World Scientific
www.worldscientific.com

ON THE GENUS OF THE INTERSECTION GRAPH
OF IDEALS OF A COMMUTATIVE RING

ZORAN S. PUCANOVIĆ

Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73
Belgrade, Serbia
pucanovic@grf.bg.ac.rs

MARKO RADOVANOVIĆ

Faculty of Mathematics
University of Belgrade
Studentski trg 16
Belgrade, Serbia
markor@matf.bg.ac.rs

ALEKSANDRA LJ. ERIĆ

Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73
Belgrade, Serbia
eric@grf.bg.ac.rs

To each commutative ring R one can associate the graph $G(R)$, called the intersection graph of ideals, whose vertices are nontrivial ideals of R. In this paper, we try to establish some connections between commutative ring theory and graph theory, by study of the genus of the intersection graph of ideals. We classify all graphs of genus 2 that are intersection graphs of ideals of some commutative rings and obtain some lower bounds for the genus of the intersection graph of ideals of a nonlocal commutative ring.

Keywords: Intersection graph; commutative rings; graph embeddings.

Mathematics Subject Classification: 13A15, 05C10, 13H99, 05E40

Z. Pucanović, M. Radovanović & A. Erić

1. Introduction

In order to get a better understanding of a given algebraic structure A, one can associate to it a graph G and study an interplay of algebraic properties of A and combinatorial properties of G. In this paper, we try to establish some connections between commutative ring theory and graph theory.

Throughout this paper all rings are commutative.

Let R be a ring with identity and $I^*(R)$ the set of its nontrivial ideals. Since the ideal structure reflects ring properties, several graphs that are based on the ideals were defined (see, e.g. [2, 9, 12, 24]). In this paper, we study *the intersection graph of ideals $G(R)$*, which is defined as follows:

$$V(G(R)) := I^*(R), \quad E(G(R)) := \{\{I_1, I_2\} : I_1 \cap I_2 \neq 0\},$$

where $V(G(R))$ (respectively, $E(G(R))$) denotes the set of vertices (respectively, edges) of the graph $G(R)$. This graph was introduced in [12], where Chakrabarty *et al.* studied planarity of intersection graphs of the ring $\mathbb{Z}_n$ and characterized those rings R for which the graph $G(R)$ is connected.

Intersection graphs of certain sets that are not necessarily ideals are studied in [13, 14]. The graphs in [13, 14] are related to the total graph of a commutative ring. For more details, we recommend reading [1, 3–5, 8].

One of the most important topological properties of a graph is its genus. Finding the genus of a given graph is a very hard problem, it is in fact NP-complete (see [28]). The problem of finding the genus of a graph associated with a ring have been studied by many authors; see [6, 10, 11, 15, 16, 21, 29–31], etc. In [18], the authors studied planar graphs which may occur as intersection graphs of some commutative rings. In [23] characterization of planar graphs that are intersection graphs of some rings was completed and all toroidal graphs that are intersection graphs of some rings were classified.

The present work continues research in this area. In Sec. 3, we classify all genus 2 graphs that are intersection graphs of some rings. In Sec. 4, we give some lower bounds on the genus of the intersection graphs of nonlocal rings. In Sec. 5, we prove that for every $g \geq 1$, there are only finitely many nonisomorphic genus g graphs that are intersection graphs of some rings.

2. Preliminaries

For the algebraic part of this paper, notation and terminology are standard and one may find it in, e.g. [7], or in [20]. For the graph theoretical part, notation and terminology may be found in [17], for the classical graph theory and [22], for the topological graph theory. We briefly recollect some basic notions and results from graph theory which we are going to use in this paper.

A graph G is an ordered pair (V, E), where $E \subseteq [V]^2$ and $[V]^2$ is the set of all 2-element subsets of V. Then $V(G) := V$ is the set of vertices of G and $E(G) := E$

is the set of edges of G. If G and G' are graphs, then G' is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. If G' contains all edges $xy \in E$ for $x, y \in V'$ (we denote the edge $\{x, y\}$ by xy), we say that G' is an *induced* subgraph of G. We use the notation $G[V']$ to denote the induced subgraph spanned by V'. If $F \subseteq [V]^2$, then $G - F := (V, E \backslash F)$. Similarly, if $W \subseteq V$, then $G - W := G[V \backslash W]$.

The degree of a vertex x, denoted by $\deg(x)$, is the number of vertices adjacent to x. If $E(G) = [V]^2$, then G is a *complete* graph. If $|V| = n$ we denote this graph by K^n. If the set $V(G)$ is a disjoint union of two nonempty sets A and B, such that two vertices are adjacent if and only if they belong to different sets, then the graph G is a *complete bipartite* graph. If $|A| = m$ and $|B| = n$, we denote this graph by $K_{m,n}$.

By a *surface* we mean a compact connected topological space such that each point has a neighborhood homeomorphic to an open disc in $\mathbb{R}^2$. We denote by $\mathbb{S}_n$ the surface obtained from the sphere $\mathbb{S}_0$ by adding n handles. It is known that every orientable surface is homeomorphic to precisely one of the surfaces $\mathbb{S}_n$ ($n \geq 0$). The number n is called the *genus* of the surface $\mathbb{S}_n$. The purpose of this paper is to study the question of embeddings of the intersection graphs in *double torus* $\mathbb{S}_2$. An *embedding* of a graph G into some topological space S is a homeomorphism between the geometric realization of G and a subspace of S. One may think of an embedding of a graph G into S as a drawing of G on S with no edge crossings. An embedding of G into S is *cellular* if each component of $S - G$ (i.e. each *face*) is homeomorphic to an open disc in $\mathbb{R}^2$. An embedding in which all faces have boundary consisting of exactly three edges is called a *triangulation*. *Genus* of a graph G is minimum n such that G can be embedded in $\mathbb{S}_n$ and embeddings of G in $\mathbb{S}_n$ are called *minimum genus embeddings*. Note that every minimum genus embedding of G is cellular ([22, Proposition 3.4.1]). This is one of the reasons why in the remaining of this paper when we say that a graph is embedded in a surface, we will assume that it is cellularly embedded. Note that graphs of genus 0 are planar graphs and graphs of genus 1 are toroidal graphs. One of the most remarkable theorems in topological graph theory, known as Euler's formula, states that if G is a finite connected graph with n vertices, e edges and of genus g, then

$$n - e + f = 2 - 2g,$$

where f is the number of faces obtained when G is cellularly embedded in $\mathbb{S}_g$.

Euler's formula can be used in combination with some combinatorial identities and other inequalities to show the nonexistence of certain embeddings. Since G is a simple graph, every face has at least three boundary edges and every edge is a boundary of two faces; so, $2e \geq 3f$, with equality if and only if G is a triangulation of the surface. We will often use this feature.

Let us recall two well-known results on the genera of complete and complete bipartite graphs. In both propositions $\lceil x \rceil$ denote the smallest integer greater than or equal to x.

Proposition 2.1 (Ringel [25]).

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \quad m, n \geq 2.$$

Proposition 2.2 (Ringel and Youngs [26]).

$$\gamma(K^n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad n \geq 3.$$

According to the previous proposition, $\gamma(K^n) = 0$ for $n \leq 4$, $\gamma(K^n) = 1$ for $5 \leq n \leq 7$ and $\gamma(K^n) = 2$ if and only if $n = 8$. So, embeddings of K^8 on $\mathbb{S}_2$ are minimum genus embedding and therefore cellular. If H is a subgraph of G, then $\gamma(H) \leq \gamma(G)$. If $\gamma(G) > n$, we will say that G is a forbidden subgraph of $\mathbb{S}_n$. Since complete graphs are often subgraphs of the graph $G(R)$, their genus is relevant for this work. We will repeatedly use the fact that K^9 is a forbidden subgraph of $\mathbb{S}_2$.

Let $\delta(\mathbb{S}_n)$ be the number of triangles in a minimal triangulation of $\mathbb{S}_n$ (triangulation of $\mathbb{S}_n$ is called minimal if the number of triangles is minimal). Jungerman and Ringel determined $\delta(\mathbb{S}_n)$ for each orientable surface $\mathbb{S}_n$ [19]. We will use this result in the special case $n = 2$.

Proposition 2.3 ([19]). *Let $\delta(\mathbb{S}_2)$ be the number of triangles in a minimal triangulation of a double torus. Then*

$$\delta(\mathbb{S}_2) = 24.$$

Length of a R-module M is the maximum n such that there exists submodules $N_0, N_1, \ldots, N_n$ of M such that $N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n$. If this maximum does not exist we say that M has infinite length. We will denote the length of a module M by $l(M)$.

Proposition 2.4. *If $G(R)$ has finite genus, then R is an Artinian ring. If we view R as a R-module, then $l(R) \leq 9$ if $\gamma(G(R)) = 2$.*

Proof. If R is not Artinian, then there exists an infinite descending chain $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of ideals in R. Hence, $G(R)$ contain K^n as a subgraph (for any n), a contradiction. Therefore, R is Artinian and hence Noetherian. So, it has finite length as a R-module and that length has the stated upper bound. $\square$

So, we can concentrate on Artinian rings. By the structure theorem for Artinian rings [7, Theorem 8.7], an Artinian ring R is isomorphic to the product of local Artinian rings R_i with maximal ideals M_i:

$$R \cong R_1 \times \cdots \times R_n. \tag{2.1}$$

In the following, we assume that the condition that a ring is Noetherian is included in the assumption that it is local. For a local ring R with unique maximal ideal M, the quotient M^k/M^{k+1} is a vector space over the field R/M for any

non-negative integer k. By Nakayama's lemma, the dimension of this vector space determines the minimal number of generators for M^k. The following proposition corresponds to the fact that M/M^2 has dimension n as a vector space over R/M. This proposition directly follows from [20, Theorem 158], so we omit the proof. We will use it often to check whether certain ideals are equal.

Proposition 2.5. *If n is the minimal number of generators of M, where M is the maximal ideal in a local ring R and if $\{x_1, \ldots, x_n\}$ is any generating set of M, then none of the elements x_i belongs to M^2.*

We end this section with the following result obtained in [23].

Lemma 2.1 ([23]). *Let R be a local ring with maximal ideal M which is minimally generated by two elements such that $M^2 \neq 0$. If the field $F(= R/M)$ is infinite, then the graph $G(R)$ does not have a finite genus.*

3. Rings with Genus 2 Intersection Graphs of Ideals

We begin this section by presenting graphs Γ' and Γ'', embedded in a double torus (see Figs. 3 and 4 at the end of this section), which are crucial for our discussion since all intersection graphs $G(R)$ of genus 2 are either Γ' or subgraphs of Γ''.

Proposition 3.1. *Let R be a commutative Artinian ring. If in the product (2.1) one has $n \geq 3$, then $\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to Γ'.*

Proof. Let R be a commutative ring such that $\gamma(G(R)) = 2$.

First, let us assume that $n \geq 4$. Then $G(R)$ contains a subgraph induced by $0 \times 0 \times 0 \times R_4 \times \cdots \times R_n$, $R_1 \times 0 \times 0 \times R_4 \times \cdots \times R_n$, $0 \times R_2 \times 0 \times R_4 \times \cdots \times R_n$, $0 \times 0 \times R_3 \times R_4 \times \cdots \times R_n$, $R_1 \times R_2 \times 0 \times R_4 \times \cdots \times R_n$, $R_1 \times 0 \times R_3 \times R_4 \times \cdots \times R_n$, $0 \times R_2 \times R_3 \times R_4 \times \cdots \times R_n$, $R_1 \times R_2 \times R_3 \times 0 \times \cdots \times R_n$, $R_1 \times R_2 \times 0 \times 0 \times \cdots \times R_n$, $R_1 \times 0 \times R_3 \times 0 \times \cdots \times R_n$. This graph has $v = 10$ vertices, $e \geq 40$ edges and genus $g \leq 2$. So, by Euler's formula $f = e - v + 2 - 2g$, i.e. $f \geq 28$ and therefore $0 \geq 3f - 2e = f + 2(f - e) \geq 4$, a contradiction.

So, $n = 3$. If rings R_i, for $1 \leq i \leq 3$, are fields, then by [18] $G(R)$ is planar, a contradiction. So, at least one of R_i, $1 \leq i \leq 3$, is not a field.

First, let us assume that R_1 and R_2 are local rings with maximal ideals $M_1 \neq 0$, $M_2 \neq 0$. Then ideals $I_1 = M_1 \times 0 \times 0$, $I_2 = R_1 \times 0 \times 0$, $I_3 = M_1 \times M_2 \times 0$, $I_4 = M_1 \times R_2 \times 0$, $I_5 = M_1 \times 0 \times R_3$, $I_6 = R_1 \times 0 \times R_3$, $I_7 = M_1 \times M_2 \times R_3$, $I_8 = M_1 \times R_2 \times R_3$, $I_9 = R_1 \times M_2 \times R_3$ induce a K^9 contained in $G(R)$, a contradiction.

So, w.l.o.g. we may assume that $M_1 \neq 0$ and $M_2 = M_3 = 0$. If $M_1^2 \neq 0$ (by Nakayama's lemma $M_1^2 \neq M_1$), similarly as in the previous case, we find that the ideals $M_1^2 \times 0 \times 0$, $M_1^2 \times R_2 \times 0$, $M_1^2 \times R_2 \times R_3$, $M_1^2 \times 0 \times R_3$, $M_1 \times 0 \times 0$, $M_1 \times 0 \times R_3$, $M_1 \times R_2 \times 0$, $M_1 \times R_2 \times R_3$, $R_1 \times 0 \times 0$ induce a K^9 contained in $G(R)$, a contradiction.

So, $M_1^2 = 0$. Let $\{x_1, x_2, \ldots, x_n\}$ be a minimal set of generators of M_1. If $n \geq 2$, then ideals $R_1 \times 0 \times 0$, $R_1 \times R_2 \times 0$, $R_1 \times 0 \times R_3$, $M_1 \times 0 \times 0$, $M_1 \times R_2 \times 0$, $M_1 \times 0 \times R_3$, $M_1 \times R_2 \times R_3$, $\langle x_1 \rangle \times 0 \times 0$, $\langle x_1 \rangle \times R_2 \times 0$ induce a K^9 contained in $G(R)$, a contradiction. Therefore, M_1 is a principal ideal, say $M_1 = \langle x \rangle$ and $x^2 = 0$. The intersection graph $G(R)$ then contains 10 vertices: $v_1 = \langle x \rangle \times 0 \times 0$, $v_2 = \langle x \rangle \times R_2 \times 0$, $v_3 = \langle x \rangle \times 0 \times R_3$, $v_4 = \langle x \rangle \times R_2 \times R_3$, $v_5 = R_1 \times 0 \times 0$, $v_6 = R_1 \times R_2 \times 0$, $v_7 = R_1 \times 0 \times R_3$, $v_8 = 0 \times R_2 \times 0$, $v_9 = 0 \times 0 \times R_3$ and $v_{10} = 0 \times R_2 \times R_3$. It is easy to see that $G(R)$ is isomorphic to Γ' (isomorphism is given by $v_i \mapsto i$), which can be embedded in $\mathbb{S}_2$ as shown in Fig. 3. □

Proposition 3.2. *Suppose that $R \cong R_1 \times R_2$ is a product of two local rings. Then $\gamma(G(R)) = 2$ if and only if one of the rings, say R_2 is a field and (R_1, M_1) is a local ring such that: $|R_1/M_1| = 2$, $\dim(M_1/M_1^2) = 2$ and $M_1^2 = 0$.*

Proof. Let $R \cong R_1 \times R_2$ be such that $\gamma(G(R)) = 2$. Then at least one of R_1 and R_2 is not a field, since otherwise $\gamma(G(R)) = 0$.

So, w.l.o.g. let us assume that R_1 is not a field and let $\{x_1, x_2, \ldots, x_n\}$ be a minimal set of generators of M_1. If $n \geq 3$, the ideals $\langle x_1 \rangle \times 0$, $\langle x_1 \rangle \times R_2$, $\langle x_1, x_2 \rangle \times 0$, $\langle x_1, x_2 \rangle \times R_2$, $\langle x_1, x_3 \rangle \times 0$, $\langle x_1, x_3 \rangle \times R_2$, $R_1 \times 0$, $\langle x_1, x_2, x_3 \rangle \times 0$, $\langle x_1, x_2, x_3 \rangle \times R_2$ induce a K^9 contained in $G(R)$, a contradiction. So, $n \leq 2$.

Case 1. $n = 2$.
If R_2 is not a field ideals $0 \times R_2$, $R_1 \times M_2$, $0 \times M_2$, $\langle x_1 \rangle \times R_2$, $\langle x_1 \rangle \times M_2$, $\langle x_2 \rangle \times R_2$, $\langle x_2 \rangle \times M_2$, $M_1 \times M_2$, $M_1 \times R_2$ induce a K^9 contained in $G(R)$, a contradiction. Hence, R_2 is a field. Suppose that $M_1^2 \neq 0$. If $x_1^2 \neq 0$, then the ideals $\langle x_1 \rangle \times 0$, $\langle x_1 \rangle \times R_2$, $R_1 \times 0$, $\langle x_1^2 \rangle \times 0$, $\langle x_1^2 \rangle \times R_2$, $M_1 \times 0$, $M_1 \times R_2$, $\langle x_1^2, x_2 \rangle \times 0$, $\langle x_1^2, x_2 \rangle \times R_2$ induce a K^9 contained in $G(R)$, a contradiction. Similarly, if $x_1 x_2 \neq 0$, ideals $\langle x_1 \rangle \times 0$, $\langle x_1 \rangle \times R_2$, $R_1 \times 0$, $\langle x_2 \rangle \times 0$, $\langle x_2 \rangle \times R_2$, $M_1 \times 0$, $M_1 \times R_2$, $\langle x_1 x_2 \rangle \times 0$, $\langle x_1 x_2 \rangle \times R_2$ induce a K^9, a contradiction. Thus, $x_1^2 = x_2^2 = x_1 x_2 = 0$, i.e. $M_1^2 = 0$. Such a ring has a simple structure of ideals. Besides M_1, all proper ideals of R_1 are principal and of the form:

$$\langle x_1 \rangle, \langle x_2 \rangle, \langle x_1 + \alpha x_2 \rangle, \quad \text{where } \alpha \in U(R_1).$$

Furthermore, if $\alpha, \beta \in U(R_1)$, then $\langle x_1 + \alpha x_2 \rangle = \langle x_1 + \beta x_2 \rangle$ if and only if $\alpha - \beta \in M_1$ (for details see [23, Proposition 3]). These ideals are minimal and have trivial intersections. We conclude that there are $|R_1/M_1| + 1$ such ideals; so, $|I^*(R_1)| = |R_1/M_1| + 2$.

Let us first examine the minimal case, when $|R_1/M_1| = 2$. Then $G(R)$ have 10 vertices, the ideals $v_1 = \langle x_2 \rangle \times 0$, $v_2 = M_1 \times 0$, $v_3 = \langle x_1 \rangle \times R_2$, $v_4 = R_1 \times 0$, $v_5 = \langle x_1 + x_2 \rangle \times 0$, $v_6 = M_1 \times R_2$, $v_7 = \langle x_1 + x_2 \rangle \times R_2$, $v_8 = \langle x_2 \rangle \times R_2$, $v_9 = \langle x_1 \rangle \times 0$ and $v_{10} = 0 \times R_2$. It is easy to see that $G(R)$ is isomorphic to $\Gamma''[1, 2, 3, 4, 5, 6, 7, 8, w_5, w_6] - \{13, 17\}$, where the isomorphism is given by $v_i \mapsto i$, for $1 \leq i \leq 8$, $v_9 \mapsto w_6$ and $v_{10} \mapsto w_5$. Therefore, as shown in Fig. 4, $G(R)$ can be embedded in $\mathbb{S}_2$.

If $|R_1/M_1| = 3$, we have 12 distinct ideals in $I^*(R)$ (in addition to the ideals from the previous case, there are new ideals $v_{11} = \langle x_1 + \alpha x_2 \rangle \times 0$ and $v_{12} = \langle x_1 + \alpha x_2 \rangle \times R_2$, for some $\alpha \in U(R) \setminus \{1\}$). Let us assume that $\gamma(G(R)) = 2$, i.e. that $G(R)$ can be embedded on $\mathbb{S}_2$. Note that vertex set $S = \{v_2, v_3, v_4, v_6, v_7, v_8, v_{12}\}$ induces a K^7 and that the only neighbors of vertices from $I^*(R) \setminus S$ are in $G[S]$. Let $H = G[S \cup \{v_{10}\}]$. This graph has $v' = 8$ vertices and $e' = 26$ edges. If this graph were embeddable in $\mathbb{S}_1$, then, by Euler's formula, the number of faces in that embedding is $f' = 18$. But $54 = 3f' > 2 \cdot e' = 52$, a contradiction. So, embedding of H obtained from embedding of $G(R)$ in $\mathbb{S}_2$ is minimum genus embedding, therefore cellular. By Euler's formula, the number of faces in this embedding of H is $f' = 16$. Note that for $v \in I^*(R) \setminus (S \cup \{v_{10}\})$ all neighbors of v are some vertices of the face of H in which v is contained. Also, all vertices from $I^*(R) \setminus S$ are adjacent to v_2, v_4 and v_6, so no two of them are contained in the same face of the embedding of H (two vertices inside of disc cannot be adjacent to the same three points on the boundary of this disc such that these edges do not cross). So, there are at least four faces of H that are of size at least 4 and therefore $52 = 2 \cdot e' \geq (f' - 4) \cdot 3 + 4 \cdot 4 = 52$. Hence, embedding of H has exactly four faces of size 4 and 12 of size 3. These four faces contain vertices v_2, v_4 and v_6, so one of these vertices, w.l.o.g. v_2, is adjacent to the other two in at least two of these quadrilaterals. This implies that the only neighbors of v_2 (in H) are v_4 and v_6, a contradiction.

If $|R_1/M_1| > 3$, then in addition to the ideals from the previous two cases, there are at least two new ideals $\langle x_1 + \beta x_2 \rangle \times 0$, $\langle x_1 + \beta x_2 \rangle \times R_2$, where $\beta \in U(R) \setminus \{1, \alpha\}$. Note that a subgraph of $G(R)$ induced by these ideals has $v = 14$ vertices and $e = 52$ edges. Since the genus of this graph is $g \leq 2$, we have that $f \geq 36$ and therefore $3f > 2e$, a contradiction.

Case 2. $n = 1$.
We have that R_1 is a local Artinian ring with the maximal ideal $M_1 = \langle x_1 \rangle$, $x_1 \neq 0$. So, $I^*(R_1) = \{\langle x_1^k \rangle \mid 1 \leq k \leq r - 1\}$ where r is the smallest number such that $x_1^r = 0$. Depending on r and the ring R_2, we have several subcases.

2.1. $r = 2$ and R_2 is a field. Then $|I^*(R)| = 4$, so $G(R)$ is planar, a contradiction.

2.2. $r = 3$ or $r = 4$ and R_2 is a field. Then $G(R)$ is toroidal (see [23, Proposition 5]), a contradiction.

2.3. $r \geq 5$. In this case ideals $\langle x_1 \rangle \times 0$, $R_1 \times 0$, $\langle x_1^2 \rangle \times 0$, $\langle x_1^3 \rangle \times 0$, $\langle x_1^4 \rangle \times 0$, $\langle x_1 \rangle \times R_2$, $\langle x_1^2 \rangle \times R_2$, $\langle x_1^3 \rangle \times R_2$ and $\langle x_1^4 \rangle \times R_2$ induce a K^9 contained in $G(R)$, a contradiction.

2.4. R_1 and R_2 are local rings with maximal ideals $M_1 = \langle x_1 \rangle$, $M_2 = \langle y_1 \rangle$, such that $x_1^2 = 0$, $y_1^2 = 0$. Then the intersection graph $G(R)$ is toroidal (see [23, Proposition 5]), a contradiction.

2.5. R_1 and R_2 are local rings with maximal ideals $M_1 = \langle x_1 \rangle$, $M_2 = \langle y_1 \rangle$, such that $x_1^2 \neq 0$ or $y_1^2 \neq 0$. W.l.o.g. let us assume that $x_1^2 \neq 0$. Then the graph induced by the vertices $\langle x_1 \rangle \times \langle y_1 \rangle$, $\langle x_1 \rangle \times R_2$, $\langle x_1^2 \rangle \times R_2$, $\langle x_1^2 \rangle \times \langle y_1 \rangle$, $R_1 \times \langle y_1 \rangle$, $\langle x_1 \rangle \times 0$, $\langle x_1^2 \rangle \times 0$, $R_1 \times 0$, $0 \times \langle y_1 \rangle$ and $0 \times R_2$ is a subgraph of $G(R)$. This

graph has $v = 10$ vertices, $e = 39$ edges and genus $g \leq 2$. So, by Euler's formula $f \geq 27$ and therefore $3f > 2e$, a contradiction. □

It remains to consider the case of a local Artinian ring R with maximal ideal $M \neq 0$. Since M is finitely generated, we study this with respect to the minimal number of generators.

Proposition 3.3. *Let (R, M) be a local Artinian commutative ring such that M is minimally generated by n elements. If $n \geq 3$, then $\gamma(G(R)) > 2$.*

Proof. It is enough to consider the minimal case, i.e. $M = \langle x, y, z \rangle$. The ideals $I_1 = \langle x + y + z \rangle$, $I_2 = \langle x, y \rangle$, $I_3 = \langle x, z \rangle$, $I_4 = \langle y, z \rangle$, $I_5 = \langle x, y, z \rangle$, $I_6 = \langle x, y + z \rangle$, $I_7 = \langle y, x + z \rangle$, $I_8 = \langle z, x + y \rangle$, $I_9 = \langle x \rangle$ and $I_{10} = \langle x + y \rangle$ are different. Let us prove that the ideal $I_{11} = \langle x + y, y + z \rangle$ is different from ideals I_k, $1 \leq i \leq 10$. Suppose that this is not the case and let $I_{11} = I_k$, for some $1 \leq k \leq 10$. By our assumption M is minimally generated by three elements, so $I_{11} \neq I_5$ and $I_{11} \neq I_{10}$ and we only need to examine the following cases:

(1) $I_{11} = I_1$. Then $x + y + z \in I_{11}$. Hence $z = x + y + z - (x + y)$, $x = x + y + z - (y + z)$, are in I_{11}. Finally $x + y - x = y \in I_{11}$ and therefore $I_{11} = M$, a contradiction. Similarly, we can prove that $I_{11} \neq I_9$.
(2) $I_{11} = I_2$. Then $x, y \in I_{11}$ and therefore $z = y + z - y \in I_{11}$, i.e. $I_{11} = M$, a contradiction. Similarly, we can prove that $I_{10} \notin \{I_3, I_4, I_6, I_7, I_9\}$.

Let us now examine the subgraph of $G(R)$ induced by the ideals I_k, $1 \leq k \leq 11$. This subgraph has $v = 11$ vertices and $e \geq 40$ edges. If $\gamma(G(R)) \leq 2$, then genus of this subgraph is $g \leq 2$ and hence $f \geq 27$. Therefore $3f - 2e = f + 2(f - e) = f + 2(2 - 2g - v) \geq 1$, a contradiction. □

We now concentrate on the case when M is minimally generated by two elements. First of all, if $M^2 = 0$, then $G(R)$ is planar (see [18], or [23, Proposition 3]); so we assume that $M^2 \neq 0$. Furthermore, if $\gamma(G(R))$ is finite, then according to Lemma 2.1, field $F(= R/M)$ is finite. Hence, in the following we assume that F is a finite field and $M^2 \neq 0$.

Proposition 3.4. *If R is a local ring with maximal ideal M minimally generated with two generators and $\gamma(G(R)) = 2$, then M^2 is a principal ideal.*

Proof. Let us suppose that $M^2(\neq 0)$ is not principal. Then there exists some elements $u, v \in M^2$ such that $u \notin \langle v \rangle$ and $v \notin \langle u \rangle$. It is clear that the ideals $\langle u \rangle$, $\langle v \rangle$, $\langle u + v \rangle$ and M^2 are different. Note that M/M^2 is a union of one-dimensional subspaces. Since $|M/M^2| = |F|^2$, we conclude that there are at least $|F| + 1$ one-dimensional subspaces of M/M^2.

Case 1. $|F| \geq 4$.

There are (at least) five ideals I_1, I_2, I_3, I_4 and I_5 which contain $M^2(\neq 0)$. So, R has the following 10 ideals: M, I_1, I_2, I_3, I_4, I_5, M^2, $\langle u \rangle$, $\langle v \rangle$, $\langle u+v \rangle$. The first seven ideals all have degree 9 and the last three have degree (at least) 7. Therefore, the subgraph of $G(R)$ induced by these ideals has $e \geq 42$ edges and hence $2e - 3f = 36 - e < 0$, a contradiction.

Case 2. $|F| = 3$.

There are (at least) four ideals I_1, I_2, I_3 and I_4 which contain $M^2(\neq 0)$. So, R has the following nine ideals: M, I_1, I_2, I_3, I_4, M^2, $\langle u \rangle$, $\langle v \rangle$, $\langle u+v \rangle$. The first six ideals all have degree 8 and the last three have degree (at least) 6 and therefore the subgraph of $G(R)$ induced by these ideals has $e \geq 33$ edges. If $\gamma(G(R)) = 2$, then the genus of this subgraph is $g \leq 2$ and therefore $f \geq 22$. Now, $3f - 2e \geq 0$, so $e = 33$, $f = 22$ and $g = 2$. Therefore, this subgraph induces a triangulation of $\mathbb{S}_2$ with 22 triangles. By Proposition 2.3 this is not possible.

Case 3. $|F| = 2$ and $\dim(M^2/M^3) = 3$.

Since M^2/M^3 is a vector space over F, we have three linearly independent vectors u, v and w in M^2/M^3. If V is a subspace spanned by these vectors, then $\dim(V/\langle u \rangle) = 2$. Hence $V/\langle u \rangle$ is a union of three one-dimensional subspaces which contain $u \neq 0$. So, we have four two-dimensional subspaces and therefore four different ideals J_1, J_2, J_3 and J_4 such that $J_k \subset M^2$ and $J_k \cap J_l \neq 0$. Thus, the ideals M, I_1, I_2, I_3, M^2, J_1, J_2, J_3 and J_4 induce a K^9 contained in $G(R)$, a contradiction.

Case 4. $|F| = 2$, $\dim(M^2/M^3) = 2$ and $M^3 \neq 0$.

Since $\dim(M^2/M^3) = 2$, we have that M^2/M^3 is a union of three one-dimensional subspaces, so we get three different ideals J_1, J_2 and J_3 (contained in M^2) which contain M^3. On the other hand $\dim(M/M^2) = 2$, so we have three different ideals I_1, I_2 and I_3 which properly contain M^2. Hence, R has (at least) nine ideals M, M^2, M^3, I_1, I_2, I_3, J_1, J_2 and J_3. The first six ideals all have degree 8 (in this subgraph) and the last three have degree (at least) 6. Similarly as in the Case 2, this contradicts Proposition 2.3.

Case 5. $|F| = 2$, $\dim(M^2/M^3) = 2$ and $M^3 = 0$.

In this case we need to examine the following possibilities.

5.1. $M^2 = \langle x^2, y^2 \rangle$, $xy = 0$. Let us examine the subgraph induced by the ideals: M, M^2, $\langle x, y^2 \rangle$, $\langle x^2, y \rangle$, $\langle x+y \rangle$, $\langle x \rangle$, $\langle y \rangle$, $\langle x^2 \rangle$, $\langle y^2 \rangle$ and $\langle x^2 + y^2 \rangle$. The first five ideals all have degree 9 in this subgraph (note that $xy = 0$, hence $x^2, y^2 \in \langle x+y \rangle$), the last one has degree 5, while the others have degree 6. If $\gamma(G(R)) = 2$, then the genus of this subgraph is $g \leq 2$ and therefore $f \geq 25$. So $3f - 2e \geq 1$, a contradiction.

5.2. $M^2 = \langle x^2, y^2 \rangle$, $xy \neq 0$. We have $xy = \alpha x^2 + \beta y^2$, for some $\alpha, \beta \in R$. If $\alpha, \beta \in M$, since $M^3 = 0$, we have $xy = 0$, a contradiction. If exactly one of α, β is invertible, w.l.o.g. $\alpha \in U(R)$ and $\beta \in M$, we have $xy = \alpha x^2$; so, $x(y - \alpha x) = 0$. If we choose generators $u = x$ and $v = y - \alpha x$ for M, we get

that $M^2 = \langle u^2, v^2 \rangle$ and $uv = 0$. Thus, we have reduced this to the previous case. If $\alpha, \beta \in U(R)$, let us examine the subgraph induced by the ideals M, M^2, $\langle x, y^2 \rangle$, $\langle x^2, y \rangle$, $\langle x \rangle$, $\langle y \rangle$, $\langle xy \rangle$, $\langle x^2 \rangle$, $\langle y^2 \rangle$ and $\langle x + y \rangle$. Similarly, estimating degrees of these ideal (as in the previous cases), we have $e \geq 42$ and therefore $3f - 2e \geq 6$, a contradiction.

5.3. $M^2 = \langle x^2, xy \rangle$, $y^2 = 0$. Let us examine the subgraph induced by the ideals: M, M^2, $\langle x^2, y \rangle$, $\langle x \rangle$, $\langle x + y \rangle$, $\langle y \rangle$, $\langle xy \rangle$, $\langle x^2 + y \rangle$ and $\langle x^2 + xy \rangle$. Note that the first eight ideals induce a K^8, while the last ideal has degree 5 in this subgraph. If this subgraph were embeddable in a double torus, we would get a triangulation of $\mathbb{S}_2$ with 22 triangles. This contradicts Proposition 2.3.

5.4. $M^2 = \langle x^2, xy \rangle$, $y^2 \neq 0$. We have $y^2 = \alpha x^2 + \beta xy$, for some $\alpha, \beta \in R$. If $\alpha, \beta \in M$, since $M^3 = 0$, then $y^2 = 0$, a contradiction. If $\beta \in U(R)$, then $M^2 = \langle x^2, y^2 \rangle$, which was treated in 5.2. If $\alpha \in U(R)$, $\beta \in M$, we have $y^2 = \alpha x^2$. Let us examine the subgraph of $G(R)$ induced by the ideals: M, M^2, $\langle x^2, y \rangle$, $\langle x \rangle$, $\langle x^2 + y \rangle$, $\langle y \rangle$, $\langle xy \rangle$, $\langle x + y \rangle$ and $\langle x^2 + xy \rangle$. It is not difficult to check that these ideals are different. The first seven of them contain $xy \neq 0$, so they have nontrivial intersections. Also, note the following: $\langle x \rangle \cap \langle x + y \rangle \neq 0$, since $x^2 + xy \neq 0$ (if not, then M^2 is principal); $\langle x^2 + xy \rangle \cap \langle y \rangle \neq 0$, since $0 \neq x^2 + xy = \alpha^{-1} y^2 + xy$; and finally, since $y(x^2 + y) = y^2 = \alpha x^2$, we get $x^2 + xy \in \langle x^2 + y \rangle$. Therefore, $\langle x^2 + y \rangle \cap \langle x^2 + xy \rangle \neq 0$. So, the first six ideals have degree 8 (in this subgraph), the last two have degree 7, while $\langle xy \rangle$ has degree 6. If this subgraph were embeddable in $\mathbb{S}_2$, then $f = 23$. But then $3f = 69 > 68 = 2e$, a contradiction. $\qquad\square$

Proposition 3.5. *Let (R, M) be a local ring such that maximal ideal M is minimally generated by two generators and $\gamma(G(R)) = 2$. Then $M^3 = 0$ and one can choose generators u, v for M in such a way that $M^2 = \langle uv \rangle$, where $u^2 = v^2 = 0$, or $M^2 = \langle u^2 \rangle$, where $uv = 0$.*

Proof. Let $M = \langle x, y \rangle$. As in [23, Proposition 8] one can prove that one of x^2, xy, y^2 is a generator for M^2.

Case 1. $M^2 = \langle x^2 \rangle$.
First we prove that there exists u and v, such that $M^2 = \langle u^2 \rangle$ and $uv = 0$. If $xy = 0$, we are done (take $u = x$). If not, $xy = ax^2$, for some $a \in R$. Then we can choose generators $u = x$ and $v = y - ax$. One has $M^2 = \langle u^2 \rangle$ and $v^2 = ru^2$, for some $r \in R$. Since $uv = 0$, we get $v^3 = uv^2 = u^2 v = 0$.

To finish the proof it is enough to prove that $u^3 = 0$. Assume that $u^3 \neq 0$. It is not difficult to see that nontrivial ideals of R are: $v_1 = \langle u \rangle$, $v_2 = \langle u^2 \rangle$, $v_3 = \langle u^3 \rangle$, $v_4 = \langle u + v \rangle$, $v_5 = \langle u^2 + v \rangle$, $v_6 = \langle u, v \rangle$, $v_7 = \langle u^2, v \rangle$, $v_8 = \langle u^3, v \rangle$, $v_9 = \langle v \rangle$ and $v_{10} = \langle u^3 + v \rangle$. For details why they are different, one can look up the proof in [23, Proposition 8]. Note that the first eight ideals contain $u^3 \neq 0$ since $uv = 0$ and that the last two ideals have degree 3 and are adjacent to v_6, v_7 and v_8.

Claim. *Graph G induced by* $\{v_i \mid 1 \le i \le 10\}$ *is not embeddable in* $\mathbb{S}_2$.

Proof. Let us assume that G can be embedded in $\mathbb{S}_2$. This embedding induce a cellular embedding of the graph H induced by $\{v_i \mid 1 \le i \le 8\}$, since every embedding of K^8 in $\mathbb{S}_2$ is cellular. Note that H has $e = 28$ edges and therefore, by Euler's theorem, $f = 18$ faces. Let f_i be the number of faces in embedding of H of size i. Then $56 = 2e = \sum_{i \ge 3} i f_i \ge 3f + f_4 + 2f_5$ and therefore embedding of H has one pentagonal face and 17 triangular, or two quadrilateral faces and 16 triangular. Let us suppose that H has a pentagonal face. By adding a vertex v inside this face and edges between v and vertices on the boundary of this face (see Fig. 1), we get a triangulation of $\mathbb{S}_2$ with 22 triangle, which contradicts Proposition 2.3.

So, embedding of H has two quadrilateral and 16 triangular faces. Let us assume that these two quadrilateral faces share an edge. Then, deleting that edge will produce an embedding in $\mathbb{S}_2$ with 16 triangular faces and one face F of size at most six (size is maximal when quadrilateral faces have exactly two vertices in common). Now, similarly as in the previous part of the proof, adding a vertex v inside the face F and edges between v and vertices on the boundary of F (see Fig. 2), we get a triangulation of $\mathbb{S}_2$ with at most 22 triangles, a contradiction. So, the two quadrilateral faces of H do not share an edge and therefore have at most two vertices in common.

Vertices v_9 and v_{10} have three neighbors in common, so they are in different faces of the embedding of H. Since no two triangular faces have three vertices in common, by the previous remark, one of them w.l.o.g. v_9 is in a triangular face $u_1 u_2 u_3$ and the other in a quadrilateral face $u_1 u_2 u_3 u_4$. W.l.o.g. we may assume
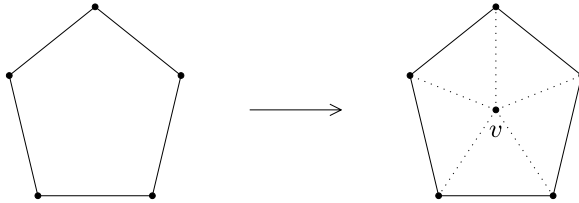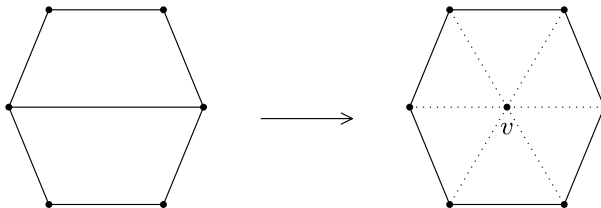

Fig. 1.   Pentagonal face.

Fig. 2.   Hexagonal face.

that u_1u_2 and u_2u_3 are edges of this quadrilateral. Then the only neighbors of u_2 (in H) are u_1 and u_3, a contradiction. This completes the proof of the claim. $\square$

So, $M^3 = 0$ and we can choose generators for M as required.

Case 2. $M^2 = \langle xy \rangle$.
This case follows from [23, Proposition 8].

Remark 3.1. In the previous proposition we proved that no $\mathbb{S}_2$ embedding of K^8 has a pentagonal face, nor two quadrilateral faces that share an edge. Note that this result was proven in [27, Theorems 2 and 3], but we opted to include our proof, since it is much shorter.

Theorem 3.1. *Let R be a local ring and $M = \langle x, y \rangle$, where $\{x, y\}$ is a minimal set of generators of M. Then, $\gamma(G(R)) = 2$ if and only if M^2 is a principal ideal and $|R/M| = 5$. The graph $G(R)$ is then isomorphic to one of the graphs K^8, or $\Gamma''[1, 2, 3, 4, 5, 6, 7, 8, w_1, w_2, w_3, w_4, w_5] - \{7w_5, 3w_5\}$.*

Proof. According to the previous results, we only need to consider two cases: $M^2 = \langle x^2 \rangle \neq 0$, where $xy = 0$ and $M^2 = \langle xy \rangle \neq 0$, where $x^2 = y^2 = 0$.

Case 1.1. $M^2 = \langle x^2 \rangle$, $xy = 0$, $y^2 = 0$.
First, we will describe the structure of principal ideals in such a ring. Let $I = \langle ax + by \rangle$, for some $a, b \in R$ be a principal ideal. If $a, b \in M$, since $xy = y^2 = 0$, we have that $I = \langle x^2 \rangle$ or $I = 0$. If $a, b \in U(R)$, we get $I = \langle x + \alpha y \rangle$, where $\alpha \in U(R)$. If $a \in U(R)$ and $b \in M$, then obviously $I = \langle x \rangle$. So let us assume that $a \in M$ and $b \in U(R)$. Then we have that $I = \langle y + \beta x^2 \rangle$, for some $\beta \in R$ (in particular, $I = \langle y \rangle$, if $\beta \in M$). So, all the principal ideals are as follows:

$$\langle x \rangle, \langle y \rangle, \langle x^2 \rangle, \langle x + \alpha y \rangle, \langle y + \beta x^2 \rangle, \quad \text{for some } \alpha, \beta \in U(R).$$

Let us show that $\langle y + \beta x^2 \rangle = \langle y + \gamma x^2 \rangle$ if and only if $\beta - \gamma \in M$.

Suppose that $\langle y + \beta x^2 \rangle = \langle y + \gamma x^2 \rangle$. Then $y + \beta x^2 = r(y + \gamma x^2)$, for some $r \in R$; so, $(1 - r)y = (r\gamma - \beta)x^2$. Now $1 - r \in M$ (if not, $\langle y \rangle \in \langle x \rangle$, a contradiction) and therefore $(1 - r)y = 0$. Since $x^2 \neq 0$, we get that $r\gamma - \beta \in M$. Hence, $(1 - r)\gamma + r\gamma - \beta = \gamma - \beta \in M$. On the other hand, if $\beta - \gamma \in M$, then obviously $y + \beta x^2 = y + (\beta - \gamma)x^2 + \gamma x^2 = y + \gamma x^2$ (note that actually $y + \beta x^2 = y + \gamma x^2$ if and only if $\beta - \gamma \in M$). Similarly it can be shown that $\langle x + \alpha y \rangle = \langle x + \delta y \rangle$ if and only if $\alpha - \delta \in M$.

Let $I = \langle ax + by, cx + dy \rangle$ be an ideal which is not principal. If $a, b, c, d \in U(R)$, then clearly $I = \langle x, y \rangle = M$. If $b, d \in M$, then I is principal, so at least one of them is invertible. Suppose that $b \in M$ and $d \in U(R)$, i.e. $I = \langle ax, y + rx \rangle$, for some $r \in R$. If $a \in U(R)$, we have that $I = M$; if not, then I is principal if $r \in U(R)$, or $I = \langle x^2, y \rangle$, if $r \in M$. If $b, d \in U(R)$, then $I = \langle ab^{-1}x + y, cd^{-1}x + y \rangle = \langle ab^{-1}x + y, (ab^{-1} - cd^{-1})x \rangle$, which was considered in the previous part of the proof.
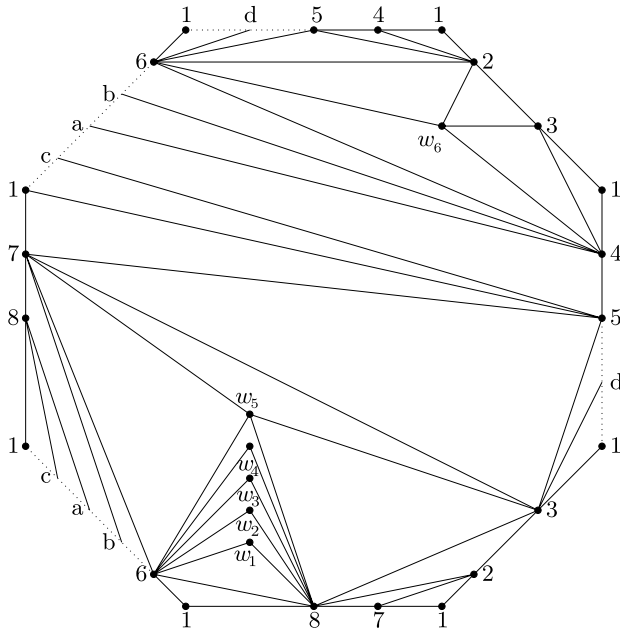
So, all the nonzero ideals in R are:

$$\langle x \rangle, \langle x^2 \rangle, \langle x + \alpha y \rangle, \langle x^2, y \rangle, \langle x, y \rangle, \langle y \rangle, \langle y + \beta x^2 \rangle (\alpha, \beta \in U(R)).$$

The last two ideals have nontrivial intersection with $\langle x, y \rangle$ and $\langle x^2, y \rangle$, while the other ideals contain $x^2 \neq 0$ and there are $3 + |R/M|$ of them. If $|R/M| \geq 6$, then $G(R)$ contains a K^9 and therefore $\gamma(G(R)) > 2$. If $|R/M| \leq 4$, then $G(R)$ is toroidal ([23, Theorem 1]). Finally, if $|R/M| = 5$, then $G(R)$ is isomorphic to $\Gamma''[1, 2, 3, 4, 5, 6, 7, 8, w_1, w_2, w_3, w_4, w_5] - \{7w_5, 3w_5\}$, so we can embed it in $\mathbb{S}_2$, as shown in Fig. 4.

Case 1.2. $M^2 = \langle x^2 \rangle, xy = 0, y^2 = ax^2$, for some $a \in U(R)$.
Using similar analysis, one gets that all the nonzero ideals in R are:

$$\langle x, y \rangle, \langle x \rangle, \langle y \rangle, \langle x^2 \rangle, \langle x + \alpha y \rangle (\alpha \in U(R)).$$

Also, as before, $\langle x + \alpha y \rangle = \langle x + \beta y \rangle$ if and only if $\alpha - \beta \in M$. All these ideals contain $x^2 \neq 0$ and there are $3 + |R/M|$ of them. Therefore, $G(R)$ is a complete graph on $3 + |R/M|$ vertices. It follows from Proposition 2.2 that $\gamma(G(R)) = 2$ if and only if $|R/M| = 5$.

Case 2. $M^2 = \langle xy \rangle$, $x^2 = y^2 = 0$.
This case does not differ from the previous one (see [23, Theorem 1]). One gets that all the nonzero ideals in R are:

$$\langle x, y \rangle, \langle x \rangle, \langle y \rangle, \langle xy \rangle, \langle x + \alpha y \rangle (\alpha \in U(R)).$$

Fig. 3.   Graph Γ'.

Fig. 4.   Graph Γ''.

All these ideals contain $xy(\neq 0)$ and there are $3+|R/M|$ of them. So, the conclusion is the same as in the previous case $(\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to $\Gamma''[1, 2, 3, 4, 5, 6, 7, 8])$. □

Proposition 3.6. *Let R be a local ring with the maximal ideal $M = \langle x \rangle$. Then, $\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to K^8.*

Proof. This ring is a principal ideal ring and all of its ideals are of the form $\langle x^k \rangle$, for $k \geq 1$. Hence, $\gamma(G(R)) = 2$ if and only if $x^9 = 0$ and $x^8 \neq 0$, i.e. $G(R)$ is isomorphic to K^8. □

We can summarize the previously obtained results in the form of the following theorem.

Theorem 3.2. *Let R be a commutative ring with identity. Then, $\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to one of the following graphs:*

$$K^8, \Gamma', \Gamma''[1, \ldots, 8, w_5, w_6] - \{13, 17\}, \quad \Gamma''[1, \ldots, 8, w_1, \ldots, w_5] - \{7w_5, 3w_5\}.$$

4. Lower Bounds for the Genus of the Intersection Graphs of Nonlocal Rings

In this section, we obtain some lower bounds for the genus of the intersection graphs of nonlocal rings, which shows that intersection graphs of nonlocal rings

have quite large genera. This gives us enough evidence to believe that the main difficulty toward fully understanding the genus of intersection graphs of rings is to understand the genus of intersection graphs of local rings. Interestingly enough, when dealing with the zero-divisor graph of a ring, it is easier to find the genus if the ring is local, then when it is nonlocal (see [11]).

Throughout this section let R be a commutative Artinian ring and R_i local Artinian rings such that $R \cong R_1 \times \cdots \times R_k$, where $k \geq 2$.

Proposition 4.1. *If R_i is not a field, for $1 \leq i \leq k$, then*

$$\gamma(G(R)) \geq \min\left\{\frac{\alpha}{8} \cdot N^{\frac{2k-2}{k}} \cdot (N^{1/k} - \alpha) - \frac{N}{2} + 1, \beta \cdot N^2 - \frac{N}{2} + 1\right\},$$

where $N = |V(G(R))|$, $\alpha = 2k(\frac{1}{3})^{\frac{k-1}{k}}$ and $\beta = \frac{3^k - 2^k - 1}{4 \cdot (2 \cdot 3^k - 2^{k+1} - 1)^2}$.

Proof. Let $n_i = |I^*(R_i)|$, $1 \leq i \leq k$. Then $N = \prod_{i=1}^{k}(n_i + 2) - 2$. Let A be the set of nontrivial ideals of R in which ith coordinate is not in $\{0, R_i\}$, for $1 \leq i \leq k$ and B the set of nontrivial ideals of R in which for at least one i, $1 \leq i \leq k$, ith coordinate is R_i. Then every vertex of A is adjacent to every vertex of B and therefore $G(R)$ contains $K_{|A|,|B|}$ as a subgraph. By Proposition 2.2,

$$\gamma(G(R)) \geq \frac{(|A| - 2)(|B| - 2)}{4}. \tag{4.1}$$

Note that $|A| = n_1 \cdots n_k$ and $|B| = (n_1 + 2) \cdots (n_k + 2) - (n_1 + 1) \cdots (n_k + 1) - 1$. Let $\sigma_j = \sum_{1 \leq i_1 < \cdots < i_j \leq k} n_{i_1} \cdots n_{i_j}$ be the jth symmetric sum of $n_1, n_2, \ldots, n_k$. Then $|A| = \sigma_k$,

$$|B| = (2 - 1)\sigma_{k-1} + (2^2 - 1)\sigma_{k-2} + \cdots + (2^{k-1} - 1)\sigma_1 + 2^k - 2,$$

$$N = \sigma_k + 2\sigma_{k-1} + \cdots + 2^{k-1}\sigma_1 + 2^k - 2.$$

Since $2(2^i - 1) \geq 2^i$, for $i \in \mathbb{N}$, we have $|A| + 2|B| \geq N$ and obviously $|A| + |B| < N$.

Case 1. $|A| \geq |B|$.
Since $|B| > \sigma_{k-1}$, by inequality of arithmetic and geometric mean we have $|B| > k|A|^{\frac{k-1}{k}}$. Therefore, $N \leq |A| + 2|B| \leq 3|A| < 3(\frac{|B|}{k})^{\frac{k}{k-1}}$, i.e. $|B| > k(\frac{N}{3})^{\frac{k-1}{k}}$. Now,

$$|A| \cdot |B| \geq 2|B| \cdot \left(\frac{N}{2} - |B|\right) > 2k\left(\frac{N}{3}\right)^{\frac{k-1}{k}}\left(\frac{N}{2} - k\left(\frac{N}{3}\right)^{\frac{k-1}{k}}\right)$$

and $|A| + |B| < N$, so by (4.1)

$$\gamma(G(R)) > \frac{k}{2}\left(\frac{N}{3}\right)^{\frac{k-1}{k}}\left(\frac{N}{2} - k\left(\frac{N}{3}\right)^{\frac{k-1}{k}}\right) - \frac{N}{2} + 1.$$

Case 2. $|B| \geq |A|$.

Since $\sigma_i \leq \binom{k}{i}\sigma_k$, we have

$$|B| \leq |A| \sum_{i=1}^{k-1} \binom{k}{i}(2^{k-i} - 1) + 2^k - 2$$

$$\leq |A| \sum_{i=1}^{k-1} \binom{k}{i}(2^{k-i} - 1) + (2^k - 2)|A|$$

$$= (3^k - 2^k - 1) \cdot |A|$$

and therefore $N \leq |A| + 2|B| \leq (2 \cdot 3^k - 2^{k+1} - 1) \cdot |A|$, i.e. $|A| > \frac{N}{2 \cdot 3^k - 2^{k+1} - 1}$. Now,

$$|A| \cdot |B| \geq |A| \cdot \frac{N - |A|}{2} \geq \frac{3^k - 2^k - 1}{(2 \cdot 3^k - 2^{k+1} - 1)^2} \cdot N^2,$$

so by (4.1)

$$\gamma(G(R)) \geq \frac{3^k - 2^k - 1}{4 \cdot (2 \cdot 3^k - 2^{k+1} - 1)^2} \cdot N^2 - \frac{N}{2} + 1. \qquad \square$$

In the following proposition we prove that if at least one of $G(R_i)$, $1 \leq i \leq k$, contains a "big" clique, then $G(R)$ also contains a "big" clique and therefore by Proposition 2.2 has a "large" genus. For a graph G, we denote by $\omega(G)$ the size of the largest clique contained in G.

Proposition 4.2. *For $1 \leq i \leq k$, let $\alpha_i = \omega(G(R_i))/|V(G(R_i))|$ if R_i is not a field and $\alpha_i = 3/2$ if R_i is a field. Then*

$$\omega(G(R)) \geq \max\{\alpha_i \mid 1 \leq i \leq k\} \cdot \frac{N}{3},$$

where $N = |V(G(R))|$.

Proof. W.l.o.g. $\alpha_k = \max\{\alpha_i \mid 1 \leq i \leq k\}$. Let $n_i = |V(G(R_i))|$. Then $N = \prod_{i=1}^{k}(n_i + 2) - 2$.

Case 1. R_k is not a field.

Let $\mathcal{I}_k$ be a set of vertices of $G(R_k)$ that induce a clique of size $\omega(G(R_k))$. Then the set

$$\{I_1 \times \cdots \times I_k \mid I_i \in I^*(R_i) \cup \{0, R_i\}, 1 \leq i \leq k-1, I_k \in \mathcal{I}_k\}$$

induces a clique of size $\omega(G(R_k)) \prod_{i=1}^{k-1}(n_i + 2) = \alpha_k n_k \prod_{i=1}^{k-1}(n_i + 2)$. Since $3n_k \geq n_k + 2$ we have

$$\omega(G(R)) \geq \frac{\alpha_k}{3} \prod_{i=1}^{k}(n_i + 2) > \alpha_k \cdot \frac{N}{3}.$$

Case 2. R_k is a field.

The set

$$\{I_1 \times \cdots \times I_{k-1} \times R_k \mid I_i \in I^*(R_i) \cup \{0, R_i\}, 1 \leq i \leq k-1\} \backslash \{R_1 \times \cdots \times R_k\}$$

induces a clique of size $\prod_{i=1}^{k-1}(n_i + 2) - 1 = N/2 = \alpha_k \cdot N$. □

By Combining the previous two proposition we obtain the main theorem of this section.

Theorem 4.1. *Genus of the intersection graph of a nonlocal ring R is at least*

$$\min\left\{\frac{\alpha}{8} \cdot N^{\frac{2k-2}{k}} \cdot (N^{1/k} - \alpha) - \frac{N}{2} + 1, \beta \cdot N^2 - \frac{N}{2} + 1, \frac{(N-6)(N-8)}{48}\right\},$$

where $N = |V(G(R))|$, $\alpha = 2k(\frac{1}{3})^{\frac{k-1}{k}}$ and $\beta = \frac{3^k - 2^k - 1}{4 \cdot (2 \cdot 3^k - 2^{k+1} - 1)^2}$.

Proof. If R_i, for every $1 \leq i \leq k$, is not a field, the proof follows from Proposition 4.1. So, let us assume that at least one of R_i, $1 \leq i \leq k$, is a field. Then $\max\{\alpha_i \mid 1 \leq i \leq k\} = 3/2$, so, by Proposition 4.2, $G(R)$ contains a clique of size $N/2$ and therefore, by Proposition 2.2

$$\gamma(G(R)) \geq \frac{(N/2 - 3)(N/2 - 4)}{12} = \frac{(N-6)(N-8)}{48}.$$ □

5. Concluding Remarks

The present work suggests that finding the genus of the intersection graph of a ring is in general a very difficult problem. Notice that even for embeddings of intersection graphs in $\mathbb{S}_2$ the list of forbidden subgraphs contains graphs that are neither complete nor complete bipartite. When classifying intersection graphs of genus 2, we needed to establish the nonexistence of certain embeddings that are consistent with Euler's formula. Some of these results were obtained by study of face-size distribution of graph embeddings, which is in general very difficult. As indicated, for example by graph Γ', obtaining an embedding of the intersection graph is not always a straight-forward task. Therefore, for arbitrary g, creating the full list of nonisomorphic genus g graphs that are intersection graphs of some rings is (probably) unrealistic. The following theorem tells us that for $g > 0$ this list is at least finite. Similar results were proven for zero-divisor graphs (see [31]) and total graphs (see [21]).

Theorem 5.1. *For every $g > 0$, there are only finitely many nonisomorphic graphs of genus g that are intersection graphs of some rings.*

Proof. It is enough to prove that there are only finitely many nonplanar graphs of genus at most g that are intersection graphs of some rings.

Let R be a commutative Artinian ring such that $\gamma(G(R)) \leq g$. Then $R \cong R_1 \times \cdots \times R_k$, where R_i, $1 \leq i \leq k$, are local Artinian rings. By Proposition 2.2 there is t (depending only on g) such that $G(R)$ does not contain a clique of size t.

Case 1. R is local ($k = 1$).

Let $\{x_1, \ldots, x_n\}$ be a minimal set of generators of the maximal ideal M of R. Then ideals $\langle x_1 \rangle$, $\langle x_1, x_2 \rangle$, ..., $\langle x_1, \ldots, x_n \rangle$ induce a clique of size n and therefore $n \leq t - 1$. Let l be the smallest number such that $M^l = 0$. Then, by Nakayama's lemma ideals $M, M^2, \ldots, M^{l-1}$ are all different and induce a clique of size $l - 1$, so $l - 1 \leq t - 1$. By Lemma 2.1 $F = R/M$ is finite. Let $|F| = r$. By the previous remark, there are only finitely many intersection graphs of some rings with $n = 1$. So, we may assume that $n \geq 2$.

We will prove that r is bounded by a constant depending only on t. First, let us assume that $M^2 \neq 0$. Vector space M/M^2 is the union of one-dimensional subspaces and since $|M/M^2| = |F|^n$, there are $|F|^{n-1} + 1 \geq r + 1$ of them. For each of these subspaces there is an ideal of R that contains M^2. Therefore $G(R)$ contains a complete graph on at least $r + 1$ vertices, so $r \leq t - 2$. So, let us assume that $M^2 = 0$. Then, by [18], $n \geq 3$, since otherwise $G(R)$ is planar. Let $\alpha_i \in R$ be some representatives of the elements of R/M, for $1 \leq i \leq r$. Ideals $I_i = \langle x_1, x_2 + \alpha_i x_3 \rangle$, $1 \leq i \leq r$, have nontrivial intersection. Let us prove that they are all different. If this were not the case, then $I_i = I_j$, for some $i \neq j$. So, $x_2 + \alpha_i x_3 \in \langle x_1, x_2 + \alpha_j x_3 \rangle$ and therefore $(\alpha_i - \alpha_j)x_3 \in I_j$. Since $\alpha_i - \alpha_j \in U(R)$, $x_3 \in I_j$ and therefore $\langle x_1, x_2, x_3 \rangle \subseteq I_j$, a contradiction. Hence, the ideals I_i, $1 \leq i \leq r$ induce a clique of size r, so $r \leq t - 1$. Note that the elements of R are linear combinations (over F) of elements $x_1^{s_1} \cdots x_n^{s_n}$, where $s_1 + \cdots + s_n \leq l \leq t - 1$. These elements uniquely determines R, so $|V(G(R))|$ is bounded (by a function only depending on g) and therefore there are only finitely many intersection graphs of local rings that are nonplanar and have genus at most g.

Case 2. $k \geq 2$.

Notice that the graph $G(R_1 \times \cdots \times R_k)$ is fully determined by graphs $G(R_i)$, $1 \leq i \leq k$. Also, $R_1 \times 0 \times \cdots \times 0$, $R_1 \times R_2 \times 0 \times \cdots \times 0, \ldots, R_1 \times \cdots \times R_{k-1} \times 0$ induce a clique of size $k - 2$ and therefore $k - 2 \leq t - 1$. So, it is enough to prove that there are only finitely many graphs $G(R_i)$, for every $1 \leq i \leq k$. To prove this let us examine the subgraph of $G(R)$ induced by

$$\{R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_k \mid I_i \in I^*(R_i)\}.$$

This subgraph is complete and has $|V(G(R_i))|$ vertices. Therefore $G(R_i)$ has at most $t - 1$ vertices, for each $1 \leq i \leq k$, which completes our proof. $\qquad\square$

Note that there are infinitely many planar graphs that are intersection graphs of some rings (see [18, 23]).

Acknowledgments

References

[1] S. Akbari, D. Kiani, F. Mohammadi and S. Moradi, The total graph and regular graph of a commutative ring, *J. Pure Appl. Algebra* **213** (2009) 2224–2228.

[2] S. Akbari, R. Nikadish and M. J. Nikmehr, Some results on the intersection graphs of ideals of rings, *J. Algebra Appl.* **12**(4) (2013) 1250200.

[3] D. F. Anderson and A. Badawi, The total graph of a commutative ring, *J. Algebra* **320** (2008) 2706–2719.

[4] D. F. Anderson and A. Badawi, On the total graph of a commutative ring without the zero element, *J. Algebra Appl.* **11**(4) (2012) 1250074.

[5] D. F. Anderson and A. Badawi, The generalized total graph of a commutative ring, *J. Algebra Appl.* **12**(5) (2013) 1250212.

[6] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra* **217** (1999) 434–447.

[7] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* (Adison-Wesley Publishing Company, Reading, MA, 1969).

[8] Z. Barati, K. Khashyarmanesh, F. Mohammadi and K. Nafar, On the associated graphs to a commutative ring, *J. Algebra Appl.* **11**(2) (2012) 1250037.

[9] M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings, *I. J. Algebra Appl.* **10**(4) (2011) 727–739.

[10] R. Belshoff and J. Chapman, Planar zero-divisor graph, *J. Algebra* **316**(1) (2007) 471–480.

[11] N. Bloomfield and C. Wickham, Local rings with genus two zero divisor graph, *Comm. Algebra* **38** (2010) 2965–2980.

[12] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, Intersection graphs of ideals of rings, *Discrete Math.* **309**(17) (2009) 5381–5392.

[13] T. T. Chelvam and T. Asir, The intersection graph of gamma sets in the total graph I, *J. Algebra Appl.* **12**(4) (2013) 1250198.

[14] T. T. Chelvam and T. Asir, The intersection graph of gamma sets in the total graph II, *J. Algebra Appl.* **12**(4) (2013) 1250199.

[15] T. T. Chelvam and T. Asir, On the genus of the total graph of a commutative ring, *Comm. Algebra* **41**(1) (2013) 142–153.

[16] H.-J. Chiang-Hsieh, P.-F. Lee and H.-J. Wang, The embedding of line graph associed to the zero-divisor graphs of commutative rings, *Israel J. Math.* **180**(1) (2010) 193–220.

[17] R. Diestel, *Graph Theory*, 3rd edn., Graduate Texts in Mathematics, Vol. 173 (Springer-Verlag, Berlin, Heidelberg, 2005).

[18] S. H. Jafari and N. J. Rad, Planarity of intersection graphs of ideals of rings, *Int. Electron. J. Algebra* **8** (2010) 161–166.

[19] M. Jungerman and G. Ringel, Minimal triangulations on orientable surfaces, *Acta Math.* **145**(1–2) (1980) 121–154.

[20] I. Kaplansky, *Commutative Rings*, Rev edn. (University of Chicago Press, Chicago, 1974).

[21] H. R. Maimani, C. Wickham and S. Yassemi, Rings whose total graphs have genus at most one, *Rocky Mountain J. Math.* **42**(5) (2012) 1551–1560.

[22] B. Mohar and C. Thomassen, *Graphs on Surfaces* (Johns Hopkins, Baltimore, MD, 2001).

[23] Z. Pucanović and Z. Petrović, Toroidality of intersection graphs of ideals of commutative rings, *Graphs Combin.* (2013) 1–10, doi:10.1007/s00373-013-1292-1.

[24] S. P. Redmond, An ideal-based zero-divisor graph of a commutative ring, *Comm. Algebra* **31** (2003) 4425–4443.

[25] G. Ringel, Das Geschlecht des vollstndiger Paaren Graphen, *Abh. Math. Sem. Univ. Hamburg* **28** (1965) 139–150.

[26] G. Ringel and J. W. T. Youngs, Solution of the Heawood map-coloring problem, *Proc. Natl. Acad. Sci. USA* **60** (1968) 438–445.

[27] A. Riskin, Genus 2 crossing number of K9, *Discrete Math.* **145** (1995) 211–227.

[28] C. Thomassen, The graph genus problem is NP-complete, *J. Algorithms* **10**(4) (1989) 568–576.

[29] H. Wang, Zero-divisor graphs of genus one, *J. Algebra* **304**(2) (2006) 666–678.

[30] C. Wickham, Classification of rings with genus one zero-divisor graphs, *Comm. Algebra* **36**(2) (2008) 1–21.

[31] C. Wickham, Rings whose zero-divisor graphs have positive genus, *J. Algebra* **321**(2) (2009) 377–383.