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Injectivity and Starlikeness of Sections of a Class of Univalent Functions

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Abstract. Let $\mathcal{G}$ denote the class of locally univalent normalized analytic functions f in the unit disk $|z|<1$ satisfying the condition

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3}{2}\quad \text{for } |z|<1.$$

In this paper, we show in particular that each partial sum $s_n(z)$ of $f\in\mathcal{G}$ is starlike in the disk $|z|\le 1/2$ for $n\ge 12$. We also prove that if $f\in\mathcal{G}$ then $\mathrm{Re}\,(s_n'(z))>0$ holds in $|z|\le 1/2$ for $n\ge 13$.

1. Introduction and Preliminary Results

For $r>0$, let $\mathbb{D}_r:=\{z\in\mathbb{C}:\,|z|<r\}$ and $\mathbb{D}:=\mathbb{D}_1$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the family of all functions f that are analytic in $\mathbb{D}$ with the normalization $f(0)=0=f'(0)-1$. Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ that are univalent in $\mathbb{D}$. A domain D in $\mathbb{C}$ is called starlike (with respect to the origin) if every line segment joining the origin to any other point in D lies completely inside D. A function $f\in\mathcal{S}$ is called starlike if $f(\mathbb{D})$ is a starlike domain. The class of all starlike functions is denoted by $\mathcal{S}^*$, and functions $f\in\mathcal{S}^*$ are characterized by the condition

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\quad z\in\mathbb{D}.$$

Using the Koebe distortion theorem and the Löwner theory of univalent functions, in 1928, Szegő [**16**] proved that n-th *partial sums*/sections $s_n(z):=z+\sum_{k=2}^{n}a_kz^k$ of $f\in\mathcal{S}$, $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, are univalent in the disk $\mathbb{D}_{1/4}$ and the number $1/4$ cannot be replaced by a larger one as the Koebe function $k(z)=z/(1-z)^2$ shows. We refer to [**3**, §8.2, pp. 241–246] and the survey article of Iliev [**5**] for some related investigations. The class of convex and the class of close-to-convex mappings are some of the important well-known standard subclasses of $\mathcal{S}$, denoted by $\mathcal{C}$, and $\mathcal{K}$, respectively. These classes are well understood and are studied extensively in the literature. We refer to the books by Duren [**3**] and Goodman [**4**].

The radius of starlikeness of $s_n(z)$, $f\in\mathcal{S}^*$, was proved by Robertson [**13**].

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Theorem A. [**13**] (see also [**15**, Theorem 2, p. 1193]) *If $f \in \mathcal{S}$ is either starlike, or convex, or typically-real, or convex in the direction of imaginary axis, then there is an N such that, for $n \geq N$, the partial sum $s_n(z)$ has the same property in $\mathbb{D}_r$, where $r \geq 1 - 3(\log n)/n$.*

Later, in [**14**], Ruscheweyh proved a stronger result by showing that the partial sums $s_n(z)$ of f are indeed starlike in $\mathbb{D}_{1/4}$ for functions f belonging not only to $\mathcal{S}$ but also to the closed convex hull of $\mathcal{S}$. Robertson [**13**] further showed that sections of the Koebe function $k(z)$ are univalent in the disk $|z| < 1-3n^{-1}\log n$ for $n \geq 5$, and that the constant 3 cannot be replaced by a smaller constant. However, Bshouty and Hengartner [**2**, p. 408] pointed out that the Koebe function is not extremal for the radius of univalency of the partial sums of $f \in \mathcal{S}$. However, a well-known theorem on convolution allows us to conclude immediately that if f belongs to $\mathcal{C}$, $\mathcal{S}^*$, or $\mathcal{K}$, then its n-th section is respectively convex, starlike, or close-to-convex in the disk $|z| < 1-3n^{-1}\log n$, for $n \geq 5$. As pointed out in [**3**, Section 8.2, p. 246] (see also [**12**, Section 6.4]), the exact (largest) radius of univalence r_n of $s_n(z)$ ($f \in \mathcal{S}$) remains an open problem.

In this paper, we shall consider the partial sums of the class of functions from $\mathcal{G}$. A locally univalent function $f \in \mathcal{A}$ is said to belong to $\mathcal{G}$ if it satisfies the condition

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D}.$$

Functions in $\mathcal{G}$ are known to be in $\mathcal{S}$ (see also [**11**]). Moreover if $f \in \mathcal{G}$, then (see e.g. [**9**, Example 1, Equation (16)] and [**7**, Theorem 1]) one has

$$\frac{zf'(z)}{f(z)} \prec g(z) = \frac{2(1-z)}{2-z}, \quad z \in \mathbb{D},$$

where $\prec$ denotes the subordination. We see that the function g above is univalent in $\mathbb{D}$ and maps $\mathbb{D}$ onto the disk $|w-(2/3)| < 2/3$. Thus, functions in $\mathcal{G}$ are starlike in $\mathbb{D}$. Further, it is a simple exercise to see that g maps the circle $|z| = r$ onto the circle

$$\left|w - \frac{2(2-r^2)}{4-r^2}\right| = \frac{2r}{4-r^2}$$

and so, by a computation, we see that for $f \in \mathcal{G}$

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leq \sin^{-1}\left(\frac{r}{2-r^2}\right), \quad |z| = r < 1.$$

In particular, this gives

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leq \sin^{-1}\left(\frac{2}{7}\right) \quad \text{for } |z| \leq 1/2. \tag{1}$$

This fact will be used in the proof of Theorem 3. We now state our main results and their proofs will be given in Section 3.

Theorem 1. *Let $f \in \mathcal{G}$ and $s_n(z)$ be its n-th partial sum. Then for each $r \in (0,1)$ and $n \geq 2$, we have*

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < |z|^n\left(1+\left(\frac{\sqrt{nr(2-r)}}{(1-r)r^n}\right)\frac{|z|}{r-|z|}\right) \quad \text{for } |z| < r. \tag{2}$$

THEOREM 2. *Let $f \in \mathcal{G}$ and $s_n(z)$ be its n-th partial sum. Then,* $\operatorname{Re}\{s_n'(z)\} > 0$ *in the disk $|z| \le 1/2$ for $n \ge 13$. In particular, $s_n'(z)$ is close-to-convex (and hence univalent) in $|z| \le 1/2$ for $n \ge 13$.*

THEOREM 3. *Let $f \in \mathcal{G}$. Then for $n \ge 12$, every section $s_n(z)$ of f is starlike in the disk $|z| \le 1/2$.*

2. Lemmas

For the proofs of our theorems, we need several lemmas.

LEMMA 1. *Let $f \in \mathcal{G}$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then*

$$|a_n| \le \frac{1}{n} \quad \text{for } n \ge 2.$$

Equality for the second coefficient holds for $f_0(z) = z - (1/2)z^2$.

PROOF. By assumption, we may write

$$1 + \frac{zf''(z)}{f'(z)} = \frac{3}{2} - \frac{1}{2}p(z)$$

where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is analytic in $\mathbb{D}$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Also, we have $|p_n| \le 2$ for all $n \ge 1$. In terms of the power series expansion, the last identity is equivalent to

$$\left(\sum_{n=1}^{\infty} n a_n z^n\right)\left(1 - \frac{1}{2}\sum_{n=1}^{\infty} p_n z^n\right) = \sum_{n=1}^{\infty} n^2 a_n z^n,$$

where $a_1 = 1$. Equating the coefficients of z^n on both sides, we deduce that

$$n^2 a_n = n a_n - \frac{1}{2}\sum_{k=1}^{n-1}(n-k)a_{n-k}p_k.$$

Thus, as $|p_n| \le 2$ for $n \ge 1$, we get

$$n(n-1)|a_n| \le \sum_{k=1}^{n-1}(n-k)|a_{n-k}| = \sum_{k=1}^{n-1} k|a_k|.$$

For $n = 2$, we easily see that $|a_2| \le 1/2$, and so for $n = 3$, we have

$$6|a_3| \le 1 + 2|a_2| \le 2, \quad \text{i.e.,} \quad |a_3| \le \frac{1}{3}.$$

Therefore, if we assume $|a_k| \le \frac{1}{k}$ for $k = 2, 3, \ldots, n-1$, then we deduce that

$$n(n-1)|a_n| \quad \le \quad \sum_{k=1}^{n-1} k\frac{1}{k} = \sum_{k=1}^{n-1} 1 = n-1,$$

so that $|a_n| \le \frac{1}{n}$. The proof of the theorem is complete by induction. We remark finally that for the function $f_0(z) = z - z^2/2$, we have

$$1 + \frac{zf_0''(z)}{f_0'(z)} = 1 - \frac{z}{1-z},$$

which implies

$$\operatorname{Re}\left(1 + \frac{zf_0''(z)}{f_0'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D}.$$

Thus, $f_0 \in \mathcal{G}$ and the coefficient inequality is sharp for the second coefficient. □

REMARK 1. After this paper was completed, the present authors with K.-J. Wirths [**8**] obtained sharp estimate for $|a_n|$ for each $n \geq 2$.

LEMMA 2. *Let $f \in \mathcal{G}$. Then*

$$\left|\frac{1}{f'(z)}\right| \leq \frac{1}{1-r} := M(r) \quad \textit{for } |z| = r.$$

PROOF. Suppose that $f \in \mathcal{G}$. Then, from the definition of the class $\mathcal{G}$, we have

$$\frac{zf''(z)}{f'(z)} \prec \frac{-z}{1-z}$$

which implies that $f'(z) \prec 1-z$ (see for example [**9**, Theorem 1, Eqn. (1)] or [**10**]). Thus, we obtain that

$$1 - r \leq |f'(z)| \leq 1 + r \quad \text{for } |z| = r,$$

and the conclusion follows. □

LEMMA 3. *Suppose that $f \in \mathcal{G}$ and $s_n(z)$ is its n-th partial sum. Assume that $|1/f'(z)| \leq M$ in $\mathbb{D}$ for some $M > 1$. Then for each $n \geq 2$*

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| \leq |z|^n \left(1 + A_n \frac{|z|}{1-|z|}\right), \ |z| = r < 1,$$

where $A_n = \sqrt{n(M^2-1)}$.

PROOF. For $f \in \mathcal{G}$, we let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ so that

$$s_n(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n.$$

As $f \in \mathcal{G}$, $f'(z)$ is non-vanishing in $\mathbb{D}$ (because f is univalent) and hence $1/f'(z)$ can be represented in the form

$$\frac{1}{f'(z)} = 1 + d_1z + d_2z^2 + \cdots$$

for some complex coefficients d_n, $n \geq 1$. Note that $2a_2 = -d_1$, and we have the identity

$$(1 + 2a_2z + 3a_3z^2 + \cdots)(1 + d_1z + d_2z^2 + \cdots) \equiv 1.$$

From the last relation, we see that

$$\sum_{k=1}^{m-1} (m-k)a_{m-k}d_k + ma_m = 0 \quad (m = 2, 3, \ldots; \ a_1 = 1).$$

Using the representation for the partial sum $s_n(z)$, we obtain that

$$\begin{aligned} \frac{s_n'(z)}{f'(z)} &= (1 + 2a_2z + 2a_3z^2 + \cdots + na_nz^{n-1})(1 + d_1z + d_2z^2 + \cdots) \\ &\equiv 1 + c_nz^n + c_{n+1}z^{n+1} + \cdots, \end{aligned}$$

where

$$c_n = na_nd_1 + (n-1)a_{n-1}d_2 + \cdots + a_1d_n.$$

The previous relation for $m = n+1$ shows that $c_n = -(n+1)a_{n+1}$ and, more generally,

$$c_m = na_nd_{m-n+1} + (n-1)a_{n-1}d_{m-n+2} + \cdots + a_1d_m \ \text{ for } m = n+1, n+2, \ldots.$$

By Lemma 1, $|a_n| \leq 1/n$ for all $n \geq 2$, and therefore, we have that for $m \geq n+1$

$$|c_m| \leq \sum_{k=1}^{n} |d_{m-n+k}|. \tag{3}$$

By assumption, $|1/f'(z)| \leq M$ for $z \in \mathbb{D}$. Hence for $0 < r < 1$, we have that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{f'(re^{i\theta})} \right|^2 d\theta = 1 + \sum_{n=1}^{\infty} |d_n|^2 r^{2n} \leq M^2$$

which, by allowing $r \to 1^-$, shows that

$$\sum_{n=1}^{\infty} |d_n|^2 \leq M^2 - 1.$$

In view of the Cauchy-Schwarz inequality and the last inequality, (3) reduces to

$$|c_m| \leq \left(\sum_{k=1}^{n} 1^2 \right)^{1/2} \left(\sum_{k=1}^{n} |d_{m-n+k}|^2 \right)^{1/2} \leq \sqrt{n(M^2-1)} = A_n$$

for $m \geq n+1$. This inequality, together with the fact that $|c_n| = |(n+1)a_{n+1}| \leq 1$, gives that for $|z| = r < 1$,

$$\begin{aligned} \left| \frac{s_n'(z)}{f'(z)} - 1 \right| &= \left| c_n z^n + c_{n+1} z^{n+1} + \cdots \right| \\ &\leq |c_n|\, |z|^n + |c_{n+1}|\, |z|^{n+1} + \cdots \\ &\leq |z|^n \left(1 + A_n \frac{|z|}{1-|z|} \right) \end{aligned}$$

for $n \geq 2$. This completes the proof of Lemma 3. □

Lemma 4. *Suppose that $f \in \mathcal{G}$ and $s_n(z)$ is its n-th partial sum. Then for each $n \geq 2$*

$$\left| \frac{s_n(z)}{f(z)} - 1 \right| < |z|^n \left(\frac{1}{n+1} + R \frac{|z|}{1-|z|} \right), \ |z| = r < 1,$$

where $R = \dfrac{\pi}{3\sqrt{2}} \approx 0.74048$.

Proof. As in the proof of Lemma 3, we let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ so that $s_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$. Since the functions in $\mathcal{G}$ are univalent, each $f \in \mathcal{G}$ can be written in the form

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{4}$$

for some complex coefficients b_n $(n \geq 1)$. In view of this observation and the two different forms of representations for f, it follows that

$$(1 + a_2 z + a_3 z^2 + \cdots)(1 + b_1 z + b_2 z^2 + \cdots) \equiv 1.$$

Comparing the powers of z on both sides, we have

$$\sum_{k=1}^{m-1} b_k a_{m-k} + a_m = 0 \quad (m = 2, 3, \ldots;\ a_1 = 1). \tag{5}$$

Using the representation for the partial sum $s_n(z)$ and (4), we obtain that

$$\begin{aligned}\frac{s_n(z)}{f(z)} &= (1+a_2z+a_3z^2+\cdots+a_nz^{n-1})(1+b_1z+b_2z^2+\cdots) \\ &\equiv 1+c_nz^n+c_{n+1}z^{n+1}+\cdots,\end{aligned}$$

where

$$c_n = b_1a_n + b_2a_{n-1} + \cdots + b_na_1. \tag{6}$$

By (5), we observe that the coefficients of z^k in the above expansion for $k=1,2,\ldots,n-1$ vanish. Equation (5) for $m=n+1$ shows that $c_n=-a_{n+1}$. Also

$$c_m = b_{m-n+1}a_n + b_{m-n+2}a_{n-1} + \cdots + b_ma_1 \quad \text{for } m=n+1,n+2,\ldots. \tag{7}$$

By Lemma 1, $|a_n|\leq 1/n$ for all $n\geq 2$, and therefore, for $m\geq n+1$, we have

$$|c_m| \leq \frac{1}{n}|b_{m-n+1}| + \frac{1}{n-1}|b_{m-n+2}| + \cdots + |b_m|.$$

Using the classical Cauchy-Schwarz inequality, it follows that for $m\geq n+1$

$$|c_m|^2 \leq \left(\sum_{k=1}^n \frac{1}{(n+1-k)^2}\right)\left(\sum_{k=1}^n |b_{m-n+k}|^2\right) =: AB.$$

For $f\in\mathcal{G}$ we have $f'(z)\prec 1-z$ and therefore,

$$\frac{f(z)}{z} \prec 1-\frac{z}{2}.$$

When f is of the form (4), it is convenient to write the last subordination relation in the form

$$\frac{z}{f(z)} \prec \frac{1}{1-(1/2)z} = 1+\sum_{k=1}^\infty \frac{1}{2^k}z^k.$$

Using Rogosinski's theorem (see [**3**, Theorem 6.2, p. 192]), we obtain that

$$\sum_{k=1}^n |b_k|^2 \leq \sum_{k=1}^n \frac{1}{2^{2k}} = \frac{1}{3}\left(1-\frac{1}{4^n}\right)$$

which implies that

$$B \leq \sum_{k=1}^\infty |b_k|^2 \leq \frac{1}{3}$$

and so, $B\leq 1/3$. On the other hand, for the first sum A, we observe that for $m\geq n+1$,

$$A = \sum_{k=1}^n \frac{1}{(n+1-k)^2} = \sum_{k=1}^n \frac{1}{k^2} < \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Thus we have

$$|c_m| \leq \sqrt{AB} < \frac{\pi}{3\sqrt{2}} = R \quad \text{for } m\geq n+1.$$

This inequality, together with the fact that $|c_n| = |a_{n+1}| \le \frac{1}{n+1}$, gives that for $|z| = r < 1$,

$$\begin{aligned}\left|\frac{s_n(z)}{f(z)} - 1\right| &\le |c_n|\,|z|^n + |c_{n+1}|\,|z|^{n+1} + \cdots \\ &< \frac{1}{n+1}|z|^n + R\left(|z|^{n+1} + |z|^{n+2} + \cdots\right) \\ &= |z|^n\left(\frac{1}{n+1} + R\frac{|z|}{1-|z|}\right)\end{aligned}$$

for $n \ge 2$. The proof is complete. □

3. Proofs of the Theorems

Proof of Theorem 1. We begin with $f \in \mathcal{G}$ and follow the method of proof of Lemma 3. First, by Lemma 2, we have

$$\left|\frac{1}{f'(z)}\right| \le \frac{1}{1-r} =: M(r) \quad \text{for } |z| = r < 1. \tag{8}$$

As observed at the end of the proof of Lemma 3, it follows that

$$\sum_{k=1}^{\infty} |d_k|^2 r^{2k} \le M(r)^2 - 1.$$

Following the notation of Lemma 3, (3) may be rewritten as

$$|c_m| \le \sum_{k=1}^{n} |d_{m-n+k}| = \sum_{k=1}^{n}\left(\frac{1}{r^{m-n+k}}\right)\left(|d_{m-n+k}|r^{m-n+k}\right)$$

for any arbitrary fixed $r \in (0,1)$. Thus, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|c_m|^2 &\le \left(\sum_{k=1}^{n}\frac{1}{r^{2(m-n+k)}}\right)\left(\sum_{k=1}^{n}|d_{m-n+k}|^2 r^{2(m-n+k)}\right) \\ &\le \left(\frac{1}{r^{2m}}\sum_{k=1}^{n}1\right)\left(M(r)^2-1\right) = \frac{n}{r^{2m}}\left(M(r)^2-1\right)\end{aligned}$$

which is true for each $r \in (0,1)$ and so,

$$|c_m| \le \frac{1}{r^m}\left(\sqrt{n(M(r)^2-1)}\right) \quad \text{for } m \ge n+1.$$

As in the proof of Lemma 3, using the above estimate, we easily have

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < |z|^n\left(1 + \frac{1}{r^n}\left(\sqrt{n(M(r)^2-1)}\right)\frac{|z|/r}{1-(|z|/r)}\right) \quad \text{for } |z| < r$$

and the proof of the theorem follows if we use the expression for $M(r) = 1/(1-r)$ given by (8). □

Let us now demonstrate the use of Theorem 1 by fixing some values for r. For example, if we put $r = 2/3$, then by (8) one has

$$M(r) = 3 \quad \text{and} \quad \sqrt{M(r)^2-1} = 2\sqrt{2}.$$

Thus, for $f \in \mathcal{S}$, Theorem 1 after some computation gives the estimate

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < |z|^n\left(1 + 2\sqrt{2n}\left(\frac{3}{2}\right)^n\frac{3|z|}{2-3|z|}\right) \quad \text{for } |z| < 2/3. \tag{9}$$

This estimate helps us to discuss the disk of close-to-convexity (and hence univalency) of partial sums of functions from $\mathcal{G}$.

Proof of Theorem 2. Let $f \in \mathcal{G}$. Then $f'(z) \prec 1 - z$ (see the proof of Lemma 2). Therefore, for $|z| \leq 1/2$ (using the maximum modulus principle), we have

$$\max_{|z|=1/2} |\arg f'(z)| \leq \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}. \tag{10}$$

The inequality (9) for $|z| = 1/2$ together with the maximum modulus principle gives that

$$\left|\frac{s_n'(z)}{f'(z)} - 1\right| < \frac{1}{2^n}\left(1 + 6\sqrt{2n}\left(\frac{3}{2}\right)^n\right) = K_1 \quad \text{for } |z| < \frac{1}{2}. \tag{11}$$

It follows that

$$\max_{|z|=1/2} \left|\arg \frac{s_n'(z)}{f'(z)}\right| \leq \sin^{-1}(K_1).$$

Finally, by (10) and (11), we find that

$$|\arg s_n'(z)| \leq |\arg f'(z)| + \left|\arg \frac{s_n'(z)}{f'(z)}\right| < \frac{\pi}{6} + \sin^{-1}(K_1) \quad \text{for } |z| < \frac{1}{2}$$

and thus,

$$|\arg s_n'(z)| < \frac{\pi}{2}$$

holds if $\sin^{-1}(K_1) \leq \pi/3$. However, the last inequality is easily seen to be true for all $n \geq 13$. $\square$

Proof of Theorem 3. As remarked in the Introduction, we see from (1) that for $f \in \mathcal{G}$:

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \sin^{-1}\left(\frac{2}{7}\right) \quad \text{for } |z| \leq 1/2.$$

As in the proof of Theorem 2, we in particular have (see Lemma 4)

$$\left|\frac{s_n(z)}{f(z)} - 1\right| < \frac{1}{2^n}\left(\frac{1}{n+1} + \frac{\pi}{3\sqrt{2}}\right) =: K_2 \text{ for } |z| \leq 1/2.$$

It follows that

$$\max_{|z|=1/2} \left|\arg \frac{s_n(z)}{f(z)}\right| \leq \sin^{-1}(K_2)$$

and from the proof of Theorem 2, we have

$$\max_{|z|=1/2} \left|\arg \frac{s_n'(z)}{f'(z)}\right| \leq \sin^{-1}(K_1)$$

where K_1 is defined by (11). This shows that

$$\begin{aligned} \left|\arg \frac{zs_n'(z)}{s_n(z)}\right| &\leq \left|\arg \frac{s_n'(z)}{f'(z)}\right| + \left|\arg \frac{zf'(z)}{f(z)}\right| + \left|\arg \frac{f(z)}{s_n(z)}\right| \\ &< \sin^{-1}(K_1) + \sin^{-1}\left(\frac{2}{7}\right) + \sin^{-1}(K_2), \end{aligned}$$

for $|z| \leq 1/2$. Finally, we see that

$$\left|\arg \frac{zs_n'(z)}{s_n(z)}\right| < \frac{\pi}{2}$$

whenever

$$\sin^{-1}(K_1) + \sin^{-1}\left(\frac{2}{7}\right) + \sin^{-1}(K_2) \leq \frac{\pi}{2}.$$

However, the last inequality is easily seen to be true for $n \geq 12$. □

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