On relations between the classes $ ${\mathcal {S}}$$ S and $${\mathcal {U}}$ $ U

Milutin Obradović, Saminathan Ponnusamy & Karl-Joachim Wirths

The Journal of Analysis

ISSN 0971-3611

J Anal
DOI 10.1007/s41478-016-0023-4

J Anal
DOI 10.1007/s41478-016-0023-4

ORIGINAL RESEARCH PAPER

On relations between the classes $\mathcal{S}$ and $\mathcal{U}$

Milutin Obradović[1] · **Saminathan Ponnusamy**[2] ·
Karl-Joachim Wirths[3]

Received: 30 October 2016 / Accepted: 28 November 2016

Abstract Let $\mathcal{A}$ denote the family of all functions f analytic in the unit disk $\mathbb{D}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. We consider the subclass $\mathcal{U}$ of $\mathcal{S}$ that is defined by the condition that for its members f the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1 \text{ for } z \in \mathbb{D}$$

holds. To these relations belong striking similarities and on the other hand big differences. We show that some results about $\mathcal{S}$ can be improved for $\mathcal{U}$, while others cannot.

Keywords Analytic and univalent functions · Subordination · Koebe transforms · Grunsky inequalities

---

Dedicated to Professor David Minda on the occasion of his retirement from the University of Cincinnati. The authors have sweet memories of his friendliness in our conversation with him.

✉ Saminathan Ponnusamy
samy@isichennai.res.in; samy@iitm.ac.in

Milutin Obradović
obrad@grf.bg.ac.rs

Karl-Joachim Wirths
kjwirths@tu-bs.de

[1] Department of Mathematics, Faculty of Civil Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia

[2] Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India

[3] Institut für Analysis und Algebra, TU Braunschweig, Pockelsstr. 14, 38106 Braunschweig, Germany

Mathematics Subject Classification Primary 30C45

1 Introduction and statement of results

Let $\mathcal{A}$ denote the family of all functions f analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. We consider relationships between $\mathcal{S}$ and its subclass $\mathcal{U}$ that is defined by the condition that for its members f the condition

$$\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1 \text{ for } z \in \mathbb{D} \tag{1.1}$$

holds. It was proved by Aksentév (1958) that $\mathcal{U} \subset \mathcal{S}$. Typical members of the class $\mathcal{U}$ are

$$\frac{z}{1 \pm z}, \frac{z}{1 \pm z^2}, \frac{z}{(1 \pm z)^2}, \frac{z}{1 \pm z + z^2},$$

and their rotations. The class $\mathcal{U}$ and its various generalizations have been studied recently. In particular, the class $\mathcal{U}$ is preserved under rotation, conjugation, dilation and omitted-value transformations but is not preserved under the square-root transformation, for example, see Obradović et al. (2016) and the references therein.

In the present paper we consider some problems, where the solutions are identical for $\mathcal{S}$ and $\mathcal{U}$ and some others where there exist differences.

The first problem we address is the question for the maximum radius of the circle around the origin wherein $\mathrm{Re}(f(z)/z) > 1/2$. The solution for the class $\mathcal{S}$ was presented by Singh (1985) and Wirths (1989) as follows: If $f \in \mathcal{S}$, then

$$\mathrm{Re}\left(\frac{f(z)}{z} \right) > \frac{1}{2} \tag{1.2}$$

if $|z| < \sqrt{2} - 1$. This bound is best possible.

It is worth recalling that if $f \in \mathcal{S}$ is convex, or starlike of order 1 / 2, or $f \in \mathcal{A}$ such that the Taylor coefficients of f are real and convex decreasing, then the condition (1.2) holds in the full disk $\mathbb{D}$. Secondly, since $\mathcal{U} \subsetneq \mathcal{S}$, (1.2) holds for the class $\mathcal{U}$, too. Indeed, the Koebe function $k(z) = z/(1 - z)^2$ belongs to the class $\mathcal{U}$ and the equation $r^{-1}k(r) = 1/2$ with $r = 1 - \sqrt{2}$ as well as the considerations by Singh (1985) show that the result (1.2) is still the best possible for the class $\mathcal{U}$.

The situation changes significantly if one considers the similar problem asking where

$$\mathrm{Re}\left(\sqrt{\frac{f(z)}{z}} \right) > \frac{1}{2} \tag{1.3}$$

is valid. The following result was proved by Duren and Schober (1971).

Theorem A *For each $f \in \mathcal{S}$, the inequality* (1.3) *holds for $|z| < R$, where $R = 0.835\ldots$ is the best possible radius. Moreover, for each z* in *$|z| > R$, there exists an $f_0 \in \mathcal{S}$ for which* (1.3) *fails to hold.*

Concerning the same question for the class $\mathcal{U}$, we may recall the following result of Obradović (1995):

$$f \in \mathcal{U} \Rightarrow \frac{z}{f(z)} \prec (1-z)^2, \text{ i.e. } \mathrm{Re}\left(\sqrt{\frac{f(z)}{z}}\right) > \frac{1}{2} \tag{1.4}$$

is valid for $z \in \mathbb{D}$ (see also, Obradović et al. 2016). Here $\prec$ denotes the usual subordination (cf. Miller and Mocanu 2000; Obradović et al. 2016; Pommerenke 1975).

In the following we will generalize the implication (1.4) for the class $\mathcal{U}_n := \mathcal{A}_n \cap \mathcal{U}$, where $\mathcal{A}_n$, $n \geq 1$, denotes the class of functions $f \in \mathcal{A}$ of the form

$$f(z) = z + a_{n+1}z^{n+1} + \cdots.$$

Theorem 1 *If $f \in \mathcal{U}_n$, then*

$$\mathrm{Re}\left(\frac{f(z)}{z}\right)^{\frac{n}{2}} > \frac{1}{2} \quad \textit{for } z \in \mathbb{D}. \tag{1.5}$$

For $n = 1$, it is a simple corollary to Theorem A that this stands in contrast to the situation in the class $\mathcal{S}$. Choose the function $f_0 \in \mathcal{S}$ and the number z_0 as indicated in Theorem A and let $f_0(z) = zh_0(z)$. Then we have

$$\mathrm{Re}\left(\sqrt{h_0(z_0)}\right) = \frac{1}{2}.$$

Let further

$$g_0(z) = z\sqrt[n]{\frac{f_0(z^n)}{z^n}}$$

and choose z_1 such that $z_1^n = z_0$, where z_0 is a complex number such that $|z_0| = R = 0.835\ldots$ and thus, $|z_1| = \sqrt[n]{|z_0|} = \sqrt[n]{0.835\ldots}$. Then $g_0(z) \in \mathcal{S} \cap \mathcal{A}_n$ and

$$\mathrm{Re}\left(\frac{g_0(z_1)}{z_1}\right)^{\frac{n}{2}} = \mathrm{Re}\left(\sqrt{h_0(z_0)}\right) = \frac{1}{2}.$$

Another item where one can see as well similarities as differences between the two classes in question is the problem of Koebe transforms. For $f \in \mathcal{S}$, we define the Koebe transform with respect to the point $\zeta \in \mathbb{D}$ as

$$g(z) := g(\zeta, z) = \frac{f\left(\frac{\zeta+z}{1+\bar{\zeta}z}\right) - f(\zeta)}{f'(\zeta)(1-|\zeta|^2)}.$$

Then it is well known that these Koebe transforms as functions of the variable z are all members of the class $\mathcal{S}$.

For the class $\mathcal{U}$ we prove

Theorem 2 *Let* $f \in \mathcal{U}$. *Then the Koebe transforms of* f *with respect to any fixed* ζ, *i.e. the functions* $z \mapsto g(z)$ *as above, belong to* $\mathcal{U}$ *if and only if*

$$\left| \frac{(\zeta - u)^2 f'(\zeta) f'(u)}{(f(u) - f(\zeta))^2} - 1 \right| < 1, \quad \zeta, u \in \mathbb{D}. \tag{1.6}$$

Remarkably, the disk with center at the origin, wherein (1.6) is satisfied for all members of the class, is the same for the classes $\mathcal{S}$ and $\mathcal{U}$. Finally, we also prove

Theorem 3 *Let* $f \in \mathcal{S}$ *or* $f \in \mathcal{U}$. *Then the inequality* (1.6) *is satisfied for* $|\zeta|, |u| < \sqrt{2} - 1$. *The result is best possible in both cases.*

We note that it might be worthwhile to consider those functions that satisfy the condition of Theorem 2.

The proofs of Theorems 1, 2 and 3 will be presented in Section 2.

2 Proofs of Theorems 1, 2 and 3

The following lemma due to Miller and Mocanu (1978) is needed for the proof of Theorem 1. See the monograph of Miller and Mocanu (2000) for a general formulation of this lemma via differential subordination.

Lemma B (Miller and Mocanu 1978) *Suppose that* $\psi : \mathbb{C}^2 \to \mathbb{C}$ *is continuous in a domain* D *of* $\mathbb{C}^2$ *such that* $(1,0) \in D$, $\mathrm{Re}\,\psi(1,0) > 0$ *and*

$$\mathrm{Re}\,\psi(ix, y) \le 0 \text{ for all } (ix, y) \in D \text{ and } y \le -n(1+x^2)/2,$$

where $n \ge 1$. *Let* $p(z) = 1 + p_n z^n + \cdots$ *be analytic in* $\mathbb{D}$ *and* $p(z) \not\equiv 1$. *If* $(p(z), zp'(z)) \in D$ *for all* $z \in \mathbb{D}$ *and* $\mathrm{Re}\,\psi(p(z), zp'(z)) > 0$ *for all* $z \in \mathbb{D}$, *then* $\mathrm{Re}\,p(z) > 0$ *in* $\mathbb{D}$.

2.1 Proof of Theorem 1

For $n = 1$, the result is the content of the implication (1.4). For $n = 2$ (i.e. when $a_2 = 0$), the appropriate result is given in the paper by Obradović and Ponnusamy (2011) but the same may be obtained from the proof that follows now.

Let $f \in \mathcal{U}$. Then (1.1) holds, or equivalently

$$\mathrm{Re}\left(2\left(\frac{f(z)}{z}\right)^2\frac{1}{f'(z)}-1\right)>0 \text{ for } z\in\mathbb{D}.$$

We now introduce

$$p(z)=2\left(\frac{f(z)}{z}\right)^{\frac{n}{2}}-1. \tag{2.1}$$

Clearly, p is analytic in $\mathbb{D}$ and has the form $p(z)=1+p_nz^n+\cdots$. We shall apply Lemma B. and prove that $\mathrm{Re}\,p(z)>0$ for $z\in\mathbb{D}$. From (2.1) we have

$$\frac{f(z)}{z}=\left(\frac{p(z)+1}{2}\right)^{\frac{2}{n}}$$

and a computation gives that

$$2\left(\frac{f(z)}{z}\right)^2\frac{1}{f'(z)}-1 \;=:\; \psi(p(z),zp'(z)),$$

where

$$\psi(r,s)=\frac{2n\left(\frac{r+1}{2}\right)^{\frac{2}{n}}(r+1)}{n(r+1)+2s}-1. \tag{2.2}$$

According to Lemma B, to prove $\mathrm{Re}\,p(z)>0$ in $\mathbb{D}$, it suffices to show that

$$\mathrm{Re}\,\psi(ix,y)\le 0 \text{ for all reals } x,y \text{ with } \mathrm{y}\le-\mathrm{n}(1+\mathrm{x}^2)/2. \tag{2.3}$$

It follows that

$$\mathrm{Re}\,\psi(ix,y)=\mathrm{Re}\,\frac{2n\left(\frac{ix+1}{2}\right)^{\frac{2}{n}}(ix+1)}{n(ix+1)+2y}-1. \tag{2.4}$$

We may use the representation $ix+1=(\sqrt{1+x^2})\,e^{i\varphi}$, $|\varphi|<\frac{\pi}{2}$, where

$$\cos\varphi=\frac{1}{\sqrt{1+x^2}},\ \sin\varphi=\frac{x}{\sqrt{1+x^2}} \text{ and } \tan\varphi=x. \tag{2.5}$$

Clearly, $x\sin\varphi\ge0$ and, since $n\ge2$, $x\sin\left(\frac{2}{n}\varphi\right)\ge0$.

By using (2.4) and (2.5), after some simple transformations, we obtain that

$$\mathrm{Re}\,\psi(ix,y)=\frac{S-T}{(n+2y)^2+n^2x^2},$$

where

$$S = 2n\left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}\cos\left(\frac{2}{n}\varphi\right)(n+2y+nx^2)$$

and

$$T = 4y^2 + 4n\left(1+\left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}x\sin\left(\frac{2}{n}\varphi\right)\right)y + n^2(1+x^2). \tag{2.6}$$

Clearly $S \le 0$ for all $n \ge 2$ and for $y \le -(n/2)(1+x^2)$. Thus, we also need to prove that $T := T(y) \ge 0$ for $n \ge 2$ and for all $x \in \mathbb{R}$ and $y \le -(n/2)(1+x^2)$. The function $T(y)$ has its minimum value at the point

$$y_0 = -\frac{n}{2}\left(1+\left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}x\sin\left(\frac{2}{n}\varphi\right)\right)$$

so that $T(y) \ge T(y_0)$. Since

$$\left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}x\sin\left(\frac{2}{n}\varphi\right) \le \left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}x\sin\varphi = \frac{x^2}{2}\left(\frac{4}{1+x^2}\right)^{\frac{1}{2}-\frac{1}{n}} \le x^2,$$

for $n \ge 2$, we easily conclude that $-(n/2)(1+x^2) \le y_0$. As $T(y)$ is decreasing when $y \le y_0$, it is enough to prove that

$$T\left(-\frac{n}{2}(1+x^2)\right) = n^2(1+x^2)\left[x^2 - 2\left(\frac{1+x^2}{4}\right)^{\frac{1}{n}}x\sin\left(\frac{2}{n}\varphi\right)\right] \ge 0 \tag{2.7}$$

for all $x \in \mathbb{R}$ and $n \ge 2$. Since, by the previous consideration, $x\sin\left(\frac{2}{n}\varphi\right) \ge 0$, we can suppose that $x \ge 0$ and $0 \le \varphi < \frac{\pi}{2}$. In view of this observation, proving the inequality (2.7) is equivalent to proving the inequality

$$\sin\left(\frac{2}{n}\varphi\right) \le \frac{x}{2}\left(\frac{4}{1+x^2}\right)^{\frac{1}{n}} \text{ for } x \ge 0, 0 \le \varphi < \pi/2, \text{ and } n \ge 2. \tag{2.8}$$

For $n = 2$, we have equality in (2.8) [(by using (2.5)]. Again, from (2.5), we obtain that $\sin^2\varphi = x^2/(1+x^2)$ and $x = \tan\varphi$, and thus the inequality (2.8) is equivalent to the inequality

$$g(\varphi) \ge g(0) = 0 \quad \text{for} \quad 0 \le \varphi < \pi/2 \text{ and } \mathrm{n} \ge 2, \tag{2.9}$$

where

$$g(\varphi) = (2\cos\varphi)^{\frac{2}{n}-1}\sin\varphi - \sin\left(\frac{2}{n}\varphi\right).$$

We find that

$$g'(\varphi) = 2(2\cos\varphi)^{\frac{2}{n}-2}\left(1 - \frac{2}{n}\sin^2\varphi\right) - \frac{2}{n}\cos\left(\frac{2}{n}\varphi\right)$$

and thus,

$$g'(0) = 2\left(\frac{1}{2^{2-\frac{2}{n}}} - \frac{1}{n}\right) > 0 \quad \text{for} \quad n \geq 3.$$

Also, a computation gives that

$$g''(\varphi) = 8(2\cos\varphi)^{\frac{2}{n}-3}\left(\left(1 - \frac{3}{n}\right)\sin\varphi + \frac{2}{n^2}\sin^3\varphi\right) + \frac{4}{n^2}\sin\left(\frac{2}{n}\varphi\right) \geq 0 \quad \text{for} \quad n \geq 3.$$

It means that the function g' is an increasing function of φ and this gives

$$g'(\varphi) \geq g'(0) > 0 \quad \text{for} \quad 0 \leq \varphi < \pi/2$$

which in turn implies that the function $g(\varphi)$ is also increasing for $0 \leq \varphi < \pi/2$ and hence, (2.9) holds. This means that (2.3) holds and hence, by Lemma B, it follows that $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. The proof of the theorem is complete. ☐

For the proof of Theorem 2, we need the following lemma, which might have been known in the literature. Since we were not able to find an apt reference we give the proof for this theorem. We want to emphasize here that the functions considered in this lemma are neither conformal maps nor harmonic functions.

Lemma 1 *Let for* $z \in \mathbb{D}$,

$$u(\zeta) = \frac{z + \zeta}{1 + \overline{\zeta} z}.$$

Then $u : \mathbb{D} \to \mathbb{D}$ *and* $u : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ *are bijective.*

Proof The injectivity is easily derived from

$$u(\zeta_1) - u(\zeta_2) = \frac{(\zeta_1 - \zeta_2)(1 + \overline{\zeta_1} z) - (\overline{\zeta_1} - \overline{\zeta_2})(\zeta_1 z + z^2)}{(1 + \overline{\zeta_1} z)(1 + \overline{\zeta_2} z)}.$$

If this difference equals zero and $\zeta_1 \neq \zeta_2$, then

$$|1 + \overline{\zeta_1} z| = |\zeta_1 z + z^2|, \quad \text{i.e.,} \quad (1 - |z|^2)(1 + |z|^2 + 2\operatorname{Re}(\overline{\zeta_1} z)) = 0,$$

which is impossible for $z \in \mathbb{D}$.

Further the functional determinant

$$\left|\frac{\partial u}{\partial \zeta}\right|^2 - \left|\frac{\partial u}{\partial \overline{\zeta}}\right|^2 = \frac{|1 + \overline{\zeta} z|^2 - |\zeta z + z^2|^2}{|1 + \overline{\zeta} z|^4}$$

does not equal zero for $\zeta \in \mathbb{D}$. Hence $u(\mathbb{D})$ is open. Further it is easily seen that $u(\mathbb{D}) \subset \mathbb{D}$ and $u(\partial\mathbb{D}) = \partial\mathbb{D}$.

Now assume that $\mathbb{D} \setminus u(\mathbb{D})$ is non-void and not open. Then there exists a point $p \in \mathbb{D} \setminus u(\mathbb{D})$ and a sequence $\{\zeta_n\}_{n\geq 1}$ in $\mathbb{D}$, such that

$$p = \lim_{n\to\infty} u(\zeta_n).$$

Let $\{\zeta_{n_k}\}_{k\geq 1}$ be a convergent subsequence of $\{\zeta_n\}$. Because of the continuity of u and $u(\partial\mathbb{D}) = \partial\mathbb{D}$, we have $\lim_{k\to\infty} \zeta_{n_k} = w \in \mathbb{D}$ and

$$p = \lim_{k\to\infty} u(\zeta_{n_k}) = u(w).$$

Hence, $\mathbb{D}\setminus u(\mathbb{D})$ is void or open. The second possibility contradicts the connectivity of $\mathbb{D}$. Together with the above this proves the assertions. ☐

2.2 Proof of Theorem 2

Let f belong to $\mathcal{U}$ and for fixed $\zeta \in \mathbb{D}$, consider its Koebe transforms $g(z)$ with respect to ζ given by

$$g(z) := g(\zeta, z) = \frac{f(u(\zeta)) - f(\zeta)}{f'(\zeta)(1 - |\zeta|^2)}, \quad u(\zeta) = \frac{z + \zeta}{1 + \overline{\zeta} z}.$$

If all Koebe transforms of f belong to $\mathcal{U}$, then by (1.1) we have

$$\left|\frac{z^2 f'(\zeta)(1 - |\zeta|^2)^2}{(f(u(\zeta)) - f(\zeta))^2} \frac{f'(u(\zeta))}{(1 + \overline{\zeta} z)^2} - 1\right| = \left|\frac{(\zeta - u(\zeta))^2 f'(\zeta) f'(u(\zeta))}{(f(u(\zeta)) - f(\zeta))^2} - 1\right| < 1$$

for all $u, \zeta \in \mathbb{D}$. According to Lemma 1 this proves the necessity of the above condition. The sufficiency can be proved similarly. ☐

2.3 Proof of Theorem 3

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(z) - f(u)}{z - u} = \sum_{n,m=0}^{\infty} d_{n,m} z^n u^m. \tag{2.10}$$

The coefficients $d_{n,m}$ are called the Grunsky coefficients of the function f. From (2.10), after differentiations with respect to z and u, we have

$$\frac{f'(z) f'(u)}{(f(z) - f(u))^2} - \frac{1}{(z - u)^2} = \sum_{n,m=1}^{\infty} n m d_{n,m} z^{n-1} u^{m-1},$$

and from here

$$\frac{(z-u)^2 f'(z)f'(u)}{(f(z)-f(u))^2} - 1 = (z-u)^2 \sum_{n,m=1}^{\infty} nmd_{n,m} z^{n-1} u^{m-1}. \tag{2.11}$$

By using Grunsky's inequalities (see Pommerenke 1975, p. 62)

$$\sum_{n=1}^{\infty} n \left| \sum_{m=1}^{\infty} d_{nm} x_m \right|^2 \leq \sum_{n=1}^{\infty} \frac{|x_n|^2}{n}, \tag{2.12}$$

if the last series converges and for arbitrary $x_n,\ n = 1, 2, \ldots$ (We note that Grunsky's inequality usually is stated with the functions from the class Σ, but it is easy to prove that Grunsky's coefficients for the functions $\log \frac{f(z)-f(u)}{z-u}$ and $\log \frac{F(z^{-1})-F(u^{-1})}{z^{-1}-u^{-1}}$, where $F(\zeta) = \frac{1}{f(1/\zeta)} \in \Sigma$ for $f \in \mathcal{S}$, are the same for $n, m \geq 1$.) we can obtain that

$$\begin{aligned}
\left| \sum_{n,m=1}^{\infty} nmd_{n,m} z^{n-1} u^{m-1} \right| &= \left| \sum_{n=1}^{\infty} \sqrt{n} z^{n-1} \sqrt{n} \sum_{m=1}^{\infty} d_{n,m} m u^{m-1} \right| \\
&\leq \left(\sum_{n=1}^{\infty} n|z|^{2(n-1)} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n \left| \sum_{m=1}^{\infty} d_{n,m} m u^{m-1} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{1-|z|^2} \left(\sum_{n=1}^{\infty} n|u|^{2(n-1)} \right)^{\frac{1}{2}} \\
&= \frac{1}{(1-|z|^2)(1-|u|^2)}.
\end{aligned}$$

From this and (2.11) we finally have

$$\left| \frac{(z-u)^2 f'(z)f'(u)}{(f(z)-f(u))^2} - 1 \right| \leq \frac{|z-u|^2}{(1-|z|^2)(1-|u|^2)} \leq \left(\frac{2r}{1-r^2} \right)^2 < 1,$$

since $|z|,\ |u| \leq r < \sqrt{2}-1$.

To prove that this result is sharp for $\mathcal{U}$ and $\mathcal{S}$ we consider the Koebe function k that belongs to both classes. A simple calculation reveals that for $f = k$, (1.6) becomes

$$\left| \frac{u-\zeta}{1-u\zeta} \right| < 1.$$

For $\zeta = (\sqrt{2}-1)i$ and $u = -(\sqrt{2}-1)i$,

$$\left| \frac{u-\zeta}{1-u\zeta} \right| = 1.$$

This implies that $\sqrt{2}-1$ is best possible. □

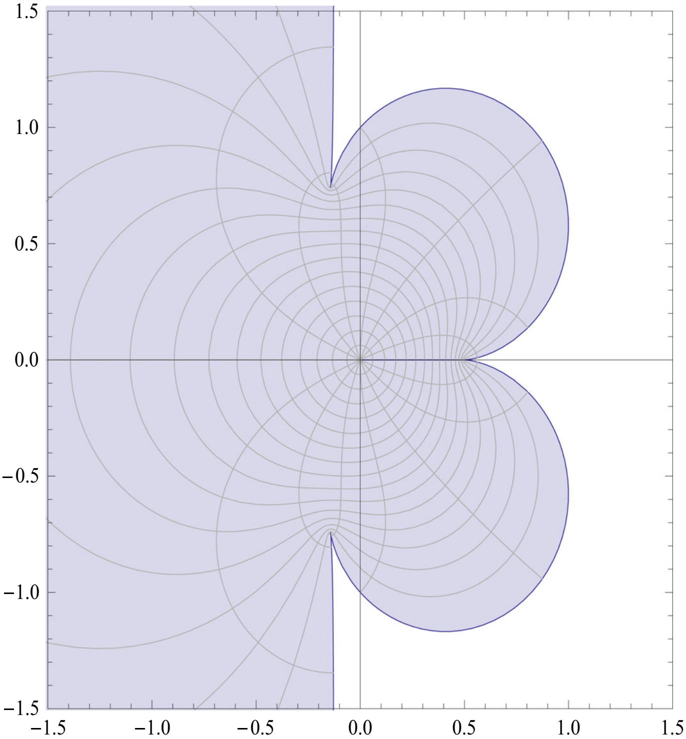

Fig. 1 The image of f_1 under $\mathbb{D}$

3 Concluding remarks

A natural question is the following: Are all functions

$$f(z) = z\prod_{k=1}^{n}(1 - e^{i\theta_k}z)^{-\alpha_k}, \theta_k \in \mathbb{R}, \alpha_k \geq 0 \text{ and } \sum_{k=1}^{n}\alpha_k = 2,$$

in the class $\mathcal{U}$? The answer is no as the function $f_0(z) = z(1 - z^3)^{-2/3}$ demonstrates. Note that

$$\frac{z}{f_0(z)} = (1 - z)^{2/3}(1 + e^{-i\pi/3}z)^{2/3}(1 + e^{i\pi/3}z)^{2/3}.$$

Moreover, if $f \in \mathcal{S}$ then $r^{-1}f(rz) \in \mathcal{U}$ for $0 < r \leq 1/\sqrt{2}$ and the result is sharp. See Obradović and Ponnusamy (2005). Furthermore, the family $\mathcal{U}$ is not a subset of the class $\mathcal{S}^\star$ of univalent starlike functions in the unit disk $\mathbb{D}$. In fact, consider the function

$$f_1(z) = \frac{z}{1 + \frac{1}{2}z + \frac{1}{2}z^3}.$$

Then it is easy to see that $f_1 \in \mathcal{U}$. On the other hand,

$$\frac{zf_1'(z)}{f_1(z)} = \frac{1 - z^3}{1 + \frac{1}{2}z + \frac{1}{2}z^3}$$

and at the boundary point $z_0 = (-1 + i)/\sqrt{2}$, we have

$$\frac{z_0 f_1'(z_0)}{f_1(z_0)} = \frac{2 - 2\sqrt{2}}{3} + \frac{1 - 2\sqrt{2}}{3} i$$

which gives that $\mathrm{Re}\,\{z_0 f_1'(z_0)/f_1(z_0)\} < 0$. Consequently, there are points in the unit disk $|z| < 1$ for which $\mathrm{Re}\,\{zf_1'(z)/f_1(z)\} < 0$ which shows that the function f_1 is not starlike in $\mathbb{D}$. The image of f_1 under $\mathbb{D}$ is shown in Fig. 1.

Acknowledgements The authors thank the referee for his/her careful reading and many useful comments. The work of the first author was supported by MNZZS Grant, No. ON174017, Serbia. The second author is on leave from the IIT Madras.

References

Aksentév, L.A. 1958. Sufficient conditions for univalence of regular functions (Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.* 1958: 3–7.

Duren, P.L., and G.E. Schober. 1971. On a class of schlicht functions. *The Michigan Mathematical Journal* 18: 353–356.

Miller, S.S., and P.T. Mocanu. 1978. Second order differential inequalities in the complex plane. *Journal of Mathematical Analysis and Applications* 65: 289–305.

Miller, S.S., and P.T. Mocanu. 2000. *Differential subordinations, theory and applications*. New York: Marcel Dekker Inc.

Obradović, M. 1995. Starlikeness and certain class of rational functions. *Mathematische Nachrichten* 175: 263–268.

Obradović, M., and S. Ponnusamy. 2005. Radius properties for subclasses of univalent functions. *Analysis (Munich)* 25: 183–188.

Obradović, M., and S. Ponnusamy. 2011. Partial sums and the radius problem for a class of conformal mappings. *Siberian Mathematical Journal* 52: 291–302.

Obradović, M., S. Ponnusamy, and K.-J. Wirths. 2016. Geometric studies on the class $\mathcal{U}(\lambda)$. *Bulletin of the Malaysian Mathematical Sciences Society* 39(3):1259–1284.

Pommerenke, Ch. 1975. *Univalent functions*. Göttingen: Vandenhoeck and Ruprecht.

Singh, R. 1985. A theorem on univalent functions. *Journal of the Indian Mathematical Society* 49: 175–177.

Wirths, K.-J. 1989. A Note on $\Re(f(z)/z)$ for univalent functions. *Journal of the Indian Mathematical Society* 54: 121–123.