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Starlike and close-to-convex functions defined by differential inequalities

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ABSTRACT

We present new sufficient conditions for an analytic function to be close-to-convex and starlike, respectively. These conditions are easy to apply and can be used to obtain functions in these classes.

1. Introduction

Let $\mathcal{H}$ denote the space of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then, we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Further, let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ having the expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f)z^n$$

and let $\mathcal{S} \subset \mathcal{A}$ be the set of univalent functions in $\mathbb{D}$. For a given $\lambda \geqslant 0$, we say that a function $f \in \mathcal{A}$ belongs to the family $\mathcal{U}(\lambda)$ if

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| \leqslant \lambda, \quad z \in \mathbb{D}. \tag{1.1}$$

It is clear that functions in $\mathcal{U}(0)$ are of the form

$$f(z) = \frac{z}{1 - bz}$$

with $|b| \leqslant 1$. If $\lambda > 0$, then we may use the strict inequality in (1.1). Also, we observe that equality in (1.1) with $\lambda = 1$ is not possible. It is well-known that $\mathcal{U}(1) \subsetneq \mathcal{S}$, and therefore functions in $\mathcal{U}(\lambda)$ are univalent if $0 \leqslant \lambda \leqslant 1$. This class has been studied by a number of authors, see [4] and the references therein.

For a given $0 \leqslant \beta < 1$, a function $f \in \mathcal{S}$ is called starlike of order β, denoted by $\mathcal{S}^*(\beta)$, if $\mathrm{Re}(zf'(z)/f(z)) > \beta$ for all $z \in \mathbb{D}$. We set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and each f in $\mathcal{S}^*$ is referred to as a starlike function and $f(\mathbb{D})$ is indeed a domain that is starlike (with respect to 0); i.e. $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ that maps the unit disk $\mathbb{D}$ onto a convex domain is called a convex function. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{S}$ that are convex. It is well-known that $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$. Finally, a function $f \in \mathcal{A}$ is close-to-convex (univalent), denoted by $f \in \mathcal{C}$, if and only if there exists a convex

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function g (not necessarily normalized) such that $\mathrm{Re}(f'(z)/g'(z)) > 0$ in $\mathbb{D}$. We remark that close-to-convex functions are necessarily univalent. We have the following strict inclusion

$$\mathscr{K} \subsetneq \mathscr{S}^*(1/2) \subsetneq \mathscr{S}^* \subsetneq \mathscr{C} \subsetneq \mathscr{S}.$$

Various aspects of these and many other special classes have been studied extensively (for details, see [1,2]).

In Section 2, we use Schwarz function version of representation of functions in $\mathscr{U}(\lambda)$ and apply convolution technique to obtain sufficiency for starlikeness, in particular. In Section 3, we present simple sufficient condition for functions to be in the class of close-to-convex functions.

In order to prove the results of Section 2, we recall that the Hadamard product $f \star g$ of two power series $f(z) := \sum_{n=0}^{\infty} a_n(f)z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g)z^n$ in $\mathscr{H}$ is the power series defined by

$$f \star g(z) := \sum_{n=0}^{\infty} a_n(f)a_n(g)z^n.$$

Clearly, $f \star g$ is also a member of $\mathscr{H}$. The following lemma, which is essentially due to Ruscheweyh [3], is crucial for the proof of Theorem 3.1.

Lemma 1.2. *Let* $c \in \mathbb{C}$ *with* $\mathrm{Re}\,(c) < 1$ *and* $F_c(z) := \sum_{n=1}^{\infty} \frac{1-c}{n-c} z^{n-1} \in \mathscr{H}$. *Then*

$$\sup_{z\in\mathbb{D}} |f \star F_c(z)| \leqslant \sup_{z\in\mathbb{D}} |f(z)|, \quad \textit{for any } f \in \mathscr{H}.$$

Let us finally consider $j \geqslant 0$ and

$$\mathscr{B}_j = \{\omega \in \mathscr{H} : |\omega(z)| \leqslant |z|^j,\ z \in \mathbb{D}\}.$$

Clearly $\mathscr{B}_j$ is a subspace of $\mathscr{H}$ and a topological space of its own.

2. Sufficient conditions for starlikeness

For our presentation, we introduce a definition. For $\alpha \in \mathbb{C}$ and $\lambda \geqslant 0$, define

$$\mathscr{G}(\alpha,\lambda) = \left\{F \in \mathscr{A} : \left|\frac{zF''(z)}{F'(z)} - (2+\alpha)\left(\frac{zF'(z)}{F(z)} - 1\right)\right| \leqslant \lambda\left|\frac{zF'(z)}{F(z)}\right|,\ z \in \mathbb{D}\right\}$$

and

$$\mathscr{U}(\lambda,\alpha) = \left\{f \in \mathscr{A} : \frac{f(z)}{z} \neq 0,\ \left|\left(\frac{z}{f(z)}\right)^2 f'(z) + \alpha\frac{z}{f(z)} - \alpha - 1\right| \leqslant \lambda,\ z \in \mathbb{D}\right\}.$$

For convenience, we set

$$\mathscr{U}_2(\lambda,\alpha) = \{f \in \mathscr{U}(\lambda,\alpha) :\ f''(0) = 0\} \quad \text{and} \quad \mathscr{G}_2(\lambda,\alpha) = \{F \in \mathscr{G}(\lambda,\alpha) :\ F''(0) = 0\}$$

and so, $\mathscr{U}(\lambda,0) \equiv \mathscr{U}(\lambda)$.

Theorem 2.1. *Let* $f \in \mathscr{U}(\alpha,\lambda)$.

(i) *If* α *is a complex number such that* $0 < \lambda \leqslant \mathrm{Re}(-\alpha)$ *and* $\delta = \lambda/\mathrm{Re}(-\alpha)$, *then we have*

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{1+\delta}, \quad z \in \mathbb{D}.$$

(ii) *If* α *is a nonzero real number with* $\alpha \leqslant -1$, *then* f *is starlike whenever* $0 < \lambda \leqslant 1/2$.

(iii) *If* α *is a complex number such that* $0 < \lambda \leqslant \min\{\mathrm{Re}(-\alpha), \lambda_0(\alpha)\}$ *then* f *is starlike, where* $\lambda_0(\alpha)$ *is given by*

$$\lambda_0(\alpha) = \frac{2|\alpha|}{|\alpha| + |1+\alpha| + \sqrt{(|\alpha| + |1+\alpha|)^2 + 4(|\alpha| - \mathrm{Re}\alpha)}}. \tag{2.2}$$

Proof. Let $f \in \mathscr{U}(\lambda,\alpha)$ and, for convenience, set $a_2 := a_2(f) (= f''(0)/2!)$. Then, as

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)}\right)^2 f'(z),$$

it follows that

$$-z\left(\frac{z}{f(z)}\right)' + (\alpha+1)\left(\frac{z}{f(z)} - 1\right) = \lambda\omega(z), \tag{2.3}$$

where ω is analytic for $|z| < 1$ such that $\omega(0) = 0$ and $|\omega(z)| \leqslant 1$. We observe that $\omega'(0) = -2a_2\alpha$. For Re $\alpha < 0$, it follows that

$$\frac{z}{f(z)} = 1 - \lambda\sum_{n=1}^{\infty}\frac{a_n(\omega)}{n-1-\alpha}z^n = 1 - \lambda\int_0^1 \frac{\omega(tz)}{t^{\alpha+2}}dt, \quad z \in \mathbb{D}, \tag{2.4}$$

for some $\omega \in \mathscr{B}_1$. Since $\omega \in \mathscr{B}_1$, we find that

$$\left|\frac{z}{f(z)} - 1\right| \leqslant \delta|z|, \quad z \in \mathbb{D}, \tag{2.5}$$

where $\delta = \lambda/|\mathrm{Re}\ \alpha|$. Therefore, we have

$$\left|\frac{f(z)}{z} - \frac{1}{1-\delta^2|z|^2}\right| \leqslant \frac{\delta|z|}{1-\delta^2|z|^2}, \quad z \in \mathbb{D}$$

which gives

$$\mathrm{Re}\ \left(\frac{f(z)}{z}\right) \geqslant \frac{1}{1+\delta|z|} > \frac{1}{1+\delta}, \quad z \in \mathbb{D}.$$

Proof of Case (i) follows.

For the proof of Case (ii), we may rewrite (2.4) in an equivalent form as

$$\frac{z}{f(z)} = 1 + \frac{\lambda}{\alpha}\omega(z) \star zF_{1+\alpha}(z), \quad z \in \mathbb{D}. \tag{2.6}$$

It follows then from (2.3) and (2.6) that

$$\frac{zf'(z)}{f(z)} = \frac{1+\alpha+\lambda\omega(z)}{1+(\lambda/\alpha)zF_{1+\alpha}(z) \star \omega(z)} - \alpha, \quad z \in \mathbb{D}. \tag{2.7}$$

By Lemma 1.2 and the maximum principle, it is enough to assert the inequality

$$\inf\left\{\mathrm{Re}\ \left(\frac{zf'(z)}{f(z)}\right) : f \in \mathscr{U}(\lambda,\alpha),\ z \in \mathbb{D}\right\} \geqslant \inf\left\{\mathrm{Re}\ \left(\frac{1+\alpha+\lambda e^{i\varphi}}{1+(\lambda/|\alpha|)e^{i\psi}} - \alpha\right) : \varphi,\psi \in \mathbb{R}\right\}. \tag{2.8}$$

Because of the analytic characterization of the class $\mathscr{S}^*$, we see that $\mathscr{U}(\lambda,\alpha) \subset \mathscr{S}^*$ if the image of $\mathbb{D}$ by any Möbius transformation of the type

$$T(z) = \frac{1+\alpha+\lambda z}{1+(\lambda/|\alpha|)e^{i\psi}z} - \alpha$$

is a disk which lies completely in the right half-plane Re $w > 0$. Now, it is a simple exercise to see that $w = T(z)$ maps the unit disk $|z| < 1$ onto the disk

$$\left|w - \frac{1+\lambda^2\left(\frac{\alpha}{|\alpha|^2} - \frac{e^{-i\psi}}{|\alpha|}\right)}{1-\lambda^2/|\alpha|^2}\right| < \lambda\frac{\left|1 - \frac{e^{-i\psi}(1+\overline{\alpha})}{|\alpha|}\right|}{1-\lambda^2/|\alpha|^2}.$$

In particular this gives

$$\mathrm{Re}\ w > \frac{1+\lambda^2\mathrm{Re}\ \left(\frac{\alpha}{|\alpha|^2} - \frac{e^{-i\psi}}{|\alpha|}\right) - \lambda\left|1 - \frac{e^{-i\psi}(1+\overline{\alpha})}{|\alpha|}\right|}{1-\lambda^2/|\alpha|^2}$$

and therefore, a sufficient condition for Re $w > 0$ to hold is that

$$1 + \lambda^2\mathrm{Re}\ \left(\frac{\alpha}{|\alpha|^2} - \frac{e^{-i\psi}}{|\alpha|}\right) \geqslant \lambda\left|1 - \frac{e^{-i\psi}(1+\overline{\alpha})}{|\alpha|}\right|. \tag{2.9}$$

From (2.8), it is clear that for the starlikeness of functions in $\mathscr{U}(\lambda,\alpha)$, it suffices to verify the condition (2.9). Clearly, this inequality (2.9) holds if

$$1 + \lambda^2\mathrm{Re}\ \left(\frac{\alpha}{|\alpha|^2}\right) \geqslant \lambda g(\psi_0) := \lambda\max_{0\leqslant\psi\leqslant 2\pi} g(\psi),$$

where

$$g(\psi) = \frac{\lambda}{|\alpha|}\mathrm{Re}\,(e^{-i\psi}) + \left|1 - \frac{e^{-i\psi}(1+\bar{\alpha})}{|\alpha|}\right|. \tag{2.10}$$

Case 1: Suppose that α is real and negative. Then, $g(\psi)$ takes the form

$$g(\psi) = -\frac{\lambda\cos\psi}{\alpha} + \left|1 + \frac{e^{-i\psi}(1+\alpha)}{\alpha}\right| = -\frac{\lambda\cos\psi}{\alpha} + \sqrt{1 + 2((1+\alpha)/\alpha)\cos\psi + ((1+\alpha)/\alpha)^2}. \tag{2.11}$$

Note that if $\alpha \leqslant -1$, then $\psi_0 = 0$ clearly gives the point of maximum for $g(\psi)$. Thus, (2.9) holds if

$$1 + \frac{\lambda^2}{\alpha} \geqslant \lambda g(0) = -\frac{\lambda^2}{\alpha} + \lambda\left(2 + \frac{1}{\alpha}\right)$$

which is equivalent to $(\lambda - \alpha)(2\lambda - 1) \leqslant 0$. This gives the condition $\lambda \leqslant 1/2$ whenever $\alpha \leqslant -1$.

Case 2: Suppose that α is a complex constant such that $\mathrm{Re}\,\alpha < 0$. In this case, by the Triangle inequality, $g(\psi)$ given by (2.10) satisfy the condition

$$|g(\psi)| \leqslant \frac{\lambda}{|\alpha|} + \left(1 + \frac{|1+\alpha|}{|\alpha|}\right).$$

Using this, we see that (2.9) holds if λ and α are related by

$$1 + \lambda^2\mathrm{Re}\left(\frac{\alpha}{|\alpha|^2}\right) \geqslant \lambda\left[\frac{\lambda}{|\alpha|} + \left(1 + \frac{|1+\alpha|}{|\alpha|}\right)\right]$$

which, by a computation, is equivalent to

$$\lambda^2(|\alpha| - \mathrm{Re}\,\alpha) + \lambda(|\alpha| + |1+\alpha|)|\alpha| - |\alpha|^2 \leqslant 0.$$

This gives the condition $\lambda \leqslant \lambda_0(\alpha)$, where

$$\lambda_0(\alpha) = \frac{-|\alpha|(|\alpha| + |1+\alpha|) + |\alpha|\sqrt{(|\alpha| + |1+\alpha|)^2 + 4(|\alpha| - \mathrm{Re}\,\alpha)}}{2(|\alpha| - \mathrm{Re}\,\alpha)}$$

which is same as the $\lambda_0(\alpha)$ given by (2.2). □

Example 2.1. When α is a real number, the condition on $\lambda_0(\alpha)$ takes a simple form. In this case, Theorem 2.1(iii) gives the following inclusion

$$\mathcal{U}(\alpha,\lambda) \subset \mathcal{S}^*$$

whenever $0 < \lambda \leqslant \frac{\sqrt{1-8\alpha}-1}{4}$ and α is a real number with $-1 \leqslant \alpha < 0$.

Corollary 2.12. *Let* $0 < \lambda \leqslant \mathrm{Re}\,(-\alpha)$ *and* $\delta = \lambda/\mathrm{Re}\,(-\alpha)$. *Then*

$$\mathcal{G}(\alpha,\lambda) \subset \mathcal{S}^*(1/(1+\delta)).$$

Proof. For $f \in \mathcal{U}(\alpha,\lambda)$, let F be defined by

$$F(z) = z\exp\left[\int_0^1 \left(\frac{f(tz)}{tz} - 1\right)\frac{dt}{t}\right].$$

Then it follows that

$$\frac{zF'(z)}{F(z)} = \frac{f(z)}{z} \tag{2.13}$$

and therefore, it is a simple exercise to see that

$$\left|\frac{zF''(z)}{F'(z)} - (2+\alpha)\left(\frac{zF'(z)}{F(z)} - 1\right)\right| \leqslant \lambda\left|\frac{zF'(z)}{F(z)}\right|$$

is equivalent to $f \in \mathcal{U}(\alpha,\lambda)$. Thus, the above correspondence $f \mapsto F$ gives a bijection from $\mathcal{U}(\alpha,\lambda)$ onto $\mathcal{G}(\alpha,\lambda)$. The result now follows from Theorem 2.1(1), and (2.13). □

Theorem 2.1 can be improved whenever $f \in \mathcal{U}(\alpha,\lambda)$ has the property that $f''(0) = 0$. For example, we have

Theorem 2.14. *Let* $f \in \mathcal{U}_2(\alpha,\lambda)$ *with* $\mathrm{Re}\,\alpha < 1$ *and* λ *be such that* $0 < \lambda \leqslant 1 - \mathrm{Re}\,\alpha$. *Then we have*

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1 - \mathrm{Re}\,\alpha}{\lambda + 1 - \mathrm{Re}\,\alpha}, \quad z \in \mathbb{D}.$$

Proof. Let $f \in \mathcal{U}_2(\lambda,\alpha)$. Because $a_2 = 0$, as in the proof of Theorem 2.1, we obtain that

$$\frac{z}{f(z)} = 1 - \lambda \sum_{n=2}^{\infty} \frac{a_n(\omega)}{n-1-\alpha} z^n = 1 - \lambda \int_0^1 \frac{\omega(tz)}{t^{\alpha+2}}\, dt, \quad |z|<1, \quad \text{for some } \omega \in \mathcal{B}_2.$$

As $|\omega(z)| \leqslant |z|^2$ and Re $\alpha < 1$, the last representation gives that

$$\left|\frac{z}{f(z)} - 1\right| \leqslant \delta|z|^2, \quad z \in \mathbb{D} \ \left(\delta = \frac{\lambda}{1 - \mathrm{Re}\ \alpha}\right)$$

which, after some computation, is seen to be equivalent to

$$\left|\frac{f(z)}{z} - \frac{1}{1-\delta^2|z|^4}\right| \leqslant \frac{\delta|z|^2}{1-\delta^2|z|^4}, \quad z \in \mathbb{D}$$

so that

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) \geqslant \frac{1}{1+\delta|z|^2} > \frac{1}{1+\delta}, \quad z \in \mathbb{D}$$

and we complete the proof. □

From the above correspondence and the fact that $f''(0) = 0$ if and only if $F''(0) = 0$, the following result is an easy consequence of Theorem 2.14. This result shows that, in the case of vanishing second coefficient of functions in $\mathcal{G}(\alpha,\lambda)$ one has an improved estimate.

Corollary 2.15. *Let* $0 < \lambda \leqslant 1 - \mathrm{Re}\ \alpha$ *and* $\delta = \lambda/(1-\mathrm{Re}\ \alpha)$. *Then*

$$\mathcal{G}_2(\alpha,\lambda) \subset \mathcal{S}^*(1/(1+\delta)).$$

The class $\mathcal{G}(-1,\lambda)$ was investigated by Silverman [5]. However, if $\alpha = -1$ then Corollary 2.12 implies the recent result of Singh [6, Theorem 1], namely the inclusion

$$\mathcal{G}(-1,\lambda) \subset \mathcal{S}^*(1/(1+\lambda))$$

for $0 < \lambda \leqslant 1$. Thus, Corollary 2.12 generalizes the result of [5,6]. Moreover, we also have an improved estimate

$$\mathcal{G}_2(-1,\lambda) \subset \mathcal{S}^*(2/(2+\lambda))$$

which holds for a larger range $0 < \lambda \leqslant 2$.

From the association between f and F above and (2.13), we observe that

$$f \in \mathcal{S}^* \Longleftrightarrow \mathrm{Re}\left(2 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}\right) > 0, \quad z \in \mathbb{D}.$$

This is similar to the Alexander transform which provides the one-to-one correspondence between convex and starlike functions.

3. Sufficient conditions for close-to-convexity

Theorem 3.1. *Suppose that* $g \in \mathcal{K} \cap \mathcal{U}(\lambda)$ *for* $0 \leqslant \lambda \leqslant 1$. *If* $f \in \mathcal{A}$ *satisfies the condition*

$$\left|f'(z)\left(\frac{z}{g(z)}\right)^2 - 1\right| \leqslant \sqrt{1-\lambda^2}, \quad z \in \mathbb{D} \tag{3.2}$$

then f is close-to-convex.

Proof. Put

$$u = f'(z)\left(\frac{z}{g(z)}\right)^2 \quad \text{and} \quad v = g'(z)\left(\frac{z}{g(z)}\right)^2$$

for a fixed $z \in \mathbb{D}$. Then, by an elementary geometry, the hypotheses gives

$$|\arg u| \leqslant \arcsin(\sqrt{1-\lambda^2}) \quad \text{and} \quad |\arg v| \leqslant \arcsin \lambda.$$

Thus, it follows that

$$\left|\arg\left(\frac{f'(z)}{g'(z)}\right)\right| = \left|\arg\left(\frac{u}{v}\right)\right| \leqslant \arcsin(\sqrt{1-\lambda^2}) + \arcsin\lambda = \frac{\pi}{2}, \quad z \in \mathbb{D}$$

which implies that the function f is close-to-convex in $\mathbb{D}$. □

Example 3.1. It is now a simple exercise to see that the function g defined by

$$\frac{z}{g(z)} = 1 + \lambda z^2$$

is in the class $\mathcal{U}(\lambda)$ whenever $0 \leqslant \lambda \leqslant 1$. Now, we consider the square root transform of the Koebe function $k(z) = z/(1+z)^2$ given by

$$s(z) = \sqrt{k(z^2)} = \frac{z}{1+z^2}.$$

Since the radius of convexity of s is known to be $r_0 = \sqrt{3-2\sqrt{2}} \approx 0.171573$, it follows that $r^{-1}s(rz)$ is a convex function for $0 < r \leqslant r_0$. This observation shows that g is convex whenever $0 < \lambda \leqslant r_0^2 \approx 0.0294373$. In particular, $g \in \mathcal{K} \cap \mathcal{U}(\lambda)$ with $\lambda \in (0, r_0^2]$. Finally, we consider a two-parameter family of functions $f_{\alpha,\lambda} := f$ defined by

$$f(z) = \int_0^z \frac{1+\alpha z}{(1+\lambda z^2)^2}\,dz,$$

where $\alpha \in \mathbb{C}$. We compute that

$$\left|f'(z)\left(\frac{z}{g(z)}\right)^2 - 1\right| = |\alpha z| \leqslant |\alpha|, \quad z \in \mathbb{D},$$

and, by Theorem 3.1, we conclude that f is close-to-convex whenever $|\alpha| \leqslant \sqrt{1-\lambda^2}$, and $\lambda \in (0, r_0^2]$. More generally, we see that

$$f(z) = z\int_0^1 \frac{1+\alpha\omega(tz)}{(1+\lambda t^2 z^2)^2}\,dt$$

is close-to-convex whenever $|\alpha| \leqslant \sqrt{1-\lambda^2}$ and $\lambda \in (0, r_0^2]$. Here $\omega(z)$ is any Schwarz function, i.e. $\omega(z)$ is any analytic map of $\mathbb{D}$ to $\mathbb{D}$ such that $\omega(0) = 0$.

Theorem 3.3. *Let $g \in \mathcal{S}^*$ such that $|(z/g(z)) - 1| \leqslant \lambda$ for $0 \leqslant \lambda \leqslant 1$. If $f \in \mathcal{A}$ satisfies the condition*

$$\left|f'(z)\left(\frac{z}{g(z)}\right)^2 - 1\right| \leqslant \sqrt{1-\lambda^2}, \quad z \in \mathbb{D},$$

then f is close-to-convex.

Proof. Put

$$u = f'(z)\left(\frac{z}{g(z)}\right)^2 \quad \text{and} \quad v = \frac{z}{g(z)}$$

for a fixed $z \in \mathbb{D}$. Again, by an elementary geometry, the hypotheses gives

$$|\arg u| \leqslant \arcsin(\sqrt{1-\lambda^2}) \quad \text{and} \quad |\arg v| \leqslant \arcsin\lambda.$$

Thus, we have

$$\left|\arg\left(\frac{zf'(z)}{g(z)}\right)\right| = \left|\arg\left(\frac{u}{v}\right)\right| \leqslant \arcsin(\sqrt{1-\lambda^2}) + \arcsin\lambda = \frac{\pi}{2}, \quad z \in \mathbb{D}$$

which implies that the function f is close-to-convex in $\mathbb{D}$. □

Example 3.2. Consider $g(z) = z/(1+\lambda z)$, where $0 \leqslant \lambda \leqslant 1$. Then $g \in \mathcal{S}^*$ and

$$|(z/g(z)) - 1| = |\lambda z| \leqslant \lambda, \quad z \in \mathbb{D}.$$

Theorem 3.3 shows that if $0 \leqslant \lambda \leqslant 1$, then every $f \in \mathcal{A}$ satisfying the condition

$$\left|f'(z)(1+\lambda z)^2 - 1\right| \leqslant \sqrt{1-\lambda^2}, \quad z \in \mathbb{D}$$

is close-to-convex in $\mathbb{D}$. Equivalently, we see that the function f defined by

$$f(z) = z\int_0^1 \frac{1+\alpha\omega(tz)}{(1+\lambda tz)^2}\,dt$$

belongs to $\mathscr{C}$ whenever $|\alpha| \leqslant \sqrt{1-\lambda^2}$ and $0 \leqslant \lambda \leqslant 1$, where $\omega(z)$ is any Schwarz function. In particular, the function

$$f(z) = z\int_0^1 \frac{1+\sqrt{1-\lambda^2}\,tz}{(1+\lambda tz)^2}\,dt$$

is univalent in $\mathbb{D}$ if $0 \leqslant \lambda \leqslant 1$. $\square$

Example 3.3. Consider the function $g(z) = z + \mu z^2$, where $0 < \mu \leqslant 1/2$. Then

$$|g''(z)| = 2\mu \leqslant 1$$

and therefore, $g \in \mathscr{S}^*$. As

$$\frac{z}{g(z)} - 1 = -\frac{\mu z}{1+\mu z}$$

and $w = -\mu z/(1+\mu z)$ maps the unit disk $\mathbb{D}$ conformally onto the disk

$$\left| w - \frac{\mu^2}{1-\mu^2} \right| < \frac{\mu}{1-\mu^2},$$

it follows that $|w| < \mu/(1+\mu)$. Now we let $\mu = \lambda/(1-\lambda)$ with $0 < \lambda \leqslant 1/3$. Then the starlike function g satisfies the condition

$$\left| \frac{z}{g(z)} - 1 \right| < \lambda, \quad z \in \mathbb{D}$$

and therefore, by Theorem 3.3, every $f \in \mathscr{A}$ such that

$$\left| f'(z) \frac{1}{\left(1+\frac{\lambda}{1-\lambda}z\right)^2} - 1 \right| < \sqrt{1-\lambda^2}, \quad z \in \mathbb{D}$$

belongs to $\mathscr{C}$. Equivalently, the function

$$f(z) = z\int_0^1 \frac{1+\sqrt{1-\lambda^2}\,\omega(tz)}{(1+(\lambda/(1-\lambda))tz)^2}\,dt$$

belongs to $\mathscr{C}$ whenever $0 < \lambda \leqslant 1/3$, and $\omega(z)$ is any Schwarz function.

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