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Radius of Univalence of Certain Combination of Univalent and Analytic Functions

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Abstract. Let $\mathcal{A}$ denote the family of all analytic functions f in the unit disk $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. Define $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$, $\mathcal{U} = \{f \in \mathcal{A} : \left|f'(z)(z/f(z))^2 - 1\right| < 1 \text{ for } z \in \mathbb{D}\}$, and $\mathcal{P}(1/2) = \{f \in \mathcal{A} : \mathrm{Re}\,(f(z)/z) > 1/2 \text{ for } z \in \mathbb{D}\}$. In this paper, we determine the radius of univalency of $F(z) = zf(z)/g(z)$ whenever $f \in \mathcal{S}$ or $\mathcal{U}$, and $g \in \mathcal{S}$ or $\mathcal{P}(1/2)$. Based on our investigations, we conjecture that F is univalent in the disk $|z| < 1/3$ whenever $f \in \mathcal{S}$ and $g \in \mathcal{P}(1/2)$. We also conjecture that F is univalent in the disk $|z| < \sqrt{5} - 2$ whenever both f and g are in $\mathcal{S}$.

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1. Introduction

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let

$$\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}.$$

We say that a function $f \in \mathcal{S}$ is starlike if $f(\mathbb{D})$ is a domain with the property that the segment $[0, w] := \{tw \mid 0 \le t \le 1\} \subset f(\mathbb{D})$ for each $w \in f(\mathbb{D})$. Let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition ([5, 6])

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| < 1 \quad \text{for } z \in \mathbb{D}.$$

Functions in $\mathcal{U}$ are known to be univalent in $\mathbb{D}$, but functions in $\mathcal{S}$ are not necessarily belong to the class $\mathcal{U}$. Moreover, functions in $\mathcal{U}$ are not necessarily starlike (see [2]). Later, several generalizations of the class $\mathcal{U}$ were investigated (see eg. [9]). It would be

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interesting to consider the harmonic analog of this class and other related classes. For some aspects of planar harmonic mappings, we refer to [1] and the references therein. On the other hand, the class $\mathscr{U}$ has many properties in common with the classical subclasses of $\mathscr{S}$, eg. Koebe function $z/(1-z)^2$ belongs to both the classes. Set

$$\mathscr{U}_2 = \{f \in \mathscr{U} : f''(0) = 0\}.$$

It is known that each function in $\mathscr{U}_2$ is included in the class $\mathscr{P}(1/2)$, where

$$\mathscr{P}(1/2) = \{f \in \mathscr{A} : \mathrm{Re}\,(f(z)/z) > 1/2 \text{ for } z \in \mathbb{D}\}.$$

We remark that $\mathscr{K} \subset \mathscr{P}(1/2)$, where $\mathscr{K}$ denotes the class of all functions $f \in \mathscr{S}$ that are convex, i.e. $f(\mathbb{D})$ is a convex domain.

Throughout the paper the function F is defined by

$$F(z) = \frac{zf(z)}{g(z)}, \tag{1.1}$$

where f will be either in $\mathscr{S}$ or in $\mathscr{U}$, and $g \in \mathscr{A}$ will be suitably chosen so that $g(z)/z$ is non-vanishing in the unit disk $\mathbb{D}$. One of the aims of this article is to find $r_0 \in (0,1]$ such that the function F defined by (1.1) is univalent in the disk $|z| < r_0$. In each case, largest value of the number r_0 satisfying the desired conclusion is an open question.

The proofs of the results rely on recent results of the first two authors, and a careful use of power series method. It seems that there exists no other method through which one can obtain the results of this paper. Finally, we state two conjectures concerning the radius of univalency of F.

We now state our main results.

Theorem 1.1. *Let $f \in \mathscr{U}$ and $g \in \mathscr{P}(1/2)$. Then the function F defined by* (1.1) *is univalent in the disk $|z| < r_0$, where*

$$r_0 = \frac{-\sqrt{6} + \sqrt{18 + \pi\sqrt{6}}}{2\sqrt{6} + \pi} \approx 0.325793.$$

Theorem 1.2. *Let $f \in \mathscr{U}$ and $g \in \mathscr{S}$. Then the function F defined by* (1.1) *is univalent in the disk $|z| < r_0$, where*

$$r_0 = \frac{2}{3 + \sqrt{25 + 8\pi/\sqrt{6}}} \approx 0.223763.$$

Theorem 1.3. *Let $f \in \mathscr{S}$ and $g \in \mathscr{P}(1/2)$. Then the function F defined by* (1.1) *is univalent in the disk $|z| < r_0$, where $r_0 = 0.315449\cdots$ is the smallest root of the equation*

$$4r^5 + 6r^4 - 8r^3 - 4r^2 + 5r - 1 = 0,$$

that lies in the interval $(0, (\sqrt{3}-1)/2)$.

It is now appropriate to consider

$$f(z) = \frac{z}{(1+z)^2} \text{ and } g(z) = \frac{z}{1-z}.$$

Then

$$\frac{z}{F(z)} = \frac{g(z)}{f(z)} = \frac{(1+z)^2}{1-z} = (1+2z+z^2)(1+z+z^2+\cdots)$$

and therefore,

$$\frac{z}{F(z)} = 1 + 3z + 4\sum_{n=2}^{\infty} z^n.$$

We see that the Taylor coefficients of $z/F(z)$ are all positive and

$$\frac{rz}{F(rz)} = 1 + 3rz + 4\sum_{n=2}^{\infty} r^n z^n.$$

According to Lemma 2.2 (below), G defined by $G(z) = r^{-1}F(rz)$ is univalent in $\mathbb{D}$ if and only if

$$4\sum_{n=2}^{\infty}(n-1)r^n = \frac{4r^2}{(1-r)^2} \le 1, \text{ i.e. } (3r-1)(r+1) \le 0.$$

This gives $r \in (0, 1/3]$ and thus, F is univalent in the disk $|z| < 1/3$. This example leads to

Conjecture 1.1. *Suppose that $f \in \mathscr{S}$ and $g \in \mathscr{P}(1/2)$ (or more generally, $g \in \mathscr{A}$ with $g(z)/z \neq 0$ with $|g^{(k)}(0)| \le k!$ for $k = 2, 3, 4, \dots$). Then the function F defined by* (1.1) *is univalent in the disk $|z| < 1/3$.*

We now state our final result.

Theorem 1.4. *Let $f, g \in \mathscr{S}$. Then the function F defined by* (1.1) *is univalent in the disk $|z| < r_0$, where $r_0 = 0.21734\cdots$ is the root of the equation*

$$20r^5 + 16r^4 - 23r^3 - 7r^2 + 7r - 1 = 0$$

in the interval $(0, 1)$.

In order to state our next conjecture, we consider

$$f(z) = \frac{z}{(1+z)^2},\ g(z) = \frac{z}{(1-z)^2} \text{ and } G(z) = r^{-1}F(rz).$$

Then, we have

$$\frac{z}{F(z)} = \frac{g(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^2 = (1 + 2z + z^2)(1 + 2z + 3z^2 + \cdots)$$

so that

$$\frac{z}{F(z)} = 1 + 4z + 4\sum_{n=2}^{\infty} nz^n.$$

Again, the Taylor coefficients of $z/F(z)$ are all positive and

$$\frac{rz}{F(rz)} = 1 + 4rz + 4\sum_{n=2}^{\infty} nr^n z^n.$$

According to Lemma 2.2, G is univalent in $\mathbb{D}$ if and only if

$$4\sum_{n=2}^{\infty}(n-1)nr^n = \frac{8r^2}{(1-r)^3} \le 1.$$

The last inequality is equivalent to

$$r^3 + 5r^2 + 3r - 1 = (r+1)(r^2 + 4r - 1) \le 0$$

which gives $0 < r \le r_0 = \sqrt{5}-2 \approx 0.236068$, where r_0 is the unique positive root of the equation $r^2+4r-1=0$ in the interval $(0,1)$. Thus, F is univalent in the disk $|z| < \sqrt{5}-2$. This example leads to

Conjecture 1.2. *Suppose that $f,g \in \mathcal{S}$ (more generally, $g \in \mathcal{A}$ with $g(z)/z \neq 0$ and $|g^{(k)}(0)| \le k(k!)$ for $k = 2,3,4,\dots$). Then the function F defined by* (1.1) *is univalent in the disk $|z| < \sqrt{5}-2$.*

It is worth reminding that the sharpness of the radius r_0 in Theorems 1.1–1.4 is open. Also, it would be interesting to investigate similar problems for holomorphic functions of several variables (see eg. [4]).

2. Preliminary lemmas

For the proofs of our results, we need the following lemmas.

Lemma 2.1. *Let $\phi(z) = 1+\sum_{n=1}^{\infty} b_n z^n$ be a non-vanishing analytic function on $\mathbb{D}$ and let f be of the form*

$$f(z) = \frac{z}{\phi(z)} = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}.$$

Then, we have the following:

- **(a)** *If $\sum_{n=2}^{\infty}(n-1)|b_n| \le 1$, then $f \in \mathcal{U}$.*
- **(b)** *If $\sum_{n=2}^{\infty}(n-1)|b_n| \le 1-|b_1|$, then $f \in \mathcal{S}^*$.*
- **(c)** *If $f \in \mathcal{U}$, then $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \le 1$.*

The conclusion **(a)** in Lemma 2.1 is from [5, 8] whereas the **(b)** is due to Reade *et al.* [10, Theorem 1]. Finally, if $f \in \mathcal{U}$, then we have

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^2 - 1 \right| = \left| -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 \right| = \left| \sum_{n=2}^{\infty}(n-1)b_n z^n \right| \le 1$$

and so the inequality

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \le 1$$

follows from Prawitz'/Parseval's theorem. We remark that Prawitz' theorem is an immediate consequence of Gronwall's area theorem. Thus, **(c)** follows.

Next we recall the following result due to Obradović and Ponnusamy [7].

Lemma 2.2. *Let $f \in \mathcal{A}$ have the form*

$$\frac{z}{f(z)} = 1+b_1 z+b_2 z^2+\cdots \quad \text{with } b_n \ge 0 \text{ for all } n \ge 2$$

and for all z in a neighborhood of $z=0$. Then we have the following equivalence:

- (a) $f \in \mathcal{S}$
- (b) $\frac{f(z)f'(z)}{z} \neq 0$ *for* $z \in \mathbb{D}$
- (c) $\sum_{n=2}^{\infty}(n-1)b_n \le 1$
- (d) $f \in \mathcal{U}$.

3. Proofs

Consider the function F defined by (1.1), where $f \in \mathcal{S}$ and $g \in \mathcal{A}$ such that $g(z)/z \neq 0$ in $\mathbb{D}$. Because $f \in \mathcal{S}$, we can express $z/f(z)$ as

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots .$$

We may let

$$g(z) = z + c_2 z^2 + c_3 z^3 + \cdots .$$

In view (1.1), we may rewrite F as

$$\frac{z}{F(z)} = \left(1 + b_1 z + b_2 z^2 + \cdots\right)\left(1 + c_2 z + c_3 z^2 + \cdots\right) = 1 + \sum_{n=1}^{\infty} B_n z^n \quad \text{(say)},$$

where

$$B_n = \sum_{k=0}^{n} b_k c_{n-k+1} \quad (b_0 = c_1 = 1)$$

and by assumption $z/F(z) \neq 0$ in $\mathbb{D}$. Also, we observe that

$$|B_n| \leq |c_{n+1}| + |b_1|\,|c_n| + \sum_{k=2}^{n} |b_k|\,|c_{n-k+1}|. \tag{3.1}$$

Now, for $r \in (0,1]$, we define G by

$$G(z) = r^{-1} F(rz) \tag{3.2}$$

and consider the function

$$\frac{rz}{F(rz)} = 1 + \sum_{n=1}^{\infty} B_n r^n z^n.$$

In order to prove that F is univalent in $|z| < r_0$, it suffices to show that $G \in \mathcal{U}$ for $0 < r \leq r_0$. According to Lemma 2.1**(a)**, $G \in \mathcal{U}$ for $0 < r \leq r_0$ if we can show

$$S(r) := \sum_{n=2}^{\infty} (n-1)|B_n| r^n \leq 1 \tag{3.3}$$

for $0 < r \leq r_0$. The estimation of the left hand side of the relation (3.3) depends on the condition on the coefficients b_n and c_n. With this setting, we now consider a number of special cases that concern our results.

Proof of Theorem 1.1. Let $f \in \mathcal{U}$ and $g \in \mathcal{P}(1/2)$. Then the Taylor coefficients c_k of g satisfy the condition

$$|c_k| \leq 1 \ \text{ for all } k = 2, 3, 4, \ldots.$$

By Lemma 2.1**(c)**, we obtain that $|b_1| = |f''(0)/2| \leq 2$ (by the Bieberbach inequality for the second coefficients of $f \in \mathcal{S}$) and

$$\sum_{k=2}^{\infty} (k-1)^2 |b_k|^2 \leq 1. \tag{3.4}$$

Using the last two inequalities, it follows from (3.1) and the Cauchy-Schwarz inequality that

$$|B_n| \leq 3 + \sum_{k=2}^{n} |b_k| \leq 3 + \left(\sum_{k=2}^{n} (k-1)^2 |b_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=2}^{n} \frac{1}{(k-1)^2}\right)^{\frac{1}{2}} \leq 3 + \frac{\pi}{\sqrt{6}}.$$

Using the bound for B_n, the sum $S(r)$ defined by (3.3) yields that

$$S(r) \le A\sum_{n=2}^{\infty}(n-1)r^n = A\frac{r^2}{(1-r)^2}, \quad A = 3+\frac{\pi}{\sqrt{6}}.$$

Finally, $S(r) \le 1$ is satisfied if $Ar^2 \le (1-r)^2$. This gives $0 < r \le r_0$, where

$$r_0 = \frac{\sqrt{A}-1}{A-1} = \frac{-\sqrt{6}+\sqrt{18+\pi\sqrt{6}}}{2\sqrt{6}+\pi} \approx 0.325793$$

is the unique positive root of equation $(A-1)r^2+2r-1=0$. Thus, $S(r) \le 1$ whenever $0 < r \le r_0$ and therefore, G_r defined by (3.2) belongs to $\mathscr{U}$ for $0 < r \le r_0$. In particular, F defined by (1.1) is univalent in the disk $|z| < r_0$. ∎

Remark 3.1. It is well-known that the class $\mathscr{S}^*(1/2)$ of starlike functions g of order $1/2$ (and hence the class of convex functions) is included in $\mathscr{P}(1/2)$. Moreover, if $g \in \mathscr{U}$ with $g''(0) = 0$, then it is known that $g \in \mathscr{P}(1/2)$ (see for example [8]). Finally, we remark that the conclusion of Theorem 1.1 continues to hold if we relax the condition $g \in \mathscr{P}(1/2)$ by $g \in \mathscr{A}$ with $g(z)/z \neq 0$ in $\mathbb{D}$ and $|g^{(k)}(0)| \le k!$ for $k = 2,3,4,\ldots$. This fact is clear from the proof of Theorem 1.1.

Proof of Theorem 1.2. Let $f \in \mathscr{U}$ and $g \in \mathscr{S}$. Then, by the de Branges theorem, the Taylor coefficients c_k of g satisfy the condition

$$|c_k| \le k \ \text{ for all } k = 2,3,4,\ldots.$$

Using the last inequality and the inequality (3.4) which holds as $f \in \mathscr{U}$, it follows from (3.1) and the Cauchy-Schwarz inequality that (since $|b_1| \le 2$)

$$\begin{aligned}
|B_n| &\le |c_{n+1}| + 2|c_n| + \sum_{k=2}^{n}|b_k|\,|c_{n-k+1}| \\
&\le (n+1) + 2n + \left(\sum_{k=2}^{n}(k-1)^2|b_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=2}^{n}\frac{(n-k+1)^2}{(k-1)^2}\right)^{\frac{1}{2}} \\
&\le 3n+1+\sqrt{C_n},
\end{aligned}$$

where

$$C_n := \sum_{k=2}^{n}\frac{(n-k+1)^2}{(k-1)^2} = \sum_{k=1}^{n-1}\frac{(n-k)^2}{k^2} = n^2\sum_{k=1}^{n-1}\frac{1}{k^2} + (n-1) - 2n\sum_{k=1}^{n-1}\frac{1}{k}.$$

The fact that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6} \ \text{ and } \ \sum_{k=1}^{n-1}\frac{1}{k} \ge 1$$

gives the crude estimate

$$C_n < \frac{\pi^2}{6}n^2 + (n-1) - 2n = \frac{\pi^2}{6}n^2 - (n+1)$$

so that

$$C_n < \frac{\pi^2}{6}n^2 \ \text{ and } \ |B_n| < 3n+1+\frac{\pi}{\sqrt{6}}n = Bn+1, \quad B = 3+\frac{\pi}{\sqrt{6}}.$$

Thus, in this case, the sum $S(r)$ given by (3.3) leads to

$$S(r) < \sum_{n=2}^{\infty}(n-1)(Bn+1)r^n = B\sum_{n=2}^{\infty}(n-1)nr^n + \sum_{n=2}^{\infty}(n-1)r^n = \frac{2Br^2}{(1-r)^3} + \frac{r^2}{(1-r)^2}$$

and therefore, the inequality (3.3) is satisfied if

$$\frac{2Br^2}{(1-r)^3} + \frac{r^2}{(1-r)^2} \leq 1, \text{ i.e. } 2(B-1)r^2 + 3r - 1 \leq 0.$$

This gives the condition $0 < r \leq r_0$, where

$$r_0 = \frac{-3+\sqrt{1+8B}}{4(B-1)} = \frac{2}{3+\sqrt{1+8B}} = \frac{2}{3+\sqrt{25+8\pi/\sqrt{6}}} \approx 0.223763$$

is the unique positive root of the equation $2(B-1)r^2 + 3r - 1 = 0$ in the interval $(0,1)$. Thus, G defined by (3.2) belongs to $\mathscr{U}$ for $0 < r \leq r_0$. In particular, F defined by (1.1) is univalent in the disk $|z| < r_0$. ∎

Remark 3.2. Clearly, the condition on $g \in \mathscr{S}$ in Theorem 1.2 can be replaced by $g \in \mathscr{A}$ with $g(z)/z \neq 0$ in $\mathbb{D}$ and $|g^{(k)}(0)| \leq k(k!)$ for $k = 2,3,4,\ldots$.

Proof of Theorem 1.3. Let $f \in \mathscr{S}$ and $g \in \mathscr{P}(1/2)$. Then f has the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

with $|b_1| \leq 2$. Moreover, from the well-known Area Theorem [3, Theorem 11 on p.193 of Vol. 2] we have

$$\sum_{k=2}^{\infty}(k-1)|b_k|^2 \leq 1,$$

and, because $g \in \mathscr{P}(1/2)$, it follows that $|c_k| \leq 1$ for $k = 2,3,4,\ldots$. As in the proof of Theorem 1.1, using the last three coefficient inequalities, it follows from (3.1) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
|B_n| &\leq |c_{n+1}| + 2|c_n| + \sum_{k=2}^{n}|b_k||c_{n-k+1}| \\
&\leq 3 + \sum_{k=2}^{n}|b_k| \\
&\leq 3 + \left(\sum_{k=2}^{n}(k-1)|b_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=2}^{n}\frac{1}{k-1}\right)^{\frac{1}{2}} \\
&\leq 3 + \sqrt{D_n},
\end{aligned}$$

where

$$D_n = \sum_{k=1}^{n-1}\frac{1}{k}.$$

Next, in view of the inequalities

$$\frac{1}{k+1} < \int_k^{k+1}\frac{dx}{x} = \log(k+1) - \log k < \frac{1}{k},$$

it follows that for $n \geq 2$,

$$D_n < \log n + \frac{n-1}{n} < \log n + 1 < n$$

and so, $|B_n| < 3 + \sqrt{n}$. Thus, in this case, the sum $S(r)$ defined by (3.3) gives

$$S(r) < \sum_{n=2}^{\infty} (n-1)\left(3+\sqrt{n}\right) r^n = \frac{3r^2}{(1-r)^2} + \sum_{n=2}^{\infty} (n-1)\sqrt{n} r^n.$$

Next, we observe that

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-1)\sqrt{n} r^n &\leq \left(\sum_{n=2}^{\infty} (n-1) r^n\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n-1) n r^n\right)^{\frac{1}{2}} \\
&= r\left(\frac{1}{(1-r)^2}\right)^{\frac{1}{2}} r\left(\frac{2}{(1-r)^3}\right)^{\frac{1}{2}} \\
&= \frac{r^2}{(1-r)^2}\sqrt{\frac{2}{1-r}}
\end{aligned}$$

and therefore, the inequality (3.3) is satisfied if

$$\phi(r) := \frac{3r^2}{(1-r)^2} + \frac{r^2}{(1-r)^2}\left(\frac{2}{1-r}\right)^{1/2} - 1 \leq 0.$$

This gives the condition $0 < r \leq r_0$, where r_0 is the smallest root of the equation $\phi(r) = 0$ in the interval $(0,1)$. If we simplify the last equation, we see that this is equivalent to

$$4r^5 + 6r^4 - 8r^3 - 4r^2 + 5r - 1 = 0,$$

and we obtain that the smallest root $r_0 \approx 0.315449$ lies in the interval $(0, (\sqrt{3}-1)/2)$. As before, the above discussion completes the proof of Theorem 1.3. ∎

Remark 3.3. Clearly, the conclusion of Theorem 1.3 continues to hold if the condition on g in Theorem 1.3 can be replaced by $g \in \mathscr{A}$ with $g(z)/z \neq 0$ in $\mathbb{D}$ and $|g^{(k)}(0)| \leq k!$ for $k = 2,3,4,\ldots$.

Proof of Theorem 1.4. Let $f, g \in \mathscr{S}$. Then, the de Branges theorem applied to g gives $|c_k| \leq k$ for $k = 2,3,4,\ldots$ and so by the Cauchy-Schwarz inequality, (3.1) gives that

$$\begin{aligned}
|B_n| &\leq 3n + 1 + \sum_{k=2}^{n} |b_k|(n-k+1) \\
&\leq 3n + 1 + \left(\sum_{k=2}^{n} (k-1)|b_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=2}^{n} \frac{(n-(k-1))^2}{k-1}\right)^{\frac{1}{2}} \\
&\leq 3n + 1 + \left(\sum_{k=2}^{n} \frac{(n-(k-1))^2}{k-1}\right)^{\frac{1}{2}} \quad \text{as } f \in \mathscr{S} \\
&= 3n + 1 + \sqrt{E_n}
\end{aligned}$$

where E_n may be written as

$$E_n = \sum_{k=1}^{n-1} \frac{(n-k)^2}{k}$$

$$
\begin{aligned}
&= n^2 \sum_{k=1}^{n-1} \frac{1}{k} - 2(n-1)n + \frac{(n-1)n}{2} \\
&= n^2 \sum_{k=1}^{n-1} \frac{1}{k} - \frac{3(n-1)n}{2} \\
&< n^2 \left(\log n + \frac{n-1}{n} \right) - \frac{3(n-1)n}{2} \\
&= n^2 \log n - \frac{(n-1)n}{2} < n^2 \log n < n^3.
\end{aligned}
$$

It follows that $|B_n| < 3n + 1 + n\sqrt{n}$ and the sum $S(r)$ defined by (3.3) takes the form

$$
\begin{aligned}
S(r) &< \sum_{n=2}^{\infty} (n-1) \left(3n + 1 + n\sqrt{n}\right) r^n \\
&\le 3 \sum_{n=2}^{\infty} (n-1)nr^n + \sum_{n=2}^{\infty} (n-1)r^n + \sum_{n=2}^{\infty} (n-1)n\sqrt{n}r^n \\
&\le \frac{6r^2}{(1-r)^3} + \frac{r^2}{(1-r)^2} + \left(\sum_{n=2}^{\infty} (n-1)nr^n \right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n-1)n^2 r^n \right)^{\frac{1}{2}} \\
&= \frac{6r^2}{(1-r)^3} + \frac{r^2}{(1-r)^2} + \frac{2r^2}{(1-r)^3} \sqrt{\frac{2+r}{1-r}} \\
&=: \psi(r), \quad \text{say} .
\end{aligned}
$$

Now, the inequality (3.3), namely, $S(r) \le 1$, is satisfied if $\psi(r) \le 1$. This gives $0 < r \le r_0$, where r_0 is the smallest root of the equation $\psi(r) - 1 = 0$, or equivalently

$$20r^5 + 16r^4 - 23r^3 - 7r^2 + 7r - 1 = 0,$$

in the interval $(0,1)$. We see that $r_0 = 0.21734$ and with this value of r_0, we complete the proof of Theorem 1.4. ∎

Remark 3.4. Again we remark that the condition $g \in \mathscr{S}$ in Theorem 1.4 may be replaced by $g \in \mathscr{A}$ with $g(z)/z \neq 0$ in $\mathbb{D}$ with $|c_k| \le k$ for $k = 2, 3, 4, \ldots$. The conclusion of Theorem 1.4 continues to hold under this weaker hypothesis.

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