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Univalence and starlikeness of certain transforms defined by convolution of analytic functions ☆

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Abstract

Let $\mathcal{U}(\lambda)$ denote the class of all analytic functions f in the unit disk Δ of the form $f(z) = z + a_2z^2 + \cdots$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant \lambda, \quad z \in \Delta.$$

In this paper we find conditions on λ and on $c \in \mathbb{C}$ with $\operatorname{Re} c \geqslant 0 \neq c$ such that for each $f \in \mathcal{U}(\lambda)$ satisfying $(z/f(z)) * F(1, c; c+1; z) \neq 0$ for all $z \in \Delta$ the transform

$$G(z) = G_f^c(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)}, \quad z \in \Delta,$$

is univalent or starlike. Here $F(a, b; c; z)$ denotes the Gauss hypergeometric function and $*$ denotes the convolution (or Hadamard product) of analytic functions on Δ.

Keywords: Univalent, starlike and convex functions

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1. Introduction

Let $\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{A}$ be the set of all functions analytic in Δ with the usual normalization $f(0) = 0 = f'(0) - 1$. Also, we let $\mathcal{S} = \{f \in \mathcal{A}\colon f \text{ is univalent in } \Delta\}$. If $f \in \mathcal{S}$ maps Δ onto a starlike domain (with respect to the origin), i.e. if $tw \in f(\Delta)$ whenever $t \in [0,1]$ and $w \in f(\Delta)$, then we say that f is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. A necessary and sufficient condition for $f \in \mathcal{A}$ to be starlike is the inequality [3,5]

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \Delta. \tag{1}$$

Let $\mathcal{U}(\lambda)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant \lambda, \quad z \in \Delta.$$

We set $\mathcal{U} = \mathcal{U}(1)$. We remark that from $f \in \mathcal{U}(\lambda)$ it follows that $f(z)/z \neq 0$ for $z \in \Delta$. It is well known that $\mathcal{U} \subsetneq \mathcal{S}$ (see [1,10]) and so, for $0 \leqslant \lambda \leqslant 1$, one has $\mathcal{U}(\lambda) \subsetneq \mathcal{S}$. In a recent paper [9, Corollary 1.1] the authors have obtained the largest $r \in (0,1]$ such that for each $f \in \mathcal{S}$ the function $z \mapsto r^{-1}f(rz)$ is included in $\mathcal{U}$. More precisely, the authors have proved that

$$\max\bigl\{r \in (0,1]\colon r^{-1}f(rz) \in \mathcal{U} \text{ for every } f \in \mathcal{S}\bigr\} = 1/\sqrt{2}. \tag{2}$$

For the proof of our results, we need the following lemmas.

Lemma 1. *(See [8].) If $f \in \mathcal{U}(\lambda)$, $a := |f''(0)|/2 \leqslant 1$ and $0 \leqslant \lambda \leqslant \frac{\sqrt{2-a^2}-a}{2}$, then $f \in \mathcal{S}^*$.*

Recently, Fournier and Ponnusamy [4] have indicated a proof for the sharpness part of Lemma 1 by stating that there exists a nonstarlike function $f \in \mathcal{U}$ such that with $a = |f''(0)|/2$ it holds that

$$0 < \frac{\sqrt{2-a^2}-a}{2} < \sup_{z\in\Delta}\left|f'(z)\left(\frac{z}{f(z)}\right)^2 - 1\right| \leqslant 1 - a.$$

A careful analysis of results in [4] implies that Lemma 1 is actually sharp (see also [15] for a detailed proof). For a general result, we refer to [13,14].

Lemma 2. *(See [12, Corollary 3.2].) If $f(z) = z + a_{n+1}z^{n+1} + \cdots$ $(n \geqslant 2)$ belongs to $\mathcal{U}(\lambda)$ and*

$$0 \leqslant \lambda \leqslant \frac{n-1}{\sqrt{(n-1)^2+1}},$$

then $f \in \mathcal{S}^*$.

We observe that for $n = 2$ (i.e. $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$), Lemma 2 gives Lemma 1.

Lemma 3. *Let $\phi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be a nonvanishing analytic function on Δ and let f be of the form*

$$f(z) = \frac{z}{\phi(z)} = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n}. \tag{3}$$

Then, we have the following:

(1) *If* $\sum_{n=2}^{\infty}(n-1)|b_n| \leqslant \lambda$, *then* $f \in \mathcal{U}(\lambda)$.
(2) *If* $\sum_{n=2}^{\infty}(n-1)|b_n| \leqslant 1-|b_1|$, *then* $f \in \mathcal{S}^*$.

The first part of Lemma 3 is from [7,8] whereas the second part is obtained from [16, Theorem 1]. At this place it is important to present the following example: Consider the function

$$f(z)=\frac{z}{1+ibz+(e^{2i\beta}/2)z^3}.$$

Then, for $|b| \leqslant 1/2$ and β a real number, we have (with $b_1=ib$, $b_2=0$, $b_3=e^{2i\beta}/2$ and $b_n=0$ for $n \geqslant 4$)

$$\operatorname{Re}\left(\frac{z}{f(z)}\right) > 1-|b|-\frac{1}{2} \geqslant 0 \quad \text{and} \quad \sum_{n=2}^{\infty}(n-1)|b_n|=1$$

and so, by Lemma 3(1), $f \in \mathcal{U} \subseteq \mathcal{S}$. On the other hand f is not in $\mathcal{S}^*$ when $0<b \leqslant 1/2$ and $0<\beta<\arctan(2b)$, because

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\bigg|_{z=1}=\frac{[\sin\beta-2b\cos\beta]\sin\beta}{|1+ib+(e^{2i\beta}/2)|^2}<0.$$

This example shows the sharpness of the condition in part (2) of Lemma 3.

2. Results

If f and g are analytic functions on Δ with $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$, then the convolution (Hadamard product) of f and g, denoted by $f*g$, is an analytic function on Δ given by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n, \quad z \in \Delta.$$

For $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a,b;c;z)*f(z):=\sum_{n=1}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_nz^n, \quad c \notin -\mathbb{N},\ z \in \Delta, \tag{4}$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_0=1$, $(a)_n:=a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$. Here $F(a,b;c;z)$ denotes the Gauss hypergeometric function which is analytic in Δ. As a special case of the Euler integral representation for the hypergeometric function, one has

$$F(1,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{1}{1-tz}t^{b-1}(1-t)^{c-b-1}\,dt, \quad z \in \Delta,\ \operatorname{Re}c>\operatorname{Re}b>0.$$

Using this representation we have, for $f \in \mathcal{A}$,

$$zF(1,c;c+1;z)*f(z)=z\left(F(1,c;c+1;z)*\frac{f(z)}{z}\right)$$

and therefore, we obtain the following form:

$$zF(1,c;c+1;z)*f(z)=zc\int_0^1\frac{f(tz)}{tz}t^{c-1}\,dt, \quad z \in \Delta,\ \operatorname{Re}c>0. \tag{5}$$

Now, we state and prove our results.

Theorem 1. *Let* $f \in \mathcal{U}(\lambda)$ *and* $c \in \mathbb{C}$ *with* $\operatorname{Re} c \geqslant 0 \neq c$ *such that*

$$\bigl(z/f(z)\bigr) * F(1, c; c+1; z) \neq 0 \quad \text{in } \Delta,$$

and $G = G_f^c$ *be the transform defined by*

$$G(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)}, \quad z \in \Delta. \tag{6}$$

Further, let A *be a nonnegative real number such that* $A = |\frac{c}{c+1}\frac{f''(0)}{2}| \leqslant 1$. *Then we have the following*:

(1) $G \in \mathcal{U}(\lambda|c|/|c+2|)$. *The result is sharp especially when* $|f''(0)/2| \leqslant 1 - \lambda$. *In particular,* $G \in \mathcal{U}$ *whenever* $0 < \lambda \leqslant |(c+2)/c|$.
(2) $G \in \mathcal{S}^*$ *whenever* $0 < \lambda \leqslant \frac{|c+2|}{2|c|}(\sqrt{2 - A^2} - A)$.
In particular, if $\lambda = 1$, $f''(0) = 0$ *and* $|c - 2| \leqslant 2\sqrt{2}$ *with* $\operatorname{Re} c \geqslant 0$, *then* $G \in \mathcal{S}^*$.

Proof. We consider the function

$$\frac{z}{G(z)} = \frac{z}{f(z)} * F(1, c; c+1; z), \quad z \in \Delta. \tag{7}$$

Differentiating $z/G(z)$ shows that

$$(c+1)\frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z) = c\frac{z}{G(z)} + z\left(\frac{z}{G(z)}\right)', \quad z \in \Delta. \tag{8}$$

Further, using the series expansion of $F(1, c; c+1; z)$ from (4), we have

$$F(1, c; c+1; z) = 1 + \sum_{n=1}^{\infty} \frac{(c)_n}{(c+1)_n} z^n = 1 + \sum_{n=1}^{\infty} \frac{c}{c+n} z^n, \quad z \in \Delta, \tag{9}$$

which yields

$$cF(1, c; c+1; z) + zF'(1, c; c+1; z) = \frac{c}{1-z}, \quad z \in \Delta,$$

from which in combination with (7) and (8), one obtains

$$(c+1)\frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z) = c\frac{z}{f(z)}, \quad z \in \Delta. \tag{10}$$

Now, we set

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z).$$

Then $p(z)$ is analytic on Δ (with $p(0) = 1$ and $p'(0) = 0$); for one has the relations (7) and, by (10),

$$p(z) = (c+1)\frac{z}{G(z)} - c\frac{z}{f(z)}, \quad z \in \Delta, \tag{11}$$

and $z \mapsto z/f(z)$ is analytic on Δ, as by assumption $f \in \mathcal{U}(\lambda)$ and so $f(z)/z \neq 0$ on Δ. From (8), (10) and (11) one then obtains that

$$\begin{aligned}
cp(z)+zp'(z) &= (c+1)c\frac{z}{G(z)}+(c+1)z\left(\frac{z}{G(z)}\right)'-c^2\frac{z}{f(z)}-cz\left(\frac{z}{f(z)}\right)' \\
&= c\left[(c+1)\frac{z}{f(z)}-c\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'\right] \\
&= c\left[\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'\right] \\
&= c\left(\frac{z}{f(z)}\right)^2 f'(z).
\end{aligned} \tag{12}$$

Now, as $f\in\mathcal{U}(\lambda)$, it follows that

$$\left|p(z)+\frac{1}{c}zp'(z)-1\right|<\lambda, \quad z\in\Delta, \tag{13}$$

and so (because $p'(0)=0$), from the work of Hallenbeck and Ruscheweyh [6] (see also [11]), we deduce that

$$\left|p(z)-1\right|\leqslant\frac{\lambda|c|}{|c+2|}|z|^2, \quad z\in\Delta.$$

The conclusion (1) follows and the bound $\lambda|c|/|c+2|$ is sharp. To prove the sharpness, we consider functions f in $\mathcal{U}(\lambda)$ of the form

$$f(z)=\frac{z}{1-a_2z+\lambda z^2}, \quad z\in\Delta,$$

where $a_2=f''(0)/2$ and $|a_2|\leqslant 1-\lambda$, so that $1-a_2z+\lambda z^2\neq 0$ for all $z\in\Delta$. Moreover, since $\operatorname{Re} c\geqslant 0$, it follows that $|c+2|>|c+1|>|c|$ and, therefore,

$$\left|1-a_2\frac{c}{c+1}z+\lambda\frac{c}{c+2}z^2\right|\neq 0$$

for all $z\in\Delta$, provided $|a_2|\leqslant 1-\lambda$. Then, by (6) and (9), a computation gives

$$G(z)=\frac{z}{1-a_2(c/(c+1))z+(\lambda c/(c+2))z^2}$$

which is analytic on Δ, $z/G(z)\neq 0$ on Δ and

$$\left(\frac{z}{G(z)}\right)^2G'(z)-1=-\frac{\lambda c}{c+2}z^2.$$

We have that $G\in\mathcal{U}(\lambda|c|/|c+2|)$.

The second part is a consequence of Lemma 1. In fact, it suffices to observe from the definition of $G(z)$ that

$$A:=\left|\frac{G''(0)}{2}\right|=\left|\frac{c}{c+1}\frac{f''(0)}{2}\right|.$$

Then, by Lemma 1, G is starlike whenever $A\leqslant 1$ and

$$0\leqslant\frac{\lambda|c|}{|c+2|}\leqslant\frac{\sqrt{2-A^2}-A}{2}$$

and the result follows from the last inequality. □

Remark. We recall first that if $|a_2| \leqslant 1-\lambda$, then it is known that [8]

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{1+|a_2|+\lambda} \geqslant \frac{1}{2} \quad \text{for } z \in \Delta. \tag{14}$$

Further, from the work of Ruscheweyh [17, Lemma 2], it follows that

$$\operatorname{Re} F(1,c;c+1;z) > \frac{1}{2}, \quad z \in \Delta,\ \operatorname{Re} c \geqslant 0. \tag{15}$$

From (14), it follows that $\operatorname{Re}(f(z)/z) > 0$, $z \in \Delta$. From this observation and (15), we obtain (using either the Herglotz representation formula for functions with positive real part or [18]) that

$$\operatorname{Re}\left(\frac{f(z)}{z} * F(1,c;c+1;z)\right) > 0, \quad z \in \Delta,\ \operatorname{Re} c \geqslant 0,$$

and so, in particular, that $(z/f(z)) * F(1,c;c+1;z) \neq 0$ for all $z \in \Delta$, $\operatorname{Re} c \geqslant 0$.

Remark. In case $\operatorname{Re} c > 0$, the formula (5) shows that the transform $G(z) = G_f^c(z)$ defined by (6) has a second representation in the form

$$G(z) = z\left(c \int_0^1 \frac{tz}{f(tz)} t^{c-1}\, dt\right)^{-1}, \quad z \in \Delta.$$

Using Lemma 2, Theorem 1 can be generalized as follows:

Theorem 2. *For a fixed $n \geqslant 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belong to $\mathcal{U}(\lambda)$ and let $c \in \mathbb{C}$ with* $\operatorname{Re} c \geqslant 0 \neq c$ *such that $(z/f(z)) * F(1,c;c+1;z) \neq 0$ in Δ, and $G = G_f^c$ be the transform defined by* (6). *Then we have the following*:

(1) *$G \in \mathcal{U}(\lambda|c|/|c+n|)$. In particular, $G \in \mathcal{U}$ whenever* $0 < \lambda \leqslant |(c+n)/c|$.
(2) *$G \in \mathcal{S}^*$ whenever* $0 < \lambda \leqslant \frac{|c+n|(n-1)}{|c|\sqrt{(n-1)^2+1}}$.

Proof. We note that

$$\frac{z}{f(z)} = \frac{1}{1+a_{n+1}z^n + \cdots} = 1 - a_{n+1}z^n + \cdots,$$

so that

$$\frac{z}{f(z)} * F(1,c;c+1;z) = 1 - a_{n+1}\left(\frac{c}{c+n}\right)z^n + \cdots.$$

Thus, G can be written in the form

$$G(z) = z + a_{n+1}\left(\frac{c}{c+n}\right)z^{n+1} + \cdots$$

and therefore, as in the proof of Theorem 1, the function p defined by

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z) = 1 + (n-1)a_{n+1}\left(\frac{c}{c+n}\right)z^n + \cdots$$

is analytic in Δ such that $p(0)=1$, $p'(0)=\cdots=p^{(n-1)}(0)=0$. As $f\in\mathcal{U}(\lambda)$, p satisfies (13). Consequently (see [6,11]),

$$\bigl|p(z)-1\bigr|\leqslant\frac{\lambda|c||z|^n}{|c+n|},\quad z\in\Delta,$$

and the proof of part (1) is complete. The second part is a consequence of Lemma 2. □

3. Sufficient conditions for functions in $\mathcal{U}$ or in $\mathcal{S}^*$

We recall that $\mathcal{U}\subsetneq\mathcal{S}$. Next we consider the following question: *Given a univalent function f, is it possible to generate functions in $\mathcal{U}$ or in $\mathcal{S}^*$?* Our next result actually provides a method of obtaining functions in $\mathcal{U}$.

Theorem 3. *Let $h(z)=1+\sum_{n=1}^{\infty}c_nz^n$ be an analytic function on Δ and $a_2\in\mathbb{C}$ such that*

$$|c_1a_2|+\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\leqslant 1\quad\text{and}\quad\lambda:=\left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{1/2}<+\infty.\tag{16}$$

Then for every function $f\in\mathcal{S}$ with $f''(0)/2=a_2$ the function H_f defined by

$$\frac{z}{H_f(z)}=\left(\frac{z}{f(z)}\right)*h(z)$$

belongs to $\mathcal{U}(\lambda)$, and thus to $\mathcal{S}$ if $\lambda\leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda\leqslant 1-|a_2c_1|$.

Proof. Let $f\in\mathcal{S}$ and be of the form (3). Then $a_2=f''(0)/2=-b_1$,

$$\frac{z}{H_f(z)}=\left(\frac{z}{f(z)}\right)*h(z)=1+\sum_{n=1}^{\infty}b_nc_nz^n$$

and from the well-known Area Theorem [5, Theorem 11, p. 193, Vol. 2] we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2\leqslant 1.\tag{17}$$

Now, by the triangle inequality, we see for all $z\in\Delta$ that

$$\begin{aligned}\left|\frac{z}{H_f(z)}\right|&\geqslant 1-|c_1b_1||z|-\sum_{n=2}^{\infty}\bigl(\sqrt{n-1}|b_n|\bigr)\left(\frac{|c_n|}{\sqrt{n-1}}\right)|z|^n\\&\geqslant 1-|c_1a_2||z|-|z|^2\sum_{n=2}^{\infty}\bigl(\sqrt{n-1}|b_n|\bigr)\left(\frac{|c_n|}{\sqrt{n-1}}\right)\\&\geqslant 1-|c_1a_2||z|-|z|^2\left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\\&\qquad\text{(by Cauchy–Schwarz inequality)}\\&\geqslant 1-|c_1a_2|-\left(\sum_{n=2}^{\infty}\frac{|c_n|^2}{n-1}\right)^{1/2}\quad\text{by (17)}\\&\geqslant 0\quad\text{by (16).}\end{aligned}$$

Using this and the first inequality in (16), we obtain that $z/H_f(z) \neq 0$ in Δ. Next we find that

$$\sum_{n=2}^{\infty}(n-1)|c_nb_n| = \sum_{n=2}^{\infty}\bigl(\sqrt{n-1}|b_n|\bigr)\bigl(\sqrt{n-1}|c_n|\bigr) \leqslant \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{1/2}\left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{1/2} \leqslant \lambda \quad \text{by (17) and (16).}$$

Thus, $H_f \in \mathcal{U}(\lambda)$ by Lemma 3(1), and, in particular, $H_f \in \mathcal{U} \subseteq \mathcal{S}$ if $\lambda \leqslant 1$. By Lemma 3(2), we obtain the last part of the conclusion. □

Example 1. Choose $h(z) = 1/(1-az)$ with $|a| = r < 1$. Then, according to (16), r has to satisfy the condition

$$|a_2|r + r\bigl(\log\bigl(1/\bigl(1-r^2\bigr)\bigr)\bigr)^{1/2} \leqslant 1 \quad \text{and} \quad \lambda = r^2/\bigl(1-r^2\bigr).$$

Then for each function $f \in \mathcal{S}$ with $f''(0)/2 = a_2$ the function $a^{-1}f(az)$ belongs to $\mathcal{U}(\lambda)$ and thus to $\mathcal{S}$ if $\lambda \leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda \leqslant 1 - |a_2|r$. In particular, it is a simple exercise to see that

$$f \in \mathcal{S} \quad \text{with } f''(0) = 0 \quad \Rightarrow \quad a^{-1}f(az) \in \mathcal{U} \cap \mathcal{S}^*$$

whenever $0 < |a| = r \leqslant 1/\sqrt{2}$. At this place it is interesting to compare with (2).

Example 2. Choose $h(z) = 1/(1-az^2)$ with $|a| = r < 1$. Then, by (16), r has to satisfy the condition

$$\frac{r}{2}\log\left(\frac{1+r}{1-r}\right) \leqslant 1 \quad \text{and} \quad \lambda = \frac{r\sqrt{1+r^2}}{1-r^2}.$$

Therefore, if $f \in \mathcal{S}$ then the function $z/((z/f(z)) * h(z))$ belongs to $\mathcal{U}(\lambda)$ and thus to $\mathcal{S}^*$ if $\lambda \leqslant 1$ (since $h'(0) = 0$). In fact, it is a simple exercise to see that the second condition $\lambda \leqslant 1$ is equivalent to $r \leqslant 1/\sqrt{3}$, while the first condition is equivalent to the inequality

$$g(r) = (1-r)e^{2/r} - 1 - r \geqslant 0$$

which holds if $r \leqslant 1/\sqrt{3}$. Thus, if ω and ω' denote the two square roots of a and if $f \in \mathcal{S}$, then the function H_f defined by

$$\frac{z}{H_f(z)} = \frac{z}{f(z)} * h(z) = \frac{1}{2}\left(\frac{\omega z}{f(\omega z)} + \frac{\omega' z}{f(\omega' z)}\right)$$

belongs to $\mathcal{S}^*$ for $r \leqslant 1/\sqrt{3}$.

Corollary 1. *Let $f \in \mathcal{S}$ be of the form* (3) *with $a_2 = f''(0)/2$, and*

$$h(z) = 1 + c_1z + a\sum_{n=2}^{\infty}\frac{1}{(n+1)\sqrt{n-1}}z^n$$

for some complex constant a, such that

$$|c_1a_2| + |a|\sqrt{\frac{\pi^2}{12} - \frac{11}{16}} \leqslant 1 \quad \textit{and} \quad \lambda = |a|\sqrt{\frac{\pi^2}{6} - \frac{5}{4}}.$$

Then the function H_f defined by $z/H_f(z) = (z/f(z)) * h(z)$ belongs to $\mathcal{U}(\lambda)$, and thus to $\mathcal{S}$ if $\lambda \leqslant 1$, and even to $\mathcal{S}^*$ if $\lambda \leqslant 1 - |c_1 a_2|$.

Proof. Set $c_n = a/((n+1)\sqrt{n-1})$ for all $n \geqslant 2$. The condition (16) takes the form

$$|c_1 a_2| + |a| \left(\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2} \right)^{1/2} \leqslant 1 \quad \text{and} \quad \lambda = |a| \left(\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} \right)^{1/2}.$$

Recall that

$$\sum_{n=2}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - \frac{5}{4}.$$

Now, if we write

$$\frac{1}{(n^2-1)^2} = \frac{1}{4} \left[\frac{1}{(n-1)^2} + \frac{1}{(n+1)^2} - \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right],$$

then it is a simple exercise to see that

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^2} = \frac{1}{4} \left[2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 1 - \frac{1}{4} - \frac{3}{2} \right] = \frac{\pi^2}{12} - \frac{11}{16}.$$

The conclusion follows from Theorem 3. □

Finally, it would be appropriate to recall the recent result of the authors in [2] in which a number of interesting applications are also derived.

Theorem 4. *(See [2, Theorem 3.9].) Let $f, g \in \mathcal{S}$ with the representations*

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \qquad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

If

$$\Phi(z) = \frac{z}{f(z)} * \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n \neq 0$$

for every $z \in \Delta$, then $F(z) = \frac{z}{\Phi(z)} \in \mathcal{U}$, and, in particular, F is univalent in Δ.

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