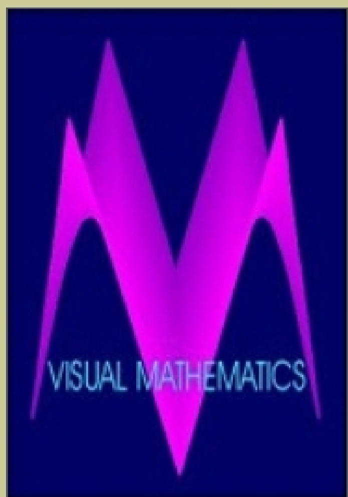

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FORMING A REGULAR PENTAGON, DECAGON AND PENTAGRAM USING ORIGAMI TECHNIQUE

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Forming a regular pentagon, decagon and pentagram using origami technique is based on the construction of approximately 36^o angle by dint of A4 paper of format. The implemented method is "fold and cut". We endeavored to obtain the polygons with the most accurate sides, i.e. with the minimal aberration of the interior angle. There is also considered an ideal paper format, which would provide the most accurate result, with negligible error. As the most suitable, we accepted the argentic rectangle.

Key words: pentagon, decagon, pentagram, origami, argentic rectangle

1. INTRODUCTION: REGULAR POLYGONS

Constructions of regular polygons (triangle, square and pentagon) start in ancient times. Also, even then it was known that it is possible to construct a polygon with twice the number of pages. The possibility of constructing a proper heptagon was analyzed by Archimedes, although the first construction using conic sections was written in the tenth century, by Arab mathematicians.

Analytic geometry made it possible to transfer the problem of geometric constructions solvability on solving algebraic equations, so

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that the construction of regular n-gon is equivalent to the solution of the cyclotomic equation:

$$x^n - 1 = 0.$$

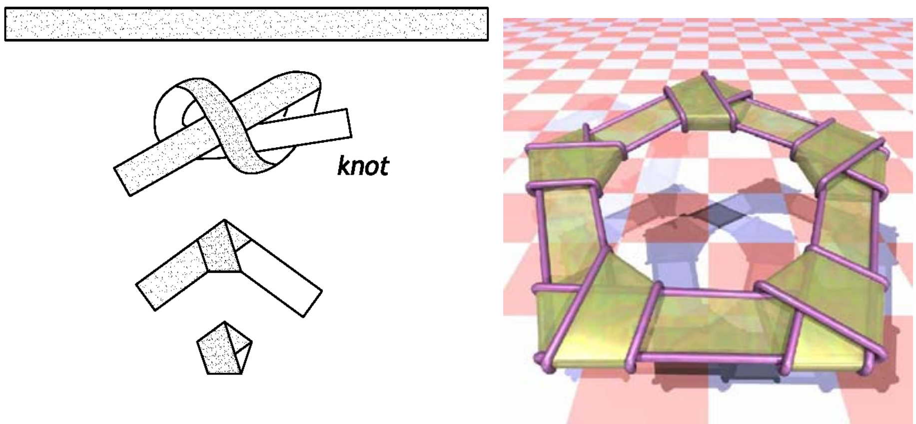
Relationship between geometry, algebra and The Number theory first was noted by German mathematician Carl Friedrich Gauss (1777 -1855) who managed to resolve the problem of geometric design of proper polygons with **n** sides.

Geometric constructions of a regular pentagon by ruler and compass (about 40 varieties) can be found in the book of regular polygons "*Pravilni poligoni*" [3].

This paper deals with one of the possible constructions of pentagon (decagon and five-pointed stars) by the method of folding and cutting paper. Origami mathematics is intensively developing in the last twenty years, linking skills of folding paper (origami) with the geometric structure of flat figure, polyhedron and 3D form.

1.1 Knot Method

By the method of folding strips of paper in a knot (**Figure 1**) it is possible to get regular pentagon:



1) 2) *the picture is taken from [9]*

Figure 1. *Knot Pentagon*

1.2. Fold-And-Cut Method

Using the method of folding paper of any format, so that a section (cut) obtains the desired image in the plane, will be the subject of this paper.

This idea was first recorded in 1721. in the Japanese book *Kan Chu Sen*, and later, in XX Century, the popular illusionist and magician Harry Houdini and Gerald Loe used it in their tricks. Mathematician Martin Gardner (1914 -) first raised the question of the formation of more complex polygons with this method, in the series "*Scientific American*".

2. FIVE-POINTED STAR CONSTRUCTION BY METHOD OF FOLD-AND-CUT

By the method of fold-and-cut, upholsterer Betsy Ross has changed the appearance of stars on the American flag, says the anecdote. In 1776. Betsy was asked by George Washington to create six pointed star that would look like the Star of David. The secret of B. Ross was in the attempt to use her art of folding paper (sized 8 ½ x 10) to fascinate the committee and to get a five-pointed star by just one cut.

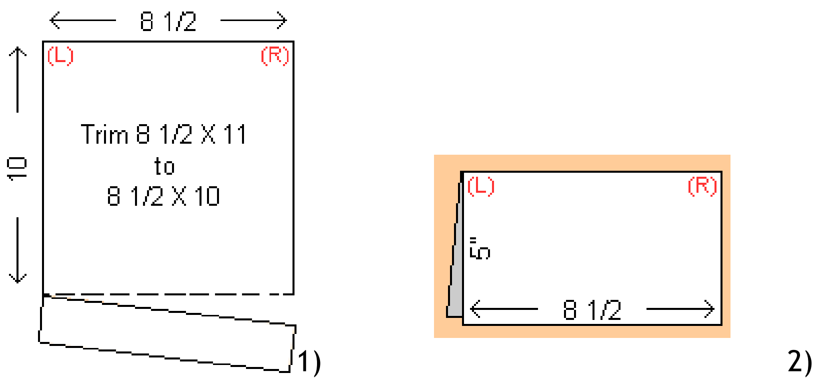


Figure 2: (*1- 2) Prepairing and folding the paper of Letter format*

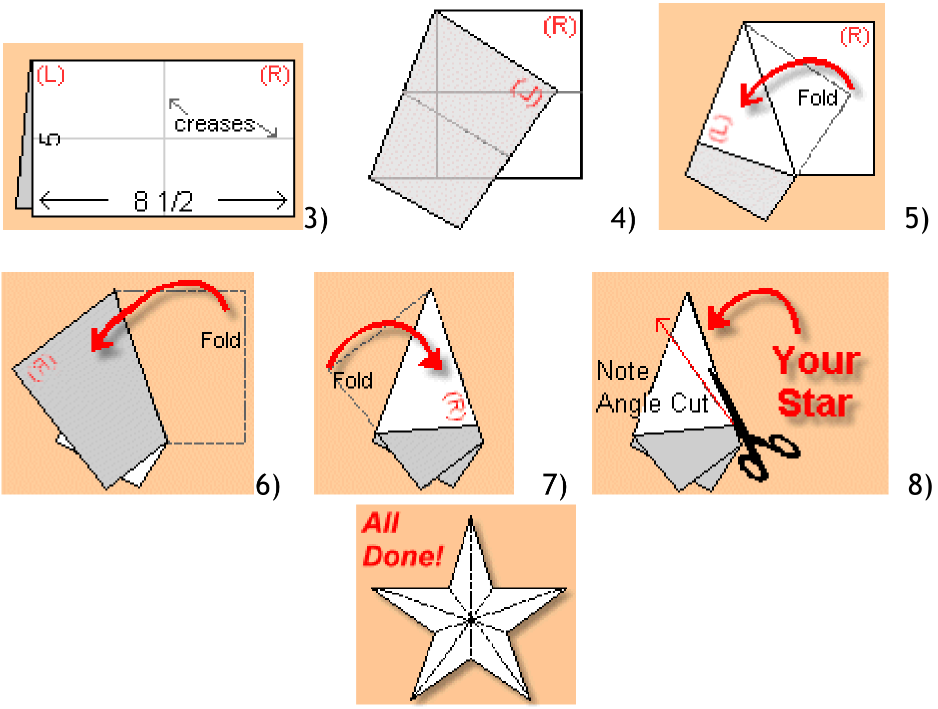


Figure 2 (3-9): *Making 5-pointed star by fold-and-cut method, the picture is taken from [10]*

2.1.The Construction of Five-Pointed Star by Fold-And-Cut Method, Using Letter Size Paper

It is also possible to bend the letter size paper (8 ½ x 11", ie. 21.59 x 27.94 cm) so that one section gets five-pointed star, pentagon or decagon, whereat it is necessary to do the following:

1° Fold a letter size paper by longer half-page; one of the angles of the paper edge is marked as **L,** (**Figure 3**).
2° Paper bent like this, now again is divided in half by a short side (to indicate the midpoint, **P**);
3° Bend upper right corner (point **K**) to pre-specified midpoint of the paper edge (point **P**).
4° Folded paper marks a point **R,** so that there is established triangle **LRP**, which values are calculated in the following remarks;

5° Now bend the top left side of the hypotenuse of the triangle **RP LRP**;
6° The folded paper now:
7° Bend back, right edge to specified left edge of the paper;

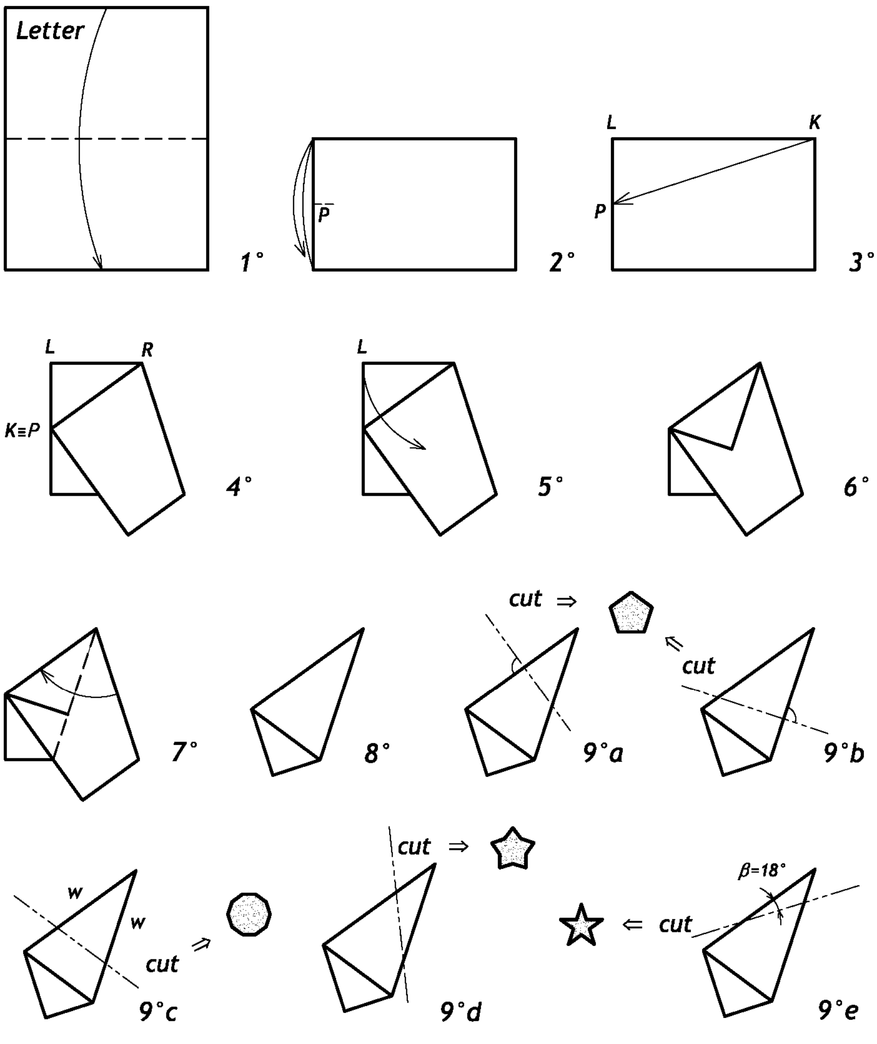


Figure 3: *Origami Pentagon, Dodecagon And Star Pentagon, Folded From A Letter Paper Format*

8° Folded like this, cut a paper by one cut!
9° (**a** or **b**) The section of the paper, such that is orthogonal to the one of its sides, will give a right-angled triangle, which will, when we develop it, emerge as right pentagon;

(**c**) if the folded paper is cut in manner to get an isosceles triangle, then the developed figure will be decagon;

(**d** or **e**) section that provides a scalene triangle will provide a variety of five-pointed stars in development, while for the angle $\beta=18^o$ we will get the regular star pentagon – pentagram.

2.2. The Construction of Regular Pentagon and Five-Pointed Star by Fold-And-Cut Method, Using A4 Paper

Using origami technique, we will make a regular pentagon for which we use standard **A4** paper (rectangle ratio is $1: \sqrt{2}$) with small modifications of the previous (fold-and-cut) method by B. Ross.

To create accurate plane figures, we need to know the characters mark of origami and also the skills of paper folding. Required accessories are: a couple of papers **A4** (21 x 29.7 cm, ie. 8.27 "x 11.69") and scissors. Following these steps (**Figure 4**) we will get the desired regular polygons:

1° We will fold **A4** paper by a longer half-page, and obtain the point **G** as the vertex on the paper edge;
2° Bent like this, paper is once more divided on halves by a longer and a shorter side (just indicate the midpoints **M** and **N**);
3° We bend the upper left corner (point **G**) by the hypotenuse **MN** of the triangle **MNG** of previously indicated midpoint of the paper. Angle **<MNG** is about **36°**, which will be proved later on;
4° We turn over the folded paper;
5° Now we bend the top left edge to the specified right edge;
6° The folded paper we now:
7° Bend back, the left side to the right, as marked;
8° Folded like this, we cut the paper by one cut!
9° (**a** or **b**) The section of the paper, such that is orthogonal to the one of its sides, will give us the right-angled triangle, which will, when we develop it, emerge as right pentagon;

(**c**) if the folded paper is cut in manner to get an isosceles triangle, then the developed figure will be decagon;

(**d** or **e**) section that provides a scalene triangle will provide a variety of five-pointed stars in development, while for the angle $\beta=18^o$ we will get the right star pentagon – pentagram.

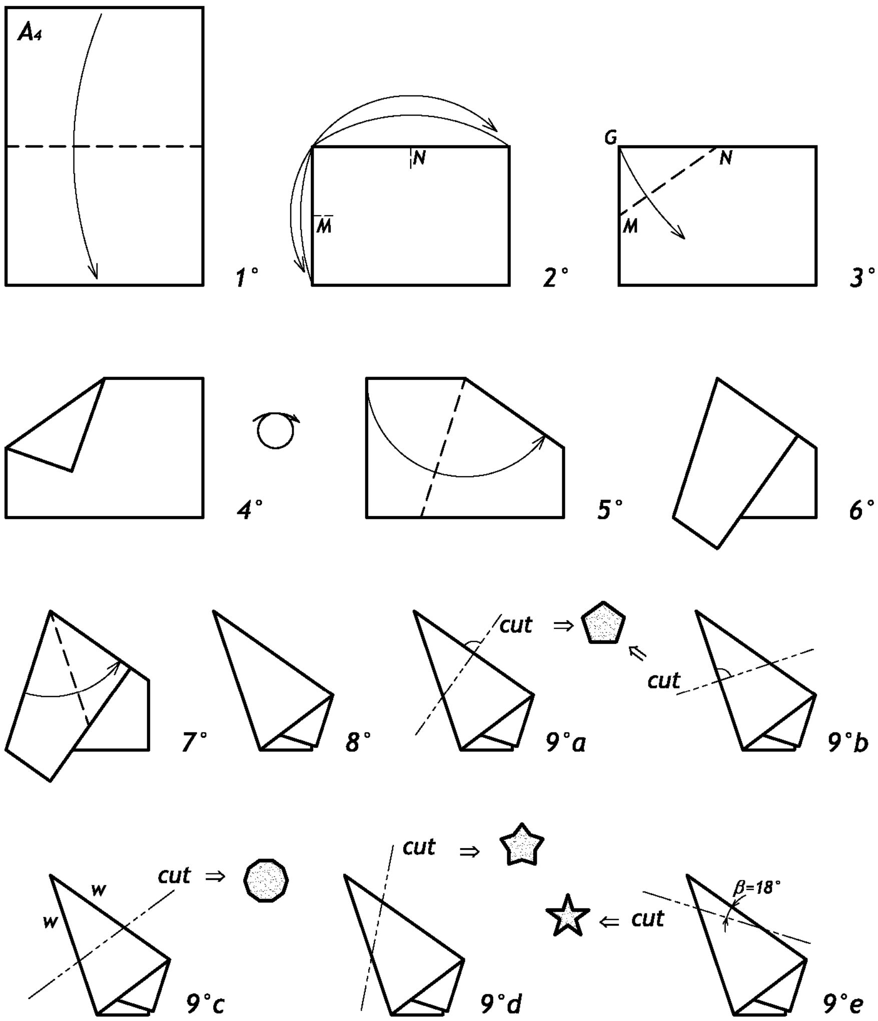


Figure 4: *Origami Pentagon, Dodecagon And Star Pentagon, Folded From* ***A4*** *Paper Format*

2.3. The Construction of Regular Pentagon and Five-Pointed Star by Fold-And-Cut Method Using Argentic Rectangle

It is also possible to bend a paper in form of a rectangle whose aspect ratio is **1:1.37638**, which is called **argentic rectangle** (21 x 28.904 cm), ie. [√ (5 +2 √ 5)] (shorter side): [2 + √ 5] (longer side), to get a regular pentagram, pentagon and decagon.

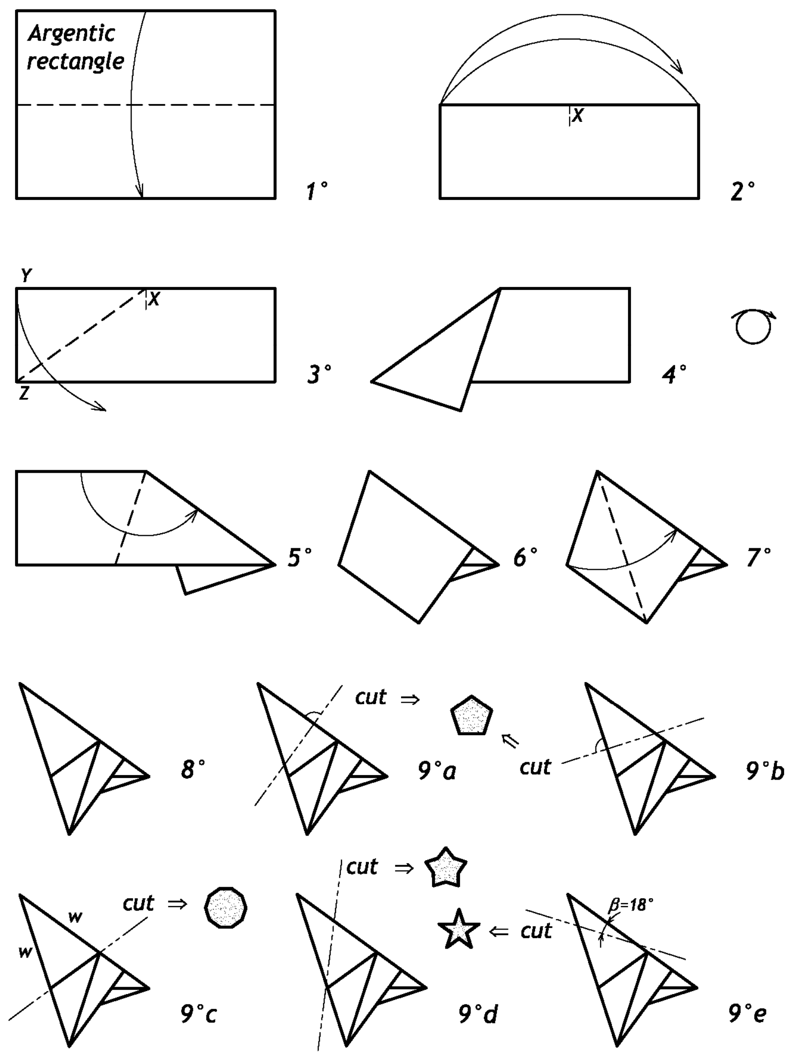


Figure 5: *Origami Pentagon, Dodecagon And Star Pentagon* By Folding Paper Of Argentic Rectangle Format

The fold-and-cut method is given in **Figure 5**:

1° We fold the argentic rectangle paper format by the short half-side, and obtain the point **Y** as the vertex of the paper edge;
2° Bent like this, paper is once more divided in half by a longer side (to indicate the midpoint **X**);
3° We bend the upper left corner (point **Y**) by the hypotenuse **XZ** of the triangle **XYZ**. Angle <**ZXY**=36°, which will be proven later on;
4° Then turn over the folded paper;
5° Now we bend the top left edge to the specified right edge;
6° The folded paper we now:
7° Bend back, left side to the right as marked;
8° Folded like this, we cut a paper by one cut!
9° (**a** or **b**) The section of the paper, such that is orthogonal to the one of its sides, will provide us the right-angled triangle, which will, when we develop it, emerge as the right pentagon;

(**c**) if the folded paper is cut in manner to get an isosceles triangle, then the developed figure will be decagon;

(**d** or **e**) section that provides a scalene triangle, will provide a variety of five-pointed stars in development, while for the angle $\beta=18^o$ we will get the regular star pentagon – pentagram.

2.4 Mathematical Analysis of The Angles of the Regular Polygon Obtained by Origami Technique

With right-angled triangles **ZXY**, **MNG** and **PRL**, which we mentioned in steps **3 °** of the Figures **3- 5**, we will calculate the angle α (as shown in the **figure 6**) and compare whether the regular polygon will be obtained by letter paper, A4 or argentic rectangle.

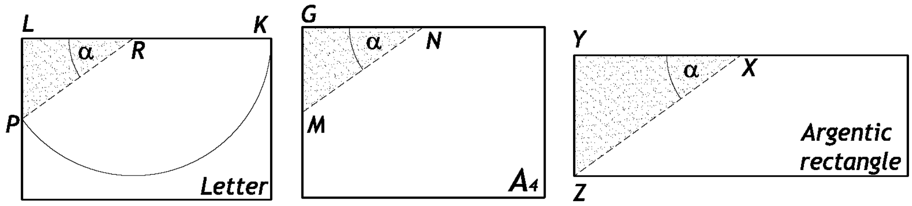


Figure 6: *Papers Of Letter, A4 And Argentic Rectangle Format*

$$LR^2 + LP^2 = RP^2$$
$$LR^2 + LP^2 = RK^2$$
$$LR^2 + 7^2 = (21{,}6 - LR)^2$$
$$\Rightarrow \quad LR = 9.66574$$

$$\tan\alpha = \frac{LP}{LR} = \frac{7}{9.66574}$$

$$= 0.7242$$

$$\Rightarrow \quad \alpha = 35.91\,^{o}$$

$$\tan\alpha = \frac{GM}{GN} = \frac{7.425}{10.5} = 0.7071$$

$$\Rightarrow \quad \alpha = 35.2657\,^{o}$$

It is clear that more accurate polygon is created from the letter formatted paper, then from A4 format. However, if we, as B. Ross did, cut A4 paper short to the edge for 0.5 cm (we will get format: 20.5 x 29.7 cm) so the polygons will be even more accurate than with the Letter paper. Certainly, due to bending of the paper that has its own thickness, it is required to take the angle of less than ***36^o***.

$$\tan\alpha = \frac{GM}{GN} = \frac{7.425}{10.25} = 0.7244$$

$$\Rightarrow \quad \alpha = 35.92\,^{o} \approx 36\,^{o}$$

In the case of format **20 x 29 cm** (cut the short side for 1 cm and the longer for 0.7 cm) we will get more accurate angle:

$$\tan\alpha = \frac{7.25}{10} = 0.725 \;\Rightarrow\; \alpha = 35.94\,^{o} \approx 36\,^{o}$$

The most accurate polygon arises from argentic rectangle format, as it can be seen from the triangle ***ZXY***:

$$\tan \alpha = \frac{YZ}{XY} = \frac{10.5}{14.452} = 0.72654$$

$$\Rightarrow \quad \alpha = 36^{o}$$

To create a regular decagon, due to the large number of bending, it is better not to reduce the format of paper A4. The error is negligible, and we can say that the polygons obtained in this manner can be considered as regular.

The methods described in sections **2.1, 2.2** and **2.3** differ very little in the folding procedure, and produce the approximate angle of ***36°***.

3. CONCLUSIONS

The resulting polygons can be inspiring and be used for further research:

- How to create heptagon, nonagon or their stelations;
- How the polygons obtained in such a manner can be used to form 3D models using origami (eg. Hyperdodecagon, **Figure 7**).

To consider application of these surfaces in various fields of engineering, design and art.

Figure 7: *Hyperdodecagon Made By Origami Technique*

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