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On the class $\mathcal{U}$

M. Obradović and S. Ponnusamy

Abstract. In this mini survey article, we present important properties of the class $\mathcal{U}$ of analytic functions f in the unit disk $|z| < 1$ which satisfy the condition

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1, \quad |z| < 1.$$

Our special emphasis is to list down few important and basic results such as characterization and necessary and sufficient coefficient conditions for functions to be in $\mathcal{U}$.

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1. Introduction and preliminaries about $\mathcal{U}$

Let $\mathcal{A}$ denote the class of all functions f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, with the normalization $f(0) = 0$ and $f'(0) = 1$. The article concerns the class $\mathcal{U}$ of all functions $f \in \mathcal{A}$ satisfying the condition

$$|U_f(z)| < 1, \quad z \in \mathbb{D},$$

where

$$U_f(z) = \left(\frac{z}{f(z)} \right)^2 f'(z) - 1. \tag{1.1}$$

According to Aksentév's theorem [1] (see also [16]) each functions in $\mathcal{U}$ belongs to $\mathcal{S}$. Here $\mathcal{S}$ denotes the class of all normalized univalent analytic functions in $\mathbb{D}$ which is indeed the main object in the theory of univalent functions. We observe that mappings $f \in \mathcal{S}$ can be associated with the mappings $F \in \Sigma$, namely univalent functions F of the form,

$$F(\zeta) = \zeta + \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad |\zeta| > 1,$$

which satisfies the condition $F(\zeta) \neq 0$ for $|\zeta| > 1$, by the correspondence

$$F(\zeta) = \frac{1}{f(1/\zeta)}, \quad |\zeta| > 1.$$

Using the change of variable $\zeta = 1/z$, the association $f(z) = 1/F(1/z)$ quickly yields the formula

$$F'(\zeta) - 1 = U_f(z),$$

where U_f is defined by (1.1). Some facts about the class $\mathcal{U}$ may now be recalled. Each function in

$$\mathcal{S}_{\mathbb{Z}} = \left\{ z,\ \frac{z}{(1 \pm z)^2},\ \frac{z}{1 \pm z},\ \frac{z}{1 \pm z^2},\ \frac{z}{1 \pm z + z^2} \right\}$$

belongs to $\mathcal{U}$. Also, it is well-known that functions in $\mathcal{S}_{\mathbb{Z}}$ are the only functions in $\mathcal{S}$ having integral coefficients in the power series expansions of $f \in \mathcal{S}$ (see [5]). From the geometric characterization of starlike functions (with respect to the origin) , it is a simple exercise to see that $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*$. Here $\mathcal{S}^* \subset \mathcal{S}$ denotes the class of all starlike (univalent) functions in $\mathbb{D}$ and every $f \in \mathcal{S}^*$ is characterized by the inequality [3, 6]

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{S}$ is said to belong to the class $\mathcal{C}$ of convex functions (i.e. $f(\mathbb{D})$ is a convex domain) if and only if $zf' \in \mathcal{S}^*$. It is worth pointing out that the Koebe function $k(z) = z/(1-z)^2$ belongs $\mathcal{U} \cap \mathcal{S}^*$. Also, the analytic characterization of starlike functions shows that $\mathcal{S}_{\mathbb{Z}} \subset \mathcal{S}^*$. We remark that functions in $\mathcal{S}_{\mathbb{Z}}$ are extremal for certain geometric subclasses of $\mathcal{S}$. In particular, it is natural to ask whether $\mathcal{U}$ is included in $\mathcal{S}^*$. In fact, $\mathcal{U}$ is not a subset of $\mathcal{S}^*$ as the function

$$f_1(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{2}z^3}$$

demonstrates. It is easy to see that $f_1 \in \mathcal{U}$. On the other hand, for this function, we have

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1 - z^3}{1 + \frac{1}{2}z + \frac{1}{2}z^3}$$

and at the boundary point $z_0 = (-1+i)/\sqrt{2}$, $|z_0| = 1$, we obtain that

$$\frac{z_0 f_1'(z_0)}{f_1(z_0)} = \frac{2 - 2\sqrt{2}}{3} + \frac{1 - 2\sqrt{2}}{3} i$$

which gives that $\mathrm{Re}\,\{z_0 f_1'(z_0)/f_1(z_0)\} < 0$. Consequently, there are points in the unit disk $|z| < 1$ for which $\mathrm{Re}\,\{zf_1'(z)/f_1(z)\} < 0$ showing that the function f_1

is not starlike in $\mathbb{D}$. More generally, the function (see [11])

$$f(z) = \frac{z}{1 + ibz + (e^{2i\beta}/2)z^3}$$

belongs $\mathcal{U}$, but is not in $\mathcal{S}^*$ when $0 < b \leq 1/2$ and $0 < \beta < \arctan(2b)$, because

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\bigg|_{z=1} = \frac{[\sin\beta - 2b\cos\beta]\sin\beta}{|1 + ib + (e^{2i\beta}/2)|^2} < 0.$$

2. Basic Properties of the class $\mathcal{U}$

Theorem 2.1. (Characterization for $\mathcal{U}$) *Every $f \in \mathcal{U}$ has the representation*

$$\frac{z}{f(z)} = 1 - a_2 z - z\int_0^z \frac{\omega(t)}{t^2}dt, \;\; a_2 = a_2(f) = \frac{f''(0)}{2},$$

where $\omega \in \mathcal{B}_1$, the class of analytic functions in the unit disk $\mathbb{D}$ such $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$.

Proof. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $\mathcal{U}$. Then one has

$$\frac{f(z)}{z} \neq 0 \;\text{ and }\; \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + (a_3 - a_2^2)z^2 + \cdots, \quad z \in \mathbb{D},$$

which may be written as

$$-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} = \left(\frac{z}{f(z)}\right)^2 f'(z) = 1 + \omega(z), \quad z \in \mathbb{D}, \tag{2.2}$$

with $\omega \in \mathcal{B}_1$. Also, by the Schwarz lemma, $|\omega(z)| \leq |z|^2$, $z \in \mathbb{D}$. From the previous relation, we obtain

$$\left(\frac{1}{f(z)} - \frac{1}{z}\right)' = -\frac{\omega(z)}{z^2},$$

and, since

$$\left(\frac{1}{f(z)} - \frac{1}{z}\right)\bigg|_{z=0} = -a_2,$$

by integration we get

$$\frac{1}{f(z)} - \frac{1}{z} - (-a_2) = -\int_0^z \frac{\omega(t)}{t^2}dt.$$

The desired representation follows. ■

This representation together with many others which follow from this led to a number of recent investigations, see for example [9, 11, 12, 13, 15]. However, because $\omega \in \mathcal{B}_1$, the Schwarz lemma gives $|\omega(z)| \leq |z|^2$ in $\mathbb{D}$. Consequently, we have

$$\left|\frac{z}{f(z)} + a_2 z - 1\right| \leq |z|^2, \quad z \in \mathbb{D}. \tag{2.3}$$

We observe that if z is fixed ($0 < |z| < 1$), then this inequality determines the range of the functional

$$\frac{z}{f(z)} + a_2 z$$

in the class $\mathcal{U}$. In particular, if $a_2 = 0$ then by a computation (2.3) gives that

$$\left|\frac{f(z)}{z} - \frac{1}{1-|z|^4}\right| \leq \frac{|z|^2}{1-|z|^4}, \quad z \in \mathbb{D}$$

so that, for every $f \in \mathcal{U}$ with $f''(0) = 0$, we have

$$\frac{|z|}{1+|z|^2} \leq |f(z)| \leq \frac{|z|}{1-|z|^2}, \quad z \in \mathbb{D}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) \geq \frac{1}{1+|z|^2} > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{2.4}$$

We now formulate

Corollary 2.5. *Let $f \in \mathcal{U}$. Then one has*

(a) $\left|\frac{z}{f(z)} - 1\right| \leq |z|(|a_2| + |z|)$, $z \in \mathbb{D}$

(b) $\mathrm{Re}\left(\frac{f(z)}{z}\right) > 0$ *for* $|z| < \frac{2}{\sqrt{4+|a_2|}+|a_2|}$

(c) $\mathrm{Re}\left(\frac{f(z)}{z}\right) > \frac{1}{2}$ *in* $\mathbb{D}$ *if* $f''(0) = 0$.

2.1. Interesting subclass of $\mathcal{U}$. Investigation on various subclasses of $\mathcal{S}$ has a long history and continues to occupy a prominent place in function theory. In [7], the authors introduced a subclass $\mathcal{P}(2)$ of $\mathcal{U}$, consisting of functions f for which

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2, \quad z \in \mathbb{D}.$$

We have the following strict inclusion (see [7]).

Theorem 2.6. $\mathcal{P}(2) \subset \mathcal{U}$.

Proof. Let $f \in \mathcal{P}(2)$ and $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. We may introduce

$$p(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 = -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1$$

so that

$$p(z) = (a_3 - a_2^2)z^2 + \cdots, \quad z \in \mathbb{D},$$

Also, we observe that $p(0) = p'(0) = 0$, and

$$zp'(z) = -z^2\left(\frac{z}{f(z)}\right)''.$$

By assumption, $|zp'(z)| < 2$ in $\mathbb{D}$ which by a well-known subordination relation gives that $|p(z)| < 1$, $z \in \mathbb{D}$. That is, $f \in \mathcal{U}$. ■

Further work on the classes $\mathcal{U}$ and $\mathcal{P}(2)$, including some interesting generalizations of these classes, may be found in [9, 17]. We remark that the constant 2 in the inclusion result of Theorem 2.6 is the best possible. For this, we consider the function

$$f(z) = \frac{z}{(1+z)^{2+\epsilon}}, \ \epsilon > 0.$$

Then we observe that

$$\left(\frac{z}{f(z)}\right)'' = (2+\epsilon)(1+\epsilon)(1+z)^{\epsilon} \ \text{ and } \ f'(z) = \frac{1-(1+\epsilon)z}{(1+z)^{3+\epsilon}}$$

from which we obtain that $f'(1/(1+\epsilon)) = 0$ and therefore, the function f is not univalent in $\mathbb{D}$.

2.2. Condition for functions to be in $\mathcal{U}$.

One of the sufficient conditions for a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to be in $\mathcal{S}^*$ is that $\sum_{n=2}^{\infty} n|a_n| \leq 1$. Moreover, this coefficient condition is also sufficient for f to belong to $\mathcal{R}$, where $\mathcal{R}$ denotes the class of normalized analytic functions f in $\mathbb{D}$ satisfying the condition

$$|f'(z) - 1| < 1 \ \text{ in } \mathbb{D}.$$

It is worth pointing out that the convex class $\mathcal{C}$ neither contained in $\mathcal{R}$ nor contains $\mathcal{R}$. In spite of the fact that neither $\mathcal{S}^*$ is included in $\mathcal{U}$ nor includes $\mathcal{U}$, we have the following interesting result (see also [4]).

Theorem 2.7. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that $\sum_{n=2}^{\infty} n|a_n| \leq 1$. Then $f \in \mathcal{U}$. The result is sharp.*

Proof. Under the assumption, we find that

$$\begin{aligned}\left|f'(z)-\left(\frac{f(z)}{z}\right)^2\right| &= \left|1+\sum_{n=2}^{\infty} na_n z^{n-1} - \left(1+\sum_{n=2}^{\infty} a_n z^{n-1}\right)^2\right| \\ &= \left|\sum_{n=2}^{\infty}(n-2)a_n z^{n-1} - \left(\sum_{n=2}^{\infty} a_n z^{n-1}\right)^2\right| \\ &= |z|^2\left|\sum_{n=3}^{\infty}(n-2)a_n z^{n-3} - \left(\sum_{n=2}^{\infty} a_n z^{n-2}\right)^2\right|\end{aligned}$$

and therefore,

$$\begin{aligned}\left|f'(z)-\left(\frac{f(z)}{z}\right)^2\right| &< \sum_{n=2}^{\infty}(n-2)|a_n| + \left(\sum_{n=2}^{\infty}|a_n|\right)^2 \\ &\leq 1-2\sum_{n=2}^{\infty}|a_n| + \left(\sum_{n=2}^{\infty}|a_n|\right)^2 \\ &\leq \left(1-\sum_{n=2}^{\infty}|a_n|\right)^2 \\ &\leq \left|\frac{f(z)}{z}\right|^2\end{aligned}$$

from which we easily obtain that $f \in \mathcal{U}$.

To see that the constant bound 1 in the coefficient estimate cannot be replaced by $1+\epsilon$, $\epsilon > 0$, we consider the function

$$f(z) = z + \frac{1+\epsilon}{n} z^n \ (n \geq 2).$$

We observe that $f'(z) = 1 + (1+\epsilon)z^{n-1}$ has a zero in $\mathbb{D}$ because $\epsilon > 0$. Thus, the result is sharp. ■

2.3. Functions in $\mathcal{U}$ of special form. In this section we focus our attention for analytic functions f in $\mathbb{D}$ of the form

$$f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}. \tag{2.8}$$

We remark that if $f \in \mathcal{S}$ then $z/f(z)$ is nonvanishing in the unit disk $\mathbb{D}$ and hence, can be represented as Taylor's series of the form (2.8) which is convenient for our investigation. Now, we recall that if $f \in \mathcal{S}$ and has the above form, then from the well-known Area Theorem [6, Theorem 11 on p.193 of Vol. 2] we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \le 1 \tag{2.9}$$

But that condition is not sufficient for the univalence of an analytic function f of the form (2.8) (see Theorem 2.13 below). In the next theorem we present a sufficient condition for the univalence in terms of the coefficients b_n of analytic functions f of the form (2.8).

Theorem 2.10. (Sufficient coefficient condition for $\mathcal{U}$) *Let $f \in \mathcal{A}$ have the form* (2.8). *If*

$$\sum_{n=2}^{\infty}(n-1)|b_n| \le 1,$$

then $f \in \mathcal{U}$ and the constant 1 *is the best possible in a sense: if*

$$\sum_{n=2}^{\infty}(n-1)|b_n| = 1+\varepsilon,$$

for some $\varepsilon > 0$, then there exists an f such that f is not univalent in $\mathbb{D}$.

Proof. The first part of the statements of the theorem follows from

$$\begin{aligned} |U_f(z| &= \left| -z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1 \right| \\ &= \left| -\sum_{n=1}^{\infty}(n-1)b_n z^n \right| \\ &\le \sum_{n=1}^{\infty}(n-1)|b_n| \le 1. \end{aligned}$$

In order to prove the sharpness part of the theorem, we consider the function $f(z) = z - qz^2$, where $q = \frac{\sqrt{1+\varepsilon}}{1+\sqrt{1+\varepsilon}}$, $\varepsilon > 0$, so that $\frac{1}{2} < q < 1$. Then, we have

$$\frac{z}{f(z)} = \frac{1}{1-qz} = 1 + \sum_{n=1}^{\infty} q^n z^n$$

and

$$\sum_{n=2}^{\infty}(n-1)|b_n| = \sum_{n=2}^{\infty}(n-1)q^n = \left(\frac{q}{1-q}\right)^2 = 1+\varepsilon.$$

Also, we see that $f'(z) = 1 - 2qz$ and therefore, $f'(\frac{1}{2q}) = 0$ showing that f is not univalent in the unit disk $\mathbb{D}$. ∎

The coefficient condition (2.10) is only a sufficient condition for f to be in the class $\mathcal{U}$. In fact it can be easily seen that the condition (2.10) is not a necessary condition for the corresponding function to be in that class. For instance, if f is given by

$$\frac{z}{f(z)} = 1 + \frac{1}{3}z^2 + \frac{\sqrt{5}}{6}iz^3 + \frac{1}{9}z^4$$

then on one hand we have

$$|U_f(z)| = \frac{1}{3}|z^2|\,|1 + \sqrt{5}iz + z^2| < 1,$$

and on the other hand,

$$\sum_{n=2}^{\infty}(n-1)|b_n| = \frac{1}{3} + \frac{\sqrt{5}}{3} + \frac{1}{3} > 1.$$

Theorem 2.11. (Necessary coefficient condition for $\mathcal{U}$) *Let $f \in \mathcal{U}$ have the form* (2.8). *Then*

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1. \tag{2.12}$$

In particular, we have $|b_1| \leq 2$ and $|b_n| \leq \frac{1}{n-1}$ for $n \geq 2$. The results are sharp.

Proof. Recall that $f \in \mathcal{U}$ if and only if

$$|U_f(z)| = \left|\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1\right| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| < 1.$$

We note that $g(z) = \sum_{n=2}^{\infty}(n-1)b_n z^n$ is analytic in $\mathbb{D}$ and therefore, with $z = re^{i\theta}$, we have

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|g(re^{i\theta})\right|^2\,d\theta < 1$$

so that, as $r \to 1^-$, we obtain the desired inequality.

Because $b_1 = -f''(0)/2$, the Bieberbach inequality gives that $|b_1| \leq 2$ and fact that the Koebe function $k(z) = z/(1-z)^2$ belongs to $\mathcal{U}$ shows that the result is the best possible. Further, the inequality (2.12) implies that for $n \geq 2$ we have that $|b_n| \leq \frac{1}{n-1}$. The functions $s_n(z)$, for $n \geq 2$, defined by

$$s_n(z) = \frac{(n-1)z}{z^n + n - 1}, \quad \text{i.e.} \quad \frac{z}{s_n(z)} = 1 + \frac{z^n}{n-1},$$

also belong to the class $\mathcal{U}$ showing that the result is sharp. ∎

We observe that the necessary coefficient condition (2.12) for the class $\mathcal{U}$ is stronger than that for the class $\mathcal{S}$, namely the inequality (2.9).

Theorem 2.13. *Let $f \in \mathcal{A}$ and have the form* (2.8) *satisfying the condition*

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \le 1.$$

Then f is univalent in the disk $|z| < \frac{1}{\sqrt{2}}$ and the result is the best possible.

Proof. Consider the function $g(z) = \frac{1}{r}f(rz)$, where $0 < r \le 1$. Then

$$\frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n r^n.$$

Because

$$\begin{aligned}\sum_{n=2}^{\infty}(n-1)|b_n|r^n &= \sum_{n=2}^{\infty}\sqrt{n-1}|b_n|\sqrt{n-1}r^n \\ &\le \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} \\ &\le \left(\sum_{n=2}^{\infty}(n-1)r^{2n}\right)^{\frac{1}{2}} = \frac{r^2}{1-r^2} \le 1\end{aligned}$$

for $0 < r \le \frac{1}{\sqrt{2}}$, it follows easily that g is in the class $\mathcal{U}$. In particular $f(z)$ is univalent in the disk $|z| < \frac{1}{\sqrt{2}}$.

For the function $f_0(z) = z - \frac{1}{\sqrt{2}}z^2$ we have

$$\frac{z}{f(z)} = \frac{1}{1-\frac{1}{\sqrt{2}}z} = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n z^n$$

and

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 = \sum_{n=2}^{\infty}(n-1)\left(\frac{1}{2}\right)^n = 1.$$

On the other hand $\operatorname{Re} f_0'(z) = \operatorname{Re}(1-\sqrt{2}z) > 0$ for $|z| < \frac{1}{\sqrt{2}}$ and $f_0'(\frac{1}{\sqrt{2}}) = 0$, which implies that $f_0(z)$ is not univalent in a larger disk. ■

Theorem 2.14. *Let $f \in \mathcal{A}$ and have the form* (2.8) *satisfying the condition*

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \le 1.$$

Then f is univalent in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}}$ and the result is the best possible.

Proof. As in the proof of Theorem 2.13, it suffices to observe that

$$\sum_{n=2}^{\infty}(n-1)|b_n|r^n \leq \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}r^{2n}\right)^{\frac{1}{2}} \leq \frac{r^2}{\sqrt{1-r^2}} \leq 1$$

whenever $r^4 + r^2 - 1 \leq 0$, that is if $0 < r \leq r_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. It means that the function g defined by $g(z) = \frac{1}{r}f(rz)$ is in the class $\mathcal{U}$ and hence, $f(z)$ is univalent in the disk $|z| < r_0 = \sqrt{\frac{\sqrt{5}-1}{2}}$.

For the function $f_0(z)$ defined by

$$\frac{z}{f_0(z)} = 1 + \sum_{n=2}^{\infty}\frac{r_0^n}{n-1}z^n = 1 - r_0 z\log(1-r_0 z)$$

we have that $\mathrm{Re}\, f_0(z) > 0$ in $\mathbb{D}$ so that $f \in \mathcal{A}$ and

$$\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 = \sum_{n=2}^{\infty}(n-1)^2\frac{r_0^{2n}}{(n-1)^2} = 1.$$

On the other hand side for $|z| < r_0$ we find that

$$\left|\left(\frac{z}{f_0(z)}\right)^2 f_0'(z) - 1\right| = \left|\frac{-r_0^2z^2}{1-r_0z}\right| < \frac{r_0^4}{1-r_0^2} = 1,$$

while for $r_0 \leq z = r < 1$:

$$\left|\left(\frac{z}{f_0(z)}\right)^2 f_0'(z) - 1\right|_{z=r} = \frac{r^4}{1-r^2} \geq 1.$$

It means that $g_0(z) = \frac{1}{r}f_0(rz)$ is in the class $\mathcal{U}$ for $r \leq r_0$, but not in a larger value of r, and hence, f is univalent in the disk $|z| < r_0$, but not in a larger disk. Moreover, a computation gives

$$f_0'(z) = \frac{1 - r_0z - r_0^2z^2}{(1-r_0z)(1-r_0zlog(1-r_0z))^2}$$

and therefore, $f_0'(r_0) = 0$. Thus, f cannot be univalent in any disk larger than the disk $|z| < r_0$. ∎

3. Function in $\mathcal{U}$ in some special situation

The class of functions $f \in \mathcal{A}$ of the form (2.8) for which $b_n \geq 0$ for all $n \geq 2$ is especially interesting and deserves a separate discussion (see [12]).

Theorem 3.1. *Let $f \in \mathcal{A}$ have the form* (2.8) *with $b_n \geq 0$ for all $n \geq 2$. Then we have the following equivalence:*

(a) $f \in \mathcal{S}$
(b) $\dfrac{f(z)f'(z)}{z} \neq 0$ *for* $z \in \mathbb{D}$
(c) $\displaystyle\sum_{n=2}^{\infty}(n-1)b_n \leq 1$
(d) $f \in \mathcal{U}$.

Proof. (a) $\Rightarrow$ (b): Let $f \in \mathcal{S}$ be of the form (2.8) with $b_n \geq 0$ for all $n \geq 2$. Then, $f'(z) \neq 0$ and $f(z)/z \neq 0$ in $\mathbb{D}$.

(b) $\Rightarrow$ (c): From the representation of f and (2.2) we quickly see that for $z \in \mathbb{D}$,

$$\left(\frac{rz}{f(rz)}\right)^2 f'(rz) = 1 - \sum_{n=2}^{\infty}(n-1)b_n r^n z^n$$

from which, as $z/f(z) \neq 0$, it follows that $f'(rz) \neq 0$ is equivalent to

$$1 - \sum_{n=2}^{\infty}(n-1)b_n r^n z^n \neq 0.$$

We claim that $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$. Suppose on the contrary that $\sum_{n=2}^{\infty}(n-1)b_n > 1$. Then, on the one hand, there exists a positive integer m such that

$$\sum_{n=2}^{m}(n-1)b_n > 1$$

and so there exists an r_0 with $0 < r_0 < 1$ and

$$\sum_{n=2}^{m}(n-1)b_n r_0^n > 1.$$

On the other hand, as $b_n \geq 0$ for $n \geq 2$, we have that

$$\left(\frac{r_0}{f(r_0)}\right)^2 f'(r_0) = 1 - \sum_{n=2}^{\infty}(n-1)b_n r_0^n \leq 1 - \sum_{n=2}^{m}(n-1)b_n r_0^n < 0$$

and, since $f'(r)$ is a continuous function of r with $f'(0) = 1$ and $f'(r_0) < 0$, there exists an r_1 $(0 < r_1 < r_0 < 1)$ such that $f'(r_1) = 0$. This is a contradiction. Consequently, we must have

$$\sum_{n=2}^{\infty}(n-1)b_n \leq 1.$$

(c) $\Rightarrow$ (d): Suppose that $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$. Then, by Theorem 2.10, it follows that $f \in \mathcal{U}$.

(d) $\Rightarrow$ (a): $\mathcal{U} \subset \mathcal{S}$ is a well-known fact. ■

The condition Theorem 3.1(c) may be used to conclude quickly that the functions

$$\frac{z}{(1+z)^2}, \quad \frac{z}{1+z}, \quad \frac{z}{1+z^2}, \quad \text{and} \quad \frac{z}{1+z+z^2}$$

are in $\mathcal{U}$.

Corollary 3.2. *If f is of the form* (2.8) *with $b_n \geq 0$ for all $n \geq 2$ and such that* $\mathrm{Re}\,(f'(z)) > 0$ *for $z \in \mathbb{D}$, then $f \in \mathcal{U}$.*

Proof. The condition $\mathrm{Re}\,(f'(z)) > 0$ for $z \in \mathbb{D}$ implies that $f \in \mathcal{S}$. The conclusion follows from Theorem 3.1 (see also Theorem 3.1). ■

Corollary 3.3. *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ is in $\mathcal{S}^*$, where $a_n \geq 0$ for $n \geq 2$, then $f \in \mathcal{U}$.*

Proof. Let $f \in \mathcal{S}^*$. Then $z/f(z)$ can be expressed as

$$\frac{z}{f(z)} = \frac{1}{1 - a_2 z - a_3 z^2 - \cdots} = 1 + b_1 z + b_2 z^2 + \cdots ,$$

where $b_n \geq 0$ for all $n \in \mathcal{N}$. Then, by Theorem 3.1, the inequality

$$\sum_{n=2}^{\infty}(n-1)b_n \leq 1$$

holds and hence, by Theorem 2.10, $f \in \mathcal{U}$. ■

Corollary 3.3 especially helpful in obtaining functions that are in $\mathcal{S}^* \cap \mathcal{U}$, as there are numerous results concerning starlike functions with negative coefficients. For example, $f_m(z) = z - z^m/m$ is in $\mathcal{S}^*$ and hence in $\mathcal{U}$. Since $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ is in $\mathcal{S}^*$ if and only if $\sum_{n=2}^{\infty} n|a_n| \leq 1$ (see [18, Theorem 2]), this result can be used to generate functions $f \in \mathcal{U}$ that are not starlike.

Example 3.4. Let

$$f(z)=\frac{z}{(1+z)^2},\ g(z)=\frac{z}{(1-z)^2}\ \text{ and }\ F(z)=\frac{zf(z)}{g(z)}.$$

Then

$$\frac{z}{F(z)}=\frac{(1+z)^2}{(1-z)^2}=(1+2z+z^2)(1+2z+3z^2+\cdot)=1+4z+4\sum_{n=2}^{\infty}nz^n$$

so that

$$\frac{rz}{F(rz)}=1+4rz+4\sum_{n=2}^{\infty}nr^nz^n$$

for $0<r<1$. By Theorem 3.1, this function F is univalent if and only if r satisfies the inequality

$$4\sum_{n=2}^{\infty}(n-1)nr^n\leq 1,$$

This gives the condition

$$\frac{8r^2}{(1-r)^3}\leq 1,\ \text{ i.e. }\ 3r+5r^2+r^3-1\leq 0,$$

which implies that $0<r\leq r_0\approx 0.23607$.

3.1. Radius property of univalent functions. If for every $f\in\mathcal{S}$ the function $\frac{1}{r}f(rz)\in\mathcal{U}$ for $0<r\leq r_0$, and r_0 is the largest number for which this hold, then we say that r_0 is the $\mathcal{U}$ radius (or the radius of $\mathcal{U}$-property) in the class $\mathcal{S}$. In this case, we may conveniently write $r_0=r_{\mathcal{U}}(\mathcal{S})$.

Theorem 3.5. $r_{\mathcal{U}}(\mathcal{S})=\frac{1}{\sqrt{2}}$.

Proof. Let $f\in\mathcal{S}$. Then every such an f has the form

$$\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}b_nz^n.$$

Then, by (2.9), we obtain that

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2\leq 1.$$

The desired conclusion clearly follows from Theorem 2.13. Moreover, to see that the number $\frac{1}{\sqrt{2}}$ is the best possible, we consider the function

$$f(z)=\frac{z(1-\frac{1}{\sqrt{2}}z)}{1-z^2}.$$

If we put $z=\rho e^{i\theta}\in\mathbb{D}$, then we have

$$\mathrm{Re}\,((1-z^2)f'(z))=\frac{(1-\rho^2)(1+\rho^2-\sqrt{2}\rho\cos\theta)}{|1-\rho^2e^{i2\theta}|^2}>0$$

for $0\le\rho<1$. Thus, f is close-to-convex in $\mathbb{D}$ and therefore, $f\in\mathcal{S}$. Next, we note that

$$\left|\left(\frac{z}{f(z)}\right)^2f'(z)-1\right|=\left|\frac{z}{\sqrt{2}-z}\right|^2$$

is less than 1 for $|z|<\frac{1}{\sqrt{2}}$, equal to 1 for $z=\frac{1}{\sqrt{2}}$ and bigger than 1 for $\frac{1}{\sqrt{2}}<z=r<1$. The sharpness part follows. ∎

Remark 3.6. In later articles, the authors (see also [4] for many related results) considered the class $\mathcal{U}(\lambda)$ defined by the condition

$$\left|\left(\frac{z}{f(z)}\right)^2f'(z)-1\right|<\lambda,\quad z\in\mathbb{D}.$$

and find that $r_{\mathcal{U}(\lambda)}(\mathcal{S})=\sqrt{\frac{\lambda}{1+\lambda}}$.

3.2. Convolution properties with $\mathcal{U}$.

Theorem 3.7. *Let $f,g\in\mathcal{S}$ with the representations*

$$\frac{z}{f(z)}=1+\sum_{n=1}^{\infty}b_nz^n,\quad \frac{z}{g(z)}=1+\sum_{n=1}^{\infty}c_nz^n.$$

If

$$\Phi(z)=\frac{z}{f(z)}*\frac{z}{g(z)}=1+\sum_{n=1}^{\infty}b_nc_nz^n\neq0$$

for every $z\in\mathbb{D}$, then

$$F(z)=\frac{z}{\Phi(z)}\in\mathcal{U}$$

and, in particular, F is univalent in $\mathbb{D}$.

Proof. For $f,g\in\mathcal{S}$ with their representations we have that

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2\le1\ \text{ and }\ \sum_{n=2}^{\infty}(n-1)|c_n|^2\le1.$$

By assumption

$$\Phi(z)=\frac{z}{f(z)}*\frac{z}{g(z)}=1+\sum_{n=1}^{\infty}b_nc_nz^n\neq0,$$

and therefore, the function F is analytic in $\mathbb{D}$. By the classical Cauchy-Schwarz inequality, we conclude that

$$\sum_{n=2}^{\infty}(n-1)|b_n c_n| \leq \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty}(n-1)|c_n|^2\right)^{\frac{1}{2}} \leq 1,$$

which, by Theorem 2.10 , $F \in \mathcal{U}$. ■

Further results on convolution may be obtained from [2, 14].

References

[1] L. A. Aksentév, Sufficient conditions for univalence of regular functions (Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **1958**(4)(1958), 3–7.

[2] R. W. Barnard, S. Naik, M. Obradović and S. Ponnusamy, Two parameter families of close-to-convex functions and convolution theorems, *Analysis*, **24**(2004), 71–94.

[3] P.L. Duren, *Univalent Functions* (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.

[4] R. Fournier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, *Complex Var. Elliptic Equ.* **52**(1)(2007), 1–8.

[5] B. Friedman, Two theorems on schlicht functions, *Duke Math. J.* **13**(1946), 171–177

[6] A.W. Goodman, *Univalent functions*, Vols. 1-2, Mariner, Tampa, Florida, 1983.

[7] M. Nunokawa, M. Obradović, and S. Owa, One criterion for univalency, *Proc. Amer. Math. Soc.* **106**(1989), 1035–1037.

[8] M. Obradović, N. N. Pascu and I. Radomir, A class of univalent functions, *Math. Japonica,* **44**(3)(1996), 565–568.

[9] M.Obradović and S. Ponnusamy, New criteria and distortion theorems for univalent functions, *Complex Variables*, **44**(2001), 173–191.

[10] M. Obradović and S. Ponnusamy, Radius properties for subclasses of univalent functions, *Analysis* **25**(2005), 183-188.

[11] M. Obradović and S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, *J. Math. Anal. and Appl.* **336**(2007), 758–767.

[12] M. Obradović and S. Ponnusamy, Coefficient characterization for certain classes of univalent functions, *Bulletin Belgian Math. Soc.-Simon Stevin* **16**(2009), 251–263.

[13] M. Obradović and S. Ponnusamy, On certain subclasses of univalent functions and radius properties, *Rev. Roumanie Math. Pures Appl.* **54**(4)(2009), 317–329.

[14] M. Obradović and S. Ponnusamy, Univalency and convolution results associated with confluent hypergeometric functions, *Houston J. Math.* **35**(4)(2009), 1313–1328.

[15] M. Obradović, S. Ponnusamy, V. Singh and P. Vasundhra, Univalency, starlikesess and convexity applied to certain classes of rational functions, *Analysis* (*Munich*) **22**(3)(2002), 225–242.

[16] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.* **33**(1972), 392–394.

[17] S. Ponnusamy and P. Vasundhra, Criteria for univalence, starlikeness and convexity, *Ann. Polon. Math.* **85**(2005), 121–133.

[18] H. SILVERMAN, Univalent functions with negative coefficicents, *Proc. Amer. Math. Soc.* **51**(1975), 109–116.

M. Obradović E-MAIL: obrad@grf.bg.ac.rs
ADDRESS:
M. Obradović,
Department of Mathematics,
Faculty of Civil Engineering,
University of Belgrade,
Bulevar Kralja Aleksandra 73, 11000,
Belgrade, Serbia.

S. Ponnusamy E-MAIL: samy@iitm.ac.in
ADDRESS:
S. Ponnusamy,
Department of Mathematics,
Indian Institute of Technology Madras,
Chennai–600 036, India.