Applied Mathematics and Computation 247 (2014) 689–694

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Univalence of quotient of analytic functions

Milutin Obradović [a], Saminathan Ponnusamy [b,*]

[a] Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia
[b] Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India

ARTICLE INFO

Keywords:
Analytic
Univalent
Radius of univalence

ABSTRACT

Let $\mathcal{A}$ denote the family of all analytic functions f in the unit disk $\mathbb{D}$ with the normalization $f(0) = 0 = f'(0) - 1$. In this note, we mainly consider the radius of univalence of F defined by $F(z) = z^2/f(z)$, where f belongs to some subclasses of $\mathcal{A}$ or $\mathcal{S}$, the class of univalent functions from $\mathcal{A}$. The results of the present article sharpen the earlier known results.

1. Introduction and main results

Let $\mathcal{A}$ denote the family of all analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$, and $\mathcal{S} = \{f \in \mathcal{A} : f$ is univalent in $\mathbb{D}\}$. Determination of the radius of univalence of many geometric subclasses has been a classical problem in geometric function theory which still has many issues to be resolved (see for example [5]). In [11] the authors considered problem of finding the radius of univalency of $A(z)$ defined by the quotient

$$A(z) = \frac{zf(z)}{g(z)}, \tag{1}$$

where f and g are chosen appropriately from some subsets of $\mathcal{A}$ such as $\mathcal{S}$. More precisely, the radius $r \in (0,1)$ was found so that the function $G(z)$ defined by $G(z) = r^{-1}A(rz)$ is in the class $\mathcal{U}$. Here

$$\mathcal{U} = \{f \in \mathcal{A} : |U_f(z)| < 1 \text{ for } z \in \mathbb{D}\},$$

where

$$U_f(z) = f'(z)\left(\frac{z}{f(z)}\right)^2 - 1, \quad z \in \mathbb{D}. \tag{2}$$

See [1] and also [4,8,9,17]. Several generalizations of the class $\mathcal{U}$ were investigated (see eg. [14]). If we choose $f(z) = z$ in (1), then the function $A(z)$ defined above reduces to the simple form

$$A(z) = \frac{z^2}{g(z)} \tag{3}$$

and from the recent paper [11] one can get a number of results for $A(z)$ given by (3). But in certain special situation such as form (3), a direct approach does provide sharp results. The aim of this paper is to deal with such cases, especially when the

* Corresponding author.
E-mail addresses: obrad@grf.bg.ac.rs (M. Obradović), samy@isichennai.res.in, samy@iitm.ac.in (S. Ponnusamy).

http://dx.doi.org/10.1016/j.amc.2014.09.028

function $A(z)$ is given by (3) with appropriate restriction on g. We remark that a general form of $A(z)$ is discussed in [7] but for a different purpose. We begin with some notations and definitions as follows:

$$\mathcal{P}(1/2) = \{f \in \mathcal{A} : \mathrm{Re}(f(z)/z) > 1/2 \text{ for } z \in \mathbb{D}\},$$
$$\mathcal{C}(-1/2) = \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \text{ for } z \in \mathbb{D}\right\} \text{ and}$$
$$\mathcal{G} = \left\{f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \text{ for } z \in \mathbb{D}\right\}.$$

Recall that if $\mathcal{U}_2 = \{f \in \mathcal{U} : f''(0) = 0\}$, then each function in $\mathcal{U}_2$ is known to be included in the class $\mathcal{P}(1/2)$. We also have $\mathcal{K} \subset \mathcal{P}(1/2)$, where $\mathcal{K}$ denotes the class of all functions $f \in \mathcal{S}$ that are convex, i.e. $f(\mathbb{D})$ is a convex domain. The classes $\mathcal{C}(-1/2)$ and $\mathcal{G}$ have been studied recently, for example in [12,15]. Functions in $\mathcal{C}(-1/2)$ are known to be close-to-convex in $\mathbb{D}$. Moreover, Ozaki [16] introduced the class $\mathcal{G}$ and proved that functions in $\mathcal{G}$ are univalent in $\mathbb{D}$. Later in [19], Umezawa discussed a general version of this class. However, functions in $\mathcal{G}$ are proved to be starlike in $\mathbb{D}$, see for eg. [6, Theorem 1] and [13 Example 1, Eq. (16)] (see also [15]).

From now onwards, for obvious reason, it is convenient to work with a change of notation, namely, with the transformation $f \mapsto F_f$ defined by

$$F(z) := F_f(z) = \frac{z^2}{f(z)}, \tag{4}$$

where f is appropriately chosen from $\mathcal{A}$.

Theorem 1. *Let F be defined by (4). Then we have the following:*

(a) $f \in \mathcal{P}(1/2)$ *implies that* $F \in \mathcal{U}$ *in the disk* $|z| < r_1 = \frac{1}{2}$*. As for the univalence the result is sharp.*
(b) $f \in \mathcal{S}$ *implies that* $F \in \mathcal{U}$ *in the disk* $|z| < r_2 = \frac{1}{3}$*, and the result is best possible (as for the univalence).*
(c) $f \in \mathcal{C}(-1/2)$ *implies that* $F \in \mathcal{U}$ *in the disk* $|z| < r_3$*, where* $r_3 = \frac{3-\sqrt{5}}{2} \approx 0.381966$*. The result is also sharp.*
(d) $f \in \mathcal{G}$ *implies that* $F \in \mathcal{U}$ *in the disk* $|z| < r_4$*, where* $r_4 \approx 0.923898$ *is the root of the equation*

$$3r + (2 - r)\log(1 - r) = 0,$$

that lies in the interval $(0, 1)$.

Proof. Consider the function F defined by (4), where we may let for convenience $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The condition on f in each case implies that $f(z)/z \neq 0$ and hence, $F \in \mathcal{A}$ and $z/F(z) \neq 0$ in $\mathbb{D}$. By the definitions of $U_F(z)$ (see (2)) and F, it follows that

$$U_F(z) = -z^2\left(\frac{1}{F(z)} - \frac{1}{z}\right)' = -z\left(\frac{f(z)}{z}\right)' + \frac{f(z)}{z} - 1 = -\sum_{n=3}^{\infty}(n-2)a_n z^{n-1}$$

and therefore,

$$|U_F(z)| \leqslant \sum_{n=3}^{\infty}(n-2)|a_n|\,|z|^{n-1}. \tag{5}$$

(a) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to $\mathcal{P}(1/2)$. Then it is well-known that $|a_n| \leqslant 1$ for $n = 2, 3, \ldots$ and, by (5), we deduce that

$$|U_F(z)| \leqslant \sum_{n=3}^{\infty}(n-2)|z|^{n-1} = \frac{r^2}{(1-r)^2} \text{ for } |z| = r.$$

We thus have, $|U_F(z)| < 1$ if $r^2 < (1-r)^2$, i.e. if $0 \leqslant r < r_1 = 1/2$. This means that $F \in \mathcal{U}$ for $|z| < 1/2$ and so, F is univalent for $|z| < 1/2$. For the function $f(z) = \frac{z}{1-z} \in \mathcal{P}(1/2)$, we see that $F(z) = z - z^2$ and so, $F'(z)$ vanishes at $z = 1/2$ showing that F cannot be univalent in the disk $|z| < r$ if $r > 1/2$. This proves that the radius 1/2 is best possible.

(b) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belong to the class $\mathcal{S}$. Then the de Branges theorem [2] gives that $|a_n| \leqslant n$ for $n = 2, 3, \ldots$. Using this coefficient inequality, the corresponding condition (5) reduces to

$$|U_F(z)| \leqslant \sum_{n=3}^{\infty}(n-2)n|z|^{n-1} = \frac{3r^2 - r^3}{(1-r)^3} \text{ for } |z| = r,$$

which is less than 1 if $0 \leqslant r < r_2 = 1/3$. Thus, $F \in \mathcal{U}$ in the disk $|z| < 1/3$. For the Koebe function $f(z) = z/(1-z)^2$, we obtain that $F(z) = z(1-z)^2 = z - 2z^2 + z^3$ and thus, $F'(z) = 1 - 4z + 3z^2$ which vanishes at $z = 1/3$. It means that the radius 1/3 is best possible.

(c) Suppose that $f \in \mathcal{C}(-1/2)$. Then $|a_n| \leqslant \frac{n+1}{2}$ holds for $n = 2, 3, \ldots$. and in view of this estimate, (5) reduces to

$$|U_F(z)| \leqslant \sum_{n=3}^{\infty}(n-2)\frac{n+1}{2}|z|^{n-1} = \frac{2r^2 - r^3}{(1-r)^3} \text{ for } |z| = r,$$

which is less than 1 if $0 \leqslant r < r_3 = \frac{3-\sqrt{5}}{2}$. Thus, $F \in \mathcal{U}$ in the disk $|z| < r_3$. In order to prove the sharpness part, we consider the function $f(z) = \frac{z(1-\frac{1}{2}z)}{(1-z)^2}$. Then it is easy to see that $f \in \mathcal{C}(-1/2)$ and for this function, we have

$$F(z) = \frac{z(1-z)^2}{1-\frac{z}{2}},$$

so that

$$F'(z) = \frac{(1-z)(1-3z+z^2)}{(1-\frac{z}{2})^2}.$$

We see that $F'(r_3) = 0$ and the desired result follows.

(d) In this case, we let $f \in \mathcal{G}$. Then from a recent result from [12], we have

$$|a_n| \leqslant \frac{1}{(n-1)n} \text{ for } n = 2, 3, \ldots$$

and thus, by (5), we obtain that for $|z| = r$

$$|U_F(z)| \leqslant \sum_{n=3}^{\infty}(n-2)\frac{1}{(n-1)n}|z|^{n-1} = -2 - \frac{2-r}{r}\log(1-r) < 1,$$

if $0 \leqslant r < r_4$, where $r_4 \approx 0.923898$ is the root of the equation

$$3r + (2-r)\log(1-r) = 0,$$

that lies in the interval $(0, 1)$. The proof is complete. □

Theorem 2. *Define*

$$F_1(z) = \frac{2z^2}{f(z) + g(z)}, \tag{6}$$

where $f, g \in \mathcal{A}$. If both f, g are in any one of $\mathcal{P}(1/2), \mathcal{S}, \mathcal{C}(-1/2)$, and $\mathcal{G}$, then the conclusions in each case of Theorem 1 continues to hold for the function F_1 defined by (6).

Proof. First we observe that

$$\begin{aligned}|U_{F_1}(z)| &= \left|-z^2\left(\frac{1}{F_1(z)} - \frac{1}{z}\right)'\right| \\ &= \frac{1}{2}\left|-z^2\left(\frac{f(z)}{z^2} - \frac{1}{z}\right)' - z^2\left(\frac{g(z)}{z^2} - \frac{1}{z}\right)'\right| \\ &= \frac{1}{2}\left|U_{F_f}(z) + U_{F_g}(z)\right| \\ &\leqslant \frac{1}{2}\left(|U_{F_f}(z)| + |U_{F_g}(z)|\right)\end{aligned}$$

and the desired conclusions follow if we apply the proof of Theorem 1. □

Theorem 3. *Let $f \mapsto G_f$ be the transformation defined by the quotient*

$$G(z) = G_f(z) = \frac{z^2}{\int_0^z \frac{t}{f(t)}\,dt}. \tag{7}$$

Then we have the following:

(a) $f \in \mathcal{U}$ implies that $G \in \mathcal{U}$ in the disk $|z| < 1$. In particular, G is univalent in $\mathbb{D}$.

(b) $f \in \mathcal{S}$ implies that $G \in \mathcal{U}$ in the disk $|z| < r_5$, where $r_5 \approx 0.969845$ is the root of the equation

$$\log(1-r^2) - 2\int_0^{r^2} \frac{\log(1-t)}{t}\, dt = 0,$$

that lies in the interval $(0,1)$.

Proof.

(a) Let $f \in \mathcal{U}$. Then f can be written as

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots \tag{8}$$

and therefore, we have

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| = \left|-z\left(\frac{z}{f(z)}\right)' + \frac{z}{f(z)} - 1\right| = \left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right| \leqslant 1.$$

Therefore, as in [10], we let $z = re^{i\theta}$ for $r \in (0,1)$ and apply the Parseval formula to obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=2}^{\infty}(n-1)b_n z^n\right|^2 d\theta = \sum_{n=2}^{\infty}(n-1)^2|b_n|^2 r^{2n} \leqslant 1.$$

Allowing $r \to 1^-$, we obtain the inequality $\sum_2^{\infty}(n-1)^2|b_n|^2 \leqslant 1$. Next, we let $g(z) = \int_0^z \frac{t}{f(t)}\, dt$. Then we have

$$g(z) = z + \sum_{n=2}^{\infty}\frac{b_{n-1}}{n} z^n,$$

so that

$$\frac{z}{G(z)} = 1 + \sum_{n=1}^{\infty}\frac{b_n}{n+1} z^n.$$

Using the method in the proof of Theorem 1 and the inequality (5), it follows easily that

$$|U_G(z)| \leqslant \sum_{n=3}^{\infty}(n-2)\frac{|b_{n-1}|}{n}|z|^{n-1} = \sum_{n=2}^{\infty}(n-1)\frac{|b_n|}{n+1}|z|^n$$

and, since

$$\sum_{n=2}^{\infty}\frac{1}{(n+1)^2} < \sum_{n=2}^{\infty}\frac{1}{(n+1)n} = \sum_{n=2}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2},$$

we deduce by the Cauchy–Schwarz inequality that

$$|U_G(z)| \leqslant \left(\sum_{n=2}^{\infty}(n-1)^2|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}\frac{1}{(n+1)^2}\right)^{\frac{1}{2}} < 1,$$

which means that $G \in \mathcal{U}$.

(b) Consider the case $f \in \mathcal{S}$. Again, f can be written in the form (8), and therefore, from the well-known Area Theorem [5, Theorem 11, p. 193, vol. 2], we have

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leqslant 1.$$

Thus, as in Case (a), Cauchy–Schwarz inequality yields that

$$\begin{aligned}|U_G(z)| &\leqslant \sum_{n=2}^{\infty}(n-1)\frac{|b_n|}{n+1}|z|^n \\ &\leqslant \left(\sum_{n=2}^{\infty}(n-1)|b_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}\frac{n-1}{(n+1)^2}|z|^{2n}\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{n=2}^{\infty}\frac{n-1}{(n+1)^2}|z|^{2n}\right)^{\frac{1}{2}}.\end{aligned}$$

Since

$$\sum_{n=1}^{\infty}\frac{1}{n^2}x^n = -\int_0^x \frac{\log(1-t)}{t}\,dt,$$

it is a simple exercise to see that

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{n-1}{(n+1)^2}|z|^{2n} &= \sum_{n=2}^{\infty}\frac{1}{n+1}|z|^{2n} - 2\sum_{n=2}^{\infty}\frac{1}{(n+1)^2}|z|^{2n}\\ &= 1-\frac{1}{|z|^2}\left(\log(1-|z|^2)-2\int_0^{|z|^2}\frac{\log(1-t)}{t}\,dt\right)\end{aligned}$$

and therefore, from the previous inequality, we find that $|U_G(z)|<1$ if

$$\log(1-|z|^2)-2\int_0^{|z|^2}\frac{\log(1-t)}{t}\,dt = \log(1-|z|^2)+2\sum_{n=1}^{\infty}\frac{1}{n^2}|z|^{2n} > 0.$$

This gives the condition $|z|<r_5$, where $r_5\approx 0.969845$ is the root of the equation

$$\log(1-r^2)-2\int_0^{r^2}\frac{\log(1-t)}{t}\,dt = 0,$$

that lies in the interval $(0,1)$ $\square$.

Our next result is related to a transform h_f of $f\in\mathcal{S}$ introduced by Danikas and Ruscheweyh [3]:

$$h_f(z) := \int_0^z \frac{tf'(t)}{f(t)}\,dt = z+\sum_{n=2}^{\infty}\frac{n}{n+1}c_n(f)z^{n+1},$$

where $c_n(f)$ $(n\geqslant 1)$ denote the logarithmic coefficients of f. It was conjectured that the transform $h_f\in\mathcal{S}$ for each $f\in\mathcal{S}$. This conjecture remains open.

Theorem 4. *Let $f\in\mathcal{S}$, $a_2=f''(0)/2!$, and let H be defined by the quotient*

$$H(z)=\frac{z^2}{\int_0^z \frac{tf'(t)}{f(t)}\,dt}. \tag{9}$$

Then $H\in\mathcal{U}$ in the disk $|z|<r_6(b)$, where $r_6(b)\geqslant r_6(0)\approx 0.557666$ is the root of the equation

$$(4(a+1)-b^2)r^6+(4(a-3)-b^2)r^4+4(3r^2-1)=0, \tag{10}$$

in $0\leqslant b\leqslant 2$ that lies in the interval $(0,1)$ and where $a=\frac{2\pi^2-12}{3}$.

For a ready reference, the roots of Eq. (10) for various values of b in the unit interval $[0,2]$ are displayed in Table 1.

Proof of Theorem 4. Consider the function $h_f(z)$ defined as above. Then, we see that

$$h_f(z)=\int_0^z\left(1+t\left(\log\frac{f(t)}{t}\right)'\right)dt = z+\sum_{n=2}^{\infty}\frac{n-1}{n}c_{n-1}(f)z^n,$$

where $c_n(f)$ $(n\geqslant 1)$ denote the logarithmic coefficients of $f\in\mathcal{S}$ defined by

Table 1
Values of $r_6(b|)$ when $b\in[0,2]$.

Values of b	Roots of Eq. (10)
0	0.557666
1/4	0.558138
1/2	0.55957
3/4	0.562015
1	0.565569
5/4	0.570383
3/2	0.57669
7/4	0.584845
2	0.595415

$$\log\frac{f(z)}{z} = \sum_{n=1}^{\infty} c_n(f) z^n.$$

Note that

$$\frac{z}{H(z)} = 1 + \sum_{n=1}^{\infty} \frac{n}{n+1} c_n(f) z^n$$

and $c_1(f) = a_2 = f''(0)/2$. Further, for the function $f \in \mathcal{S}$ the following sharp inequality is known from [18]:

$$\sum_{1}^{\infty} \left(\frac{n}{n+1}\right)^2 |c_n(f)|^2 \leqslant \frac{2\pi^2 - 12}{3} = a \approx 2.57974.$$

Again, as before, it follows easily that

$$\begin{aligned} |U_H(z)| &\leqslant \sum_{n=3}^{\infty} (n-2)\frac{n-1}{n} |c_{n-1}(f)|\, |z|^{n-1} = \sum_{n=2}^{\infty} (n-1)\frac{n}{n+1} |c_n(f)|\, |z|^n \\ &\leqslant \left(\sum_{n=2}^{\infty} \left(\frac{n}{n+1}\right)^2 |c_n(f)|^2\right)^{\frac{1}{2}} \left(\sum_{n=2}^{\infty} (n-1)^2 |z|^{2n}\right)^{\frac{1}{2}} \\ &\leqslant \left(a - \frac{1}{4}|c_1(f)|^2\right)^{\frac{1}{2}} \left(\frac{|z|^4(1+|z|^2)}{(1-|z|^2)^3}\right)^{\frac{1}{2}}, \end{aligned}$$

which is less than 1 whenever,

$$\left(a - \frac{1}{4}|c_1(f)|^2\right)|z|^4(1+|z|^2) < (1-|z|^2)^3.$$

Simplifying this inequality gives

$$B(|a_2|, r) := 4(a+1)r^6 + 4(a-3)r^4 + 4(3r^2 - 1) - (|a_2|^2 r^2 + |a_2|^2) r^4 < 0,$$

where $r = |z|$. Note that if $r_6(b)$ $(0 \leqslant b = |a_2| \leqslant 2)$ is the root of the Eq. (10), i.e. $B(b, r_6(b)) = 0$, then clearly $r_6(b) \geqslant r_6(0)$, where $r_6(0) = r_6 \approx 0.557666$ is the root of the equation

$$(a+1)r^6 + (a-3)r^4 + 3r^2 - 1 = 0,$$

that lies in the interval $(0, 1)$. The result follows. □

Acknowledgements

The work of the first author was supported by MNZZS Grant No. ON174017, Serbia. The second author is currently on leave from the Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India.

References

[1] L.A. Aksentév, Sufficient conditions for univalence of regular functions, Izv. Vysš. Učebn. Zaved. Matematika 1958 (4) (1958) 3–7 (Russian).
[2] L. de Branges, A proof of the Bieberbach conjecture, Acta Math. 154 (1985) 137–152.
[3] N. Danikas, St. Ruscheweyh, Semi-convex hulls of analytic functions in the unit disk, Analysis 4 (1999) 309–318.
[4] R. Fournier, S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex Variables Elliptic Equ. 52 (1) (2007) 1–8.
[5] A.W. Goodman, Univalent Functions, vols. 1–2, Mariner, Tampa, Florida, 1983.
[6] I. Jovanović, M. Obradović, A note on certain classes of univalent functions, Filomat 9 (1) (1995) 69–72.
[7] Y.C. Kim, T. Sugawa, On power deformations of univalent functions, Monatsh. Math. 167 (2) (2012) 231–240.
[8] M. Obradović, S. Ponnusamy, New criteria and distortion theorems for univalent functions, Complex Variables Theory Appl. 44 (2001) 173–191.
[9] M. Obradović, S. Ponnusamy, Univalence and starlikeness of certain integral transforms defined by convolution of analytic functions, J. Math. Anal. Appl. 336 (2007) 758–767.
[10] M. Obradović, S. Ponnusamy, A class of univalent functions defined by a differential inequality, Kodai Math. J. 34 (2011) 169–178.
[11] M. Obradović, S. Ponnusamy, N. Tuneski, Radius of Univalence of certain combination of univalent and analytic functions, Bull. Malays. Math. Sci. Soc. 35 (2) (2012) 325–334.
[12] M. Obradović, S. Ponnusamy, K.-J. Wirths, Coefficient characterizations and sections for some univalent functions, Sib. Math. J. 54 (4) (2013) 679–696.
[13] S. Ponnusamy, S. Rajasekaran, New sufficient conditions for starlike and univalent functions, Soochow J. Math. 21 (2) (1995) 193–201.
[14] S. Ponnusamy, P. Sahoo, Special classes of univalent functions with missing coefficients and integral transforms, Bull. Malays. Math. Sci. Soc. 28 (2) (2005) 141–156.
[15] S. Ponnusamy, A. Vasudevarao, Region of variability of two subclasses of univalent functions, J. Math. Anal. Appl. 332 (2) (2007) 1322–1333.
[16] S. Ozaki, On the theory of multivalent functions II, Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4 (1941) 45–87.
[17] S. Ozaki, M. Nunokawa, The Schwarzian derivative and univalent functions, Proc. Am. Math. Soc. 33 (1972) 392–394.
[18] O. Roth, A sharp inequality for the logarithmic coefficients of univalent functions, Proc. Am. Math. Soc. 138 (7) (2007) 2051–2054.
[19] T. Umezawa, Analytic functions convex in one direction, J. Math. Soc. Jpn. 4 (1952) 194–202.