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Sharp Bounds for Trigonometric and Hyperbolic Functions with Application to Fractional Calculus

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Abstract: Sharp bounds for $\frac{\cosh(x)}{x}$, $\frac{\sinh(x)}{x}$, and $\frac{\sin(x)}{x}$ were obtained, as well as one new bound for $\frac{e^x+\arctan(x)}{\sqrt{x}}$. A new situation to note about the obtained boundaries is the symmetry in the upper and lower boundary, where the upper boundary differs by a constant from the lower boundary. New consequences of the inequalities were obtained in terms of the Riemann–Liovuille fractional integral and in terms of the standard integral.

Keywords: polynomial bounds; L'Hôpital's rule of monotonicity; Jordan's inequality; trigonometric functions

MSC: 26D05; 26D07; 26D20

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1. Introduction and Preliminaries

Inequalities have been an ongoing topic of research since their discovery. As the proof of how interesting they are, many books were written in that field; for example, refer to the famous book [1]. The $\frac{\sin(x)}{x}$ inequality in this paper will be improved; thus, we must mention the first inequality of that nature known as Jordan's inequality.

$$\frac{2}{\pi} < \frac{\sin(x)}{x} < 1; 0 < x < \frac{\pi}{2}.$$

Multiple proofs of the Jordans inequality exist, and we refer the reader to the following papers for more detail [2–4]. Jordan's inequality was improved on the left-hand side by Mitrinović-Adamović, while the right-hand side is the known Cusa inequality. We state it here for educational purposes.

$$(\cos(x))^{\frac{1}{3}} < \frac{\sin(x)}{x} < \frac{2+\cos(x)}{3}.$$

Recently, the authors [5] sharpened Jordan's inequality further.

$$\left(1-\frac{x^2}{\pi^2}\right)e^{-\frac{\ln(2)}{\pi^2}x^2} < \frac{\sin(x)}{x} < \left(1-\frac{x^2}{\pi^2}\right)e^{(\frac{1}{\pi^2}-\frac{1}{6})x^2}; 0 < x < \pi.$$

They also provided other interesting bounds in another paper [6].

$$\left(1-\frac{x^2}{\pi^2}\right)^{\frac{\pi^4}{90}} e^{\left(\frac{\pi^2}{90}-\frac{1}{6}\right)x^2} < \frac{\sin(x)}{x}; 0 < x < \pi$$

$$\frac{\sin(x)}{x} < \frac{2}{3} + \frac{1}{3}\left(1-\frac{4x^2}{\pi^2}\right)^{\frac{\pi^4}{96}} e^{\left(\frac{\pi^2}{24}-\frac{1}{2}\right)x^2}; 0 < x < \frac{\pi}{2}.$$

In this paper, we will sharpen these bounds in a simple and efficient manner. More about such inequalities can be found in the following papers [7–11].

We provide our first definition of a fractional integral that will be used in the corollaries of the results.

Definition 1. *The generalized hypergeometric function ${}_qF_q(a;b;x)$ is defined as follows [12]:*

$${}_pF_q(a;b;x) = \sum_{k=0}^{+\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!}$$

where $(a)_k$ is the Pochhammer symbol defined as follows [12].

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1).$$

Definition 2. *The Riemann–Liouville fractional integral is defined by [13–15] where $\Re(\alpha) > 0$ and f is locally integrable.*

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)dx.$$

The functions on which we apply the Riemann–Liouville fractional integral are well defined in terms of the integral formula. We will require the following Lemma. Lemma 1 ([16], p. 10) taken below is known as L'Hôpital's rule of monotonicity. It is a very useful tool in the theory of inequalities.

Lemma 1. *Let $f, g : [m, n] \to \mathbb{R}$ be two continuous functions which are differentiable on (m, n) and $g' \neq 0$ in (m, n). If $\frac{f'}{g'}$ is increasing (or decreasing) on (m, n), then the functions $\frac{f(x)-f(m)}{g(x)-g(m)}$ and $\frac{f(x)-f(n)}{g(x)-g(n)}$ are also increasing (or decreasing) on (m, n). If $\frac{f'}{g'}$ is strictly monotone, then the monotonicity in the relationship is also strict.*

2. Main Results

We provide our first Theorem in the paper.

Theorem 1. *The following bounds hold for $x \in (0, 1)$.*

$$\frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x} < \frac{e^x + \arctan(x)}{\sqrt{x}} < e + \frac{1}{4}(\pi - 10) + \frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x}.$$

Proof. Set the following:

$$g(x) = \frac{e^x - 1 + \arctan(x) - \frac{x^2}{2} - x}{\sqrt{x}} = \frac{h_1(x)}{h_2(x)}$$

where $h_1(x) = e^x - 1 + \arctan(x) - \frac{x^2}{2} - x$ and $h_2(x) = \sqrt{x}$ with $h_1(0) = 0$ and $h_2(0) = 0$.

After differentiating, we obtain the following.

$$\frac{h_1'(x)}{h_2'(x)} = \left(-1 + e^x - x + \frac{1}{1+x^2}\right) \cdot 2\sqrt{x}.$$

Taking the following:

$$f(x) = \left(-1 + e^x - x + \frac{1}{1+x^2}\right) \cdot 2\sqrt{x}$$

and by differentiating it, we obtain the following.

$$f'(x) = \frac{(2e^x-3)x^5 + (e^x-1)x^4 + 2(2e^x-3)x^3 + (2e^x-5)x^2 + (2e^x-3)x + e^x}{\sqrt{x}(x^2+1)^2}$$

The denominator is positive for all $x \in (0,1)$. We need to show that $q(x) > 0$ where $q(x)$ denotes the numerator. Using the simple estimates $e^x \geq 1 + x, 1 > x^2$ where $x \in (0,1)$, we obtain the following.

$$q(x) > 2x^6 + 4x^4 > 0.$$

Therefore $f'(x) > 0$, which implies $f(x)$ is increasing; therefore, $\frac{h_1'(x)}{h_2'(x)}$ is increasing, which by Lemma 1 means $\frac{h_1(x)-h_1(0)}{h_2(x)-h_2(0)}$ is increasing. However, since we chose functions $h_1(x), h_2(x)$ such that $h_1(0) = 0$ and $h_2(0) = 0$, we obtain the fact that the following:

$$g(x) = \frac{e^x - 1 + \arctan(x) - x - \frac{x^2}{2}}{\sqrt{x}} = \frac{h_1(x)}{h_2(x)}$$

is increasing. Therefore, the following inequality holds:

$$g(0_+) < g(x) < g(1).$$

which provides us with the following inequality.

$$0 < \frac{e^x - 1 + \arctan(x) - x - \frac{x^2}{2}}{\sqrt{x}} < e + \frac{1}{4}(\pi - 10)$$

This is rearranged and provides us with the desired inequality.

$$\frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x} < \frac{e^x + \arctan(x)}{\sqrt{x}} < e + \frac{1}{4}(\pi - 10) + \frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x}$$

□

We provide a corollary in which we provide an estimate of the fractional inequality using the previous theorem.

Corollary 1. *The following inequality holds for* $0 < a < t$, $\alpha > t > 0$ *and* $t \in (0,1)$*:*

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sqrt{\pi}\Gamma(\alpha)t^{\alpha-\frac{1}{2}}}{\Gamma\left(\alpha+\frac{1}{2}\right)} - 2\sqrt{a}t^{\alpha-1}\,{}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; \frac{a}{t}\right) + \psi(a,t,\alpha)\right.$$

$$\left. + \frac{t^{\alpha}\left(\frac{4\left(1-\frac{a}{t}\right)^{\alpha}(2\alpha a - a + t) - 4t\,{}_2F_1\left(-\frac{1}{2}, 1-\alpha; \frac{1}{2}; \frac{a}{t}\right)}{4\alpha^2-1} + \frac{\sqrt{\pi}\left((at)^{3/2} - \sqrt{at^5}\right)\Gamma(\alpha)}{t(a-t)\Gamma\left(\alpha+\frac{3}{2}\right)}\right)}{2\sqrt{a}}\right)$$

$$< {}_aI_t^\alpha\left(\frac{e^x + \arctan(x)}{\sqrt{x}}\right) < \frac{1}{\Gamma(\alpha)}\left(\frac{(-10+4e+\pi)(t-a)^\alpha}{4\alpha}\right.$$
$$+\frac{\sqrt{\pi}\Gamma(\alpha)t^{\alpha-\frac{1}{2}}}{\Gamma\left(\alpha+\frac{1}{2}\right)} - 2\sqrt{a}t^{\alpha-1}\,{}_2F_1\left(\frac{1}{2}, 1-\alpha; \frac{3}{2}; \frac{a}{t}\right) + \psi(a,t,\alpha)$$
$$\left. + \frac{t^\alpha\left(\frac{4\left(1-\frac{a}{t}\right)^\alpha(2\alpha a - a + t) - 4t\,{}_2F_1\left(-\frac{1}{2}, 1-\alpha; \frac{1}{2}; \frac{a}{t}\right)}{4\alpha^2-1} + \frac{\sqrt{\pi}\left((at)^{3/2} - \sqrt{at^5}\right)\Gamma(\alpha)}{t(a-t)\Gamma\left(\alpha+\frac{3}{2}\right)}\right)}{2\sqrt{a}}\right)$$

where $\psi(a,t,\alpha) = {}_aI_t^\alpha\left(\frac{x^{\frac{3}{2}}}{2}\right)\Gamma(\alpha)$.

Proof. Let us first consider the convergence of the integral for the sake of completeness.

$${}_aI_t^\alpha\left(\frac{e^x + \arctan(x)}{\sqrt{x}}\right) = \int_a^t (t-x)^{\alpha-1}\frac{\arctan(x) + e^x}{\sqrt{x}}dx.$$

As we can see, the quantity that can induce a problem is $(t-x)^{\alpha-1}$ when $x \to t$. The thing to note here is that $\alpha > 0$, which means that the degree of the expression $(t-x)^{\alpha-1}$ will be between $(0,1)$, which when integrated will not proceed to the denominator; therefore, there is no division by zero. Another situation to note is that when $a = 0$, the quantity in the denominator $\sqrt{x}$ can be integrated around zero.

Similar discussions in the other corollaries lead to the same conclusion; therefore, they are omitted.

Now we are certain about applying the formula. By pplying the Riemann–Liouville integral transform:

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1} f(x)dx$$

on both sides of the inequality, we derived in the last theorem:

$$\frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x} < \frac{e^x + \arctan(x)}{\sqrt{x}} < e + \frac{1}{4}(\pi - 10) + \frac{1}{\sqrt{x}} + \frac{x^{\frac{3}{2}}}{2} + \sqrt{x}$$

and we obtain the following inequality. □

Corollary 2. *The derived inequality can be used to approximate the solution to a first-order nonlinear ordinary differential equation. Consider differential equation* $y = f(x)$ *such that* $f : (0,1) \to (0,1)$ *and* $y(t_0)$ *are defined.*

$$y' = \frac{\sqrt{y}x}{e^y + \arctan(y)}.$$

Separating the variables and integrating from t_0 *to* t*, we obtain the following.*

$$\int_{t_0}^t \frac{e^y + \arctan(y)}{\sqrt{y}}dy = \int_{t_0}^t xdx.$$

Using the inequality and solving the integral, which is then in terms of polynomials, we obtain the following solution.

The following inequality provides an estimate for $\frac{\cosh(x)}{x}$.

Theorem 2. *The following bounds hold for* $x \in (0,1)$,

$$\frac{1}{x} + \frac{x}{2} + \frac{x^3}{24} + \frac{x^5}{720} < \frac{\cosh(x)}{x} < \cosh(1) - \frac{1111}{720} + \frac{1}{x} + \frac{x}{2} + \frac{x^3}{24} + \frac{x^5}{720}.$$

Proof. Let us consider the following function.

$$g(x) = \frac{\cosh(x) - 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720}}{x} = \frac{h_1(x)}{h_2(x)}$$

where $h_1(x) = \cosh(x) - 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720}$ and $h_2(x) = x$.

Taking its derivative, we obtain the following.

$$\frac{h_1'(x)}{h_2'(x)} = \sinh(x) - x - \frac{x^3}{6} - \frac{x^5}{120}$$

Now we realize that the terms with a negative sign are exactly the terms in the $\sinh(x)$ Taylor expansion

$$\sinh(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

$$\frac{h_1'(x)}{h_2'(x)} = \sum_{n=3}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

This is obviously positive. Now, we need its increasing form. We take the following.

$$G(x) = \frac{h_1'(x)}{h_2'(x)} = \sum_{n=3}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Taking a derivative, we obtain the following:

$$G'(x) = \left(\frac{h_1'(x)}{h_2'(x)}\right)' = \sum_{n=3}^{+\infty} (2n+1)\frac{x^{2n}}{(2n+1)!} > 0$$

which means that $G(x)$ is increasing. Therefore, according to the Lemma 1, we obtain an increasing function $g(x) = \frac{h_1(x) - h_1(0)}{h_2(x) - h_2(0)}$. However, since we chose h_1, h_2 to be zero at $x = 0$, we obtain an increasing function $g(x)$. Therefore, the following inequality holds.

$$g(0) < \frac{\cosh(x) - 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720}}{x} < g(1)$$

This provides us with the following:

$$0 < \frac{\cosh(x) - 1 - \frac{x^2}{2} - \frac{x^4}{24} - \frac{x^6}{720}}{x} < \cosh(1) - \frac{1111}{720}$$

which when rearranged provides us with the desired inequality. □

The following Corollary shows how our inequality can be paired up with the fractional integral to produce an effective inequality for ${}_aI_t^\alpha(\frac{\cosh(x)}{x})$.

Corollary 3. *The following inequality holds for* $0 < a < t$ *and* $\Re(\alpha) > 0$, $t \in (0,1)$*:*

$$\frac{1}{\Gamma(\alpha)}\Big(\psi(a,t,\alpha) + \zeta(a,t,\alpha)\Big) < {}_aI_t^\alpha\left(\frac{\cosh(x)}{x}\right) <$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{(720\cosh(1)-1111)(t-a)^{\alpha}}{720\alpha}+\psi(a,t,\alpha)+\zeta(a,t,\alpha)\right)$$

where

$$\psi(a,t,\alpha)=\frac{(t-a)^{\alpha}(a\alpha+t)}{2\alpha(\alpha+1)}$$

$$+\frac{(t-a)^{\alpha}\left(\alpha(\alpha+1)(\alpha+2)a^3+3\alpha(\alpha+1)a^2t+6\alpha at^2+6t^3\right)}{24\alpha(\alpha+1)(\alpha+2)(\alpha+3)}+{}_aI_t^{\alpha}\left(\frac{x^5}{720}\right)\Gamma(\alpha)$$

$$\zeta(a,t,\alpha)=t^{\alpha-2}\left(a(\alpha-1)\,{}_3F_2\left(1,1,2-\alpha;2,2;\frac{a}{t}\right)-t(\log(a)+\psi^{(0)}(\alpha)-\log(t)+\gamma)\right)$$

Proof. Applying the Riemann–Liouville integral transform on both sides of the inequality we derived in the last Theorem and evaluating the left and right hand side, we arrive at the following inequality. □

Corollary 4. *Using similar reasoning to the Corollary 2, we can form the following differential equation, $y=f(x)$, such that $f:(0,1)\to(0,1)$ and $y(t_0)$ are defined.*

$$y'=\frac{yx}{\cosh(y)}.$$

Separating the variables and using the inequality, we can find the following solution. We omit the calculations for obvious reasons.

A similar construction of Corollaries for other Theorems can be performed, and we omit them due to obvious reasons.
The following Theorem sharpens Jordan's inequality.

Theorem 3. *The following bounds hold for $x\in(0,\frac{\pi}{2})$.*

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}<\frac{\sin(x)}{x}<$$

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\frac{x^{10}}{11!}-1+\frac{2}{\pi}+\frac{\pi^2}{24}-\frac{\pi^4}{1920}+\frac{\pi^6}{322560}-\frac{\pi^8}{92897280}+\frac{\pi^{10}}{40874803200}.$$

Proof. Let us consider the following function.

$$g(x)=\frac{\sin(x)-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\frac{x^9}{9!}+\frac{x^{11}}{11!}}{x}=\frac{h_1(x)}{h_2(x)}$$

Differentiating h_1 and h_2, respectively, we obtain the following.

$$\frac{h_1'(x)}{h_2'(x)}=\cos(x)-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\frac{x^{10}}{10!}$$

Expanding $\cos(x)$ into a Taylor series:

$$\cos(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k x^{2k}}{(2k)!}$$

we realize that the terms outside of summation are exactly the coefficients of the $\cos(x)$ expansion and, to be precise, the terms are exactly the first five terms of the $\cos(x)$ expansion, which leaves us with the following:

$$\frac{h_1'(x)}{h_2'(x)}=\sum_{k=6}^{+\infty}\frac{(-1)^k x^{2k}}{(2k)!}.$$

which is obviously positive since it is a remainder of the positive Taylor expansion.

Now, we need an increasing form. Taking the following:

$$G(x)=\frac{h_1'(x)}{h_2'(x)}=\sum_{k=6}^{+\infty}\frac{(-1)^k x^{2k}}{(2k)!}.$$

and differentiating $G(x)$, we obtain the following:

$$G'(x)=\left(\frac{h_1'(x)}{h_2'(x)}\right)'=\sum_{k=6}^{+\infty}2k\frac{(-1)^k x^{2k-1}}{(2k)!}>0.$$

which means that $G(x)$ is increasing. Therefore, we obtain the fact that $\frac{h_1'(x)}{h_2'(x)}$ is increasing in both cases; therefore, $\frac{h_1(x)-h_1(0)}{h_2(x)-h_2(0)}$ is increasing, but we chose $h_1(x), h_2(x)$ such that the following holds $h_{1,2}(0)=0$. Therefore since $g(x)$ is an increasing function, the following relation holds:

$$g(0)<g(x)<g\left(\frac{\pi}{2}\right).$$

which is evaluated at the following.

$$0<\frac{\sin(x)-x+\frac{x^3}{3!}-\frac{x^5}{5!}+\frac{x^7}{7!}-\frac{x^9}{9!}+\frac{x^{11}}{11!}}{x}<$$

$$-1+\frac{2}{\pi}+\frac{\pi^2}{24}-\frac{\pi^4}{1920}+\frac{\pi^6}{322560}-\frac{\pi^8}{92897280}+\frac{\pi^{10}}{40874803200}$$

When rearranged, it provides us with the desired inequality. □

In the following, we provide a corollary of the previously improved inequality.

Corollary 5. *The following inequality holds.*

$$1.37076216382<\int_0^{\frac{\pi}{2}}\frac{\sin(x)}{x}dx<1.37076222008$$

Proof. Integrating the inequality derived in the last Theorem from 0 to $\frac{\pi}{2}$ and integrating term by term, we obtain the following inequality. □

The next Theorem provides an estimate on the $\frac{\sinh(x)}{x}$ inequality.

Theorem 4. *The following bounds hold for $x\in(0,1)$.*

$$1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}<\frac{\sinh(x)}{x}<1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\sinh(1)-\frac{5923}{5040}.$$

Proof. Let us consider the following function.

$$g(x)=\frac{\sinh(x)-x-\frac{x^3}{3!}-\frac{x^5}{5!}-\frac{x^7}{7!}}{x}=\frac{h_1(x)}{h_2(x)}$$

Taking derivative of $h_1(x)$ and $h_2(x)$, we obtain the following.

$$\frac{h_1'(x)}{h_2'(x)}=\cosh(x)-1-\frac{x^2}{2!}-\frac{x^4}{4!}-\frac{x^6}{6!}$$

Now we expand the cosh into its Taylor series and realize that the terms outside of the sum are exactly the first four terms in the summation. Therefore, we obtain the following:

$$\frac{h_1'(x)}{h_2'(x)} = \sum_{n=4}^{+\infty} \frac{x^{2n}}{(2n)!}.$$

which is positive. We also it in increasing form. Taking the following:

$$G(x) = \frac{h_1'(x)}{h_2'(x)} = \sum_{n=4}^{+\infty} \frac{x^{2n}}{(2n)!}$$

and taking a derivative, we obtain the following:

$$G'(x) = \left(\frac{h_1'(x)}{h_2'(x)}\right)' = \sum_{n=4}^{+\infty} 2n\frac{x^{2n-1}}{(2n)!}.$$

which is positive; therefore, $G(x)$ is increasing. From the Lemma, we obtain that function $\frac{h_1(x)-h_1(0)}{h_2(x)-h_2(0)}$ is increasing too. However, since we chose functions h_1, h_2 to be zero when $x = 0$, we obtain an increasing $g(x)$. Therefore, the following inequality follows.

$$g(0) < g(x) < g(1).$$

When the expression is solved for $\frac{\sinh(x)}{x}$, we obtained the desired inequality. □

The following Corollary illustrates how the improved bounds can be used in estimating the integral.

Corollary 6. *The following bounds for the integral hold.*

$$1.05725056689 < \int_0^1 \frac{\sinh(x)}{x} dx < 1.05725334784.$$

Proof. Integrating the inequality in the previously derived Theorem from 0 to 1, we obtain the desired bounds. □

3. Conclusions

1. Sharper upper and lower bounds were obtained in terms of polynomials. New consequences of such sharper bounds are provided in the corollaries in terms of the integral estimate of $\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{x} dx$ and in terms of the fractional integral estimates of ${}_aI_t^\alpha\left(\frac{e^x+\arctan(x)}{\sqrt{x}}\right)$ and ${}_aI_t^\alpha\left(\frac{\cosh(x)}{x}\right)$.
2. Question arises with respect to which would be the lowest upper and biggest lower bound for obtained inequalities, which leaves room for further research.
3. Each of Theorem 2–4 can be easily generalized to arbitrary n as they rely on the remainder of Taylor expansion.

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