not matching the exact solution obtained using FREEVIB. Such high discrepancies are due to the simplified (beam) models used in [32-33], which are unable to predict the free vibration characteristics for plates with all edges simply supported. Better agreement is achieved for plates with higher *a*/*b* ratio. Finally, as shown in [31], better agreement would be achieved for plates with two opposite edges simply supported (SFSF).

5. CONCLUSIONS

An overview of the dynamic stiffness method and a family of novel dynamic stiffness elements developed at the Institute for Numerical Analysis and Design of Structures (INP) at the Faculty of Civil Engineering, University of Belgrade, has been presented in the paper. The dynamic stiffness elements have been implemented in the object-oriented software FREEVIB enabling free vibration analysis of a wide range of plate-like structures, considering different plate theories, isotropic or orthotropic behavior and multi-layer plates.

Free vibration study of the 5-layer rectangular CLT panel undergoing transverse vibration has demonstrated high accuracy and efficiency of the developed methodology, as well as its practical application.

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