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THE ESTIMATE OF THE DIFFERENCE OF INITIAL SUCCESSIVE COEFFICIENTS OF UNIVALENT FUNCTIONS

ZHIGANG PENG AND MILUTIN OBRADOVIĆ

(*Communicated by S. Hencl*)

Abstract. Let $\mathscr{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let S be the set of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. In this paper the sharp upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ being in several subclasses of S are presented.

1. Introduction

Let $\mathscr{A}$ denote the family of all functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = 0 = f'(0) - 1$. Let S be the set of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. Let S^* and K denote the subclasses of S consisting of starlike functions and convex functions, respectively. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ then $|a_n| \leqslant n$ and strict inequality holds for all n unless f is the Koebe function or one of its rotation. This is the famous conjecture of Bieberbach, first proposed by Bieberbach[2] in 1916 and finally proved by de Branges[1] in 1984. After Bieberbach conjecture was put forward, another coefficient problem which has attracted considerable attention is to estimate $||a_{n+1}| - |a_n||$, the difference of the moduli of successive coefficients of a function $f \in S$. Indeed, Hayman[4] proved $||a_{n+1}| - |a_n|| \leqslant A$ for $f \in S$, where $A \geqslant 1$ is an absolute constant. Pommerenke[17] conjecture that $||a_{n+1}| - |a_n|| \leqslant 1$ for $f \in S^*$ which was proved by Leung[6]. Z. Ye also estimated the difference of the moduli of successive coefficients of certain univalent functions[21, 22]. In addition to studying the bounds of $||a_{n+1}| - |a_n||$, some scholars are also interested in studying the bounds of $|a_{n+1} - a_n|$. Robertson[18] proved that $|a_{n+1} - a_n| \leqslant \frac{2n+1}{3}|a_2 - 1|$ for all $f \in K$. Recently, M. Li and T. Sugawa[7] estimated the bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for $f \in K(p)$, where $K(p) = \{f : f \in K, f''(0) = p, 0 \leqslant p \leqslant 2\}$.

In the present paper the upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ for f belonging to various subclasses of S are studied.

Mathematics subject classification (2010): 30C45.
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2. Preliminaries

Let $\mathscr{P}$ denote the class of all functions $p(z)$ analytic and having positive real part on $\mathbb{D}$, with the form

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

It is known that $|p_n| \leqslant 2$ for $p \in \mathscr{P}$ and $n = 1, 2, \cdots$[2].

In the course of the subsequent discussion, we need to make use of the following lemmas.

LEMMA 1. *Let* $-2 \leqslant p_1 \leqslant 2$ *and* $p_2, p_3 \in \mathbb{C}$. *There exists a function* $P \in \mathscr{P}$ *with*

$$P(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{1}$$

if and only if

$$2p_2 = p_1^2 + x(4 - p_1^2) \tag{2}$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)y \tag{3}$$

for some $x, y \in \mathbb{C}$ *with* $|x| \leqslant 1$ *and* $|y| \leqslant 1$.

Lemma 1 is due to Libera and Złotkiewicz[8], one can also find it in [7].

LEMMA 2. *For given real numbers* a, b, c, *let*

$$Y(a,b,c) = max_{z \in \overline{\mathbb{D}}} \left(|a + bz + cz^2| + 1 - |z|^2 \right). \tag{4}$$

If $a \geqslant 0$ *and* $c \geqslant 0$, *then*

$$Y(a,b,c) = \begin{cases} a + |b| + c, & |b| \geqslant 2(1-c) \\ 1 + a + \frac{b^2}{4(1-c)}, & |b| < 2(1-c) \end{cases}$$

The maximum in the definition of $Y(a,b,c)$ *is attained at* $z = \pm 1$ *in the first case according as* $b = \pm |b|$.

Lemma 2 is due to R. Ohno and T. Sugawa[14], one can also find it in [7].

3. Main Results

Let $\mathscr{G}$ denote the class functions f from $\mathscr{A}$ satisfying the conditions

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D},$$

It is known that $\mathscr{G} \subset S$ and $|\frac{1}{2}f''(0)| = |a_2| \leqslant \frac{1}{2}$ for $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{G}$ [19, 15, 10, 5]. Now, let

$$\mathscr{G}(p) = \{f \in \mathscr{G},\ f''(0) = p\},$$

where p is a given number satisfying $-1 \leqslant p \leqslant 1$.

THEOREM 1. *Let $0 \leqslant p \leqslant 1$ and let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be in the class $\mathcal{G}(p)$. Then the next sharp inequalities hold:*

$$|a_3 - a_2| \leqslant \frac{1}{6}(-p^2 + 3p + 1) \tag{5}$$

$$|a_4 - a_3| \leqslant \frac{1}{24}(1 - p^2)(3p + 4) \tag{6}$$

Proof. Since

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in \mathbb{D},$$

it is follows that

$$\mathrm{Re}\left(1 - 2\frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

We can put

$$1 - 2\frac{zf''(z)}{f'(z)} = P(z),$$

where P is given by (1) and satisfy $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$. From the last relation we have

$$f'(z) - 2zf''(z) = P(z)f'(z). \tag{7}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of z^n $(n = 1, 2, 3)$ in both sides of (7), we obtain

$$a_2 = -\frac{p_1}{4}, a_3 = -\frac{1}{12}p_2 - \frac{1}{6}a_2 p_1, a_4 = -\frac{1}{24}p_3 - \frac{1}{8}a_3 p_1 - \frac{1}{12}a_2 p_2. \tag{8}$$

Since, $2a_2 = f''(0) = p$, we have $p_1 = -4a_2 = -2p$ by (8). In view of these facts and Lemma 1, we have

$$\begin{aligned} p_2 &= 2(p^2 + (1 - p^2)x), \\ p_3 &= -2p^3 - 4(1 - p^2)px + 2(1 - p^2)px^2 + 2(1 - p^2)(1 - |x|^2)y, \end{aligned} \tag{9}$$

where $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
From the relations (8) and (9) and by some simple calculations, we have

$$\begin{aligned} |a_3 - a_2| &= \left|-\frac{1}{6}(1 - p^2)x - \frac{1}{2}p\right| \\ &\leqslant \frac{1}{6}(1 - p^2) + \frac{1}{2}p = \frac{1}{6}(-p^2 + 3p + 1), \end{aligned}$$

where equality occurs if $x = 1$. Also, we have

$$\begin{aligned} |a_4 - a_3| &= \left|-\frac{1}{12}(1 - p^2)(1 - |x|^2)y + [\frac{1}{24}(1 - p^2)p + \frac{1}{6}(1 - p^2)]x - \frac{1}{12}(1 - p^2)px^2\right| \\ &\leqslant \frac{1}{12}(1 - p^2)\left(1 - |x|^2 + |-\frac{1}{2}(p + 4)x + px^2|\right) \\ &\leqslant \frac{1}{12}(1 - p^2)Y(a, b, c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) and with

$$a=0,\, b=-\frac{1}{2}(p+4),\, c=p.$$

Since $0\leqslant p\leqslant 1$, we have that $|b|\geqslant 2(1-c)$. Then by using Lemma 2 we get

$$Y(a,b,c)=\frac{3}{2}p+2.$$

Therefore

$$|a_4-a_3|\leqslant \frac{1}{12}(1-p^2)Y(a,b,c)=\frac{1}{24}(1-p^2)(3p+4)$$

The equality holds for $x=-1$.

If we denote by

$$\mathscr{G}^+=\bigcup_{0\leqslant p\leqslant 1}\mathscr{G}_p=\{f: f\in\mathscr{G}, f''(0)\geqslant 0\},$$

then by using (5) and (6) and a simple calculation, we easily get

$$sup_{f\in\mathscr{G}^+}|a_3(f)-a_2(f)|=\frac{1}{2}$$

and

$$sup_{f\in\mathscr{G}^+}|a_4(f)-a_3(f)|=\frac{260+43\sqrt{43}}{2916}=0.1858...$$

where $a_n(f)\,(n=2,3,4)$ are the Taylor coefficients of $f(z)$. $\square$

As usual, let $\mathscr{U}$ denote the set of all $f\in\mathscr{A}$ satisfying the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1$$

for $z\in\mathbb{D}$. It is known that $\mathscr{U}\subset S$[16]. In recent years, the properties of $\mathscr{U}$ were studied in detail[11, 12, 13, 3]. Let

$$\mathscr{U}_p=\{f\in\mathscr{U}, f''(0)=p\},$$

where p is a given number with $-4\leqslant p\leqslant 4$ (Noticing that for $f\in\mathscr{U}$, we have $|\frac{1}{2}f''(0)|=|a_2(f)|\leqslant 2$).

THEOREM 2. *Let $0\leqslant p\leqslant 4$ and let $f(z)=z+a_2z^2+a_3z^3+\ldots$ be in the class $\mathscr{U}_p$. Then we have the following sharp inequalities :*

$$|a_3-a_2|\leqslant\begin{cases}1+\frac{p}{4}(2-p),\ 0\leqslant p\leqslant 2\\ 1+\frac{p}{4}(p-2),\ 2\leqslant p\leqslant 4.\end{cases}\tag{10}$$

$$|a_4-a_3|\leqslant\begin{cases}\frac{1}{4}(-p^3+6p^2-8p+8),\ 0\leqslant p\leqslant 2\\ \frac{1}{8}(p^3-2p^2+8p-8),\ 2\leqslant p\leqslant 4.\end{cases}\tag{11}$$

Proof. If $f \in \mathcal{U}$, then

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < 1, |z| < 1.$$

It is equivalent to

$$\operatorname{Re}\left(2\left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)} - 1\right) > 0, \quad z \in \mathbb{D}.$$

So, we can put

$$2\left(\frac{f(z)}{z}\right)^2 \frac{1}{f'(z)} - 1 = P(z),$$

where P is given by (1) and satisfy $\operatorname{Re}P(z) > 0, z \in \mathbb{D}$. From the last relation we have

$$2\left(\frac{f(z)}{z}\right)^2 - f'(z) = P(z)f'(z). \tag{12}$$

By using the relation (12) and the Taylor expansions of functions f and P, we obtain

$$p_1 = 0, a_3 = a_2^2 - \frac{1}{2}p_2, a_4 = -\frac{1}{4}p_3 - \frac{1}{2}a_2p_2 + a_2a_3. \tag{13}$$

Since $2a_2 = p$, we have $a_2 = \frac{p}{2}$. Also, since $p_1 = 0$, it follows from Lemma 1 that

$$p_2 = 2x, p_3 = 2(1-|x|^2)y \tag{14}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
By using the all previous facts, we obtain that

$$a_3 = \frac{1}{4}p^2 - x, a_4 = \frac{1}{8}p^3 - px - \frac{1}{2}(1-|x|^2)y.$$

Now, we have

$$|a_3 - a_2| = \left|-x + \frac{1}{4}p^2 - \frac{p}{2}\right| \leqslant 1 + \frac{p}{4}|p-2|,$$

or equivalently,

$$|a_3 - a_2| \leqslant \begin{cases} 1 + \frac{p}{4}(2-p), 0 \leqslant p \leqslant 2 \\ 1 + \frac{p}{4}(p-2), 2 \leqslant p \leqslant 4. \end{cases}$$

Also we have

$$\begin{aligned} |a_4 - a_3| &= \left|\frac{1}{8}p^3 - px - \frac{1}{2}(1-|x|^2)y - \frac{1}{4}p^2 + x\right| \\ &\leqslant \frac{1}{2}\left(1 - |x|^2 + \left|\frac{1}{4}p^2(p-2) + 2(1-p)x\right|\right) \\ &\leqslant \frac{1}{2}Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4). Since

$$\left|\frac{1}{4}p^2(p-2)+2(1-p)x\right| = \left|\frac{1}{4}p^2(2-p)+2(p-1)x\right|,$$

we can put $a=\frac{1}{4}p^2(2-p)$, $b=2(p-1)$, $c=0$ in case $0\leqslant p\leqslant 2$ and $a=\frac{1}{4}p^2(p-2)$, $b=2(1-p)$, $c=0$ in case $2\leqslant p\leqslant 4$. We have that $|b|\leqslant 2(1-c)$ in the first case and $|b|\geqslant 2(1-c)$ in the second case. By Lemma 2 we have

$$Y(a,b,c)=\begin{cases} 1+\frac{1}{4}p^2(2-p)+\frac{4(p-1)^2}{4}, & 0\leqslant p\leqslant 2\\ \frac{1}{4}p^2(p-2)+2(p-1), & 2\leqslant p\leqslant 4\end{cases}$$

and therefore

$$|a_4-a_3|\leqslant \begin{cases} \frac{1}{4}(-p^3+6p^2-8p+8), & 0\leqslant p\leqslant 2\\ \frac{1}{8}(p^3-2p^2+8p-8), & 2\leqslant p\leqslant 4.\end{cases}$$

Now, let

$$\mathscr{U}^+=\bigcup_{0\leqslant p\leqslant 4}\mathscr{U}_p=\{f: f\in\mathscr{U}, f''(0)\geqslant 0\}.$$

Then, in view of (10) and (11), we easily get

$$sup_{f\in\mathscr{U}^+}|a_3(f)-a_2(f)|=3$$

and

$$sup_{f\in\mathscr{U}^+}|a_4(f)-a_3(f)|=7.$$

□

For a long time, the research on Bazilevic functions has attracted the attention of many scholars[20, 9, 23]. R.Singh[20] considered a subclass $\mathscr{B}_1(\alpha)$ of Bazilevic functions. $f\in\mathscr{B}_1(\alpha)$ if $f\in\mathscr{A}$ and

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1}f'(z)\right\}>0, z\in\mathbb{D}, \alpha\geqslant 0.$$

It is well-known that $\mathscr{B}_1(\alpha)(\alpha\geqslant 0)$ is the subclass of S.
For $\alpha=1$ we have the class $\mathscr{R}$ defined by the condition

$$\mathrm{Re}\{f'(z)\}>0, z\in\mathbb{D}.$$

Further, let denote by $\mathscr{B}^{(2)}$ and $\mathscr{B}^{(3)}$ the classes given from $\mathscr{B}_1(\alpha)$ for $\alpha=2$ and $\alpha=3$, i.e. the classes of $\mathscr{A}$ satisfying the next conditions

$$\mathrm{Re}\left\{\frac{f(z)f'(z)}{z}\right\}>0, z\in\mathbb{D}$$

and

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^2 f'(z)\right\}>0, z\in\mathbb{D},$$

respectively. Also, let

$$\mathscr{R}_p = \{f \in \mathscr{R},\, f''(0) = p\},$$
$$\mathscr{B}_p^{(2)} = \{f \in \mathscr{B}^2,\, f''(0) = p\},$$
$$\mathscr{B}_p^{(3)} = \{f \in \mathscr{B}^3,\, f''(0) = p\}.$$

THEOREM 3. *Let* $0 \leqslant p \leqslant 2$ *and let* $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ *be in the class* $\mathscr{R}_p$. *Then we have the next sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{6}(4 + 3p - 2p^2). \tag{15}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{18}(13 - 4p),\ 0 \leqslant p \leqslant \frac{5}{3} \\ \frac{1}{12}(-3p^3 + 4p^2 + 9p - 8),\ \frac{5}{3} \leqslant p \leqslant 2. \end{cases} \tag{16}$$

Proof. Since $f \in \mathscr{R}_p$, we can put

$$f'(z) = P(z), \tag{17}$$

where P is given by (1) with $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$. By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1,2,3)$ in both sides of (17), we obtain

$$a_2 = \frac{1}{2}p_1,\ a_3 = \frac{1}{3}p_2,\ a_4 = \frac{1}{4}p_3. \tag{18}$$

Since $2a_2 = f''(0) = p$, it follows from (18) that $p_1 = 2a_2 = p$ and $|p| \leqslant 2$. By using Lemma 1, we have

$$\begin{aligned} p_2 &= \tfrac{1}{2}[p^2 + (4 - p^2)x], \\ p_3 &= \tfrac{1}{4}[p^3 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y] \end{aligned} \tag{19}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (18) with (19), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{1}{6}(4 - p^2)x - \frac{p}{6}(3 - p)\right| \\ &\leqslant \frac{1}{6}(4 - p^2) + \frac{p}{6}(3 - p) \\ &= \frac{1}{6}(4 + 3p - 2p^2) \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we have

$$\begin{aligned} &|a_4 - a_3| \\ &= \left|\frac{1}{16}[p^3 + 2(4 - p^2)px - (4 - p^2)px^2 + 2(4 - p^2)(1 - |x|^2)y] - \frac{1}{6}[p^2 + (4 - p^2)x]\right| \\ &\leqslant \frac{1}{8}(4 - p^2)\left[1 - |x|^2 + \left|\frac{p^2(8/3 - p)}{2(4 - p^2)} + (4/3 - p)x + \frac{p}{2}x^2\right|\right] \\ &\leqslant \frac{1}{8}(4 - p^2)Y(a, b, c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a=\frac{p^2(8/3-p)}{2(4-p^2)}, b=4/3-p, c=\frac{1}{2}p,$$

(for $p=2$, we have directly that $|a_4-a_3|=\frac{1}{6}$).
Noticing that for $p\in[0,2]$, $|b|\leqslant 2(1-c)$ is equivalent $0\leqslant p\leqslant\frac{5}{3}$, by Lemma 2 we have

$$Y(a,b,c)=\begin{cases}1+\frac{p^2(8/3-p)}{2(4-p^2)}+\frac{(4/3-p)^2}{4(1-p/2)}, & 0\leqslant p\leqslant\frac{5}{3}\\ \frac{p^2(8/3-p)}{2(4-p^2)}+p-\frac{4}{3}+\frac{1}{2}p, & \frac{5}{3}\leqslant p<2.\end{cases}$$

Hence

$$|a_4-a_3|\leqslant\begin{cases}\frac{1}{18}(13-4p), & 0\leqslant p\leqslant\frac{5}{3}\\ \frac{1}{12}(-3p^3+4p^2+9p-8), & \frac{5}{3}\leqslant p\leqslant 2.\end{cases}$$

If we denote by

$$\mathscr{R}^+=\bigcup_{0\leqslant p\leqslant 2}\mathscr{R}_p=\{f:f\in\mathscr{R},f''(0)\geqslant 0\},$$

then in view of (15) and (16), we easily get

$$sup_{f\in\mathscr{R}^+}|a_3(f)-a_2(f)|=\frac{41}{48}$$

and

$$sup_{f\in\mathscr{R}^+}|a_4(f)-a_3(f)|=\frac{13}{18}.$$

□

THEOREM 4. *Let $0\leqslant p\leqslant\frac{4}{3}$ and let $f(z)=z+a_2z^2+a_3z^3+\ldots$ be in the class $\mathscr{B}_p^2$. Then we have the next sharp inequalities:*

$$|a_3-a_2|\leqslant\frac{1}{16}(-7p^2+8p+8), 0\leqslant p\leqslant\frac{4}{3}. \tag{20}$$

$$|a_4-a_3|\leqslant\begin{cases}\frac{1}{2560}(-85p^3-400p^2-260p+1424), & 0\leqslant p\leqslant\frac{6}{5}\\ \frac{1}{80}(-5p^3-10p^2-4p+40), & \frac{6}{5}\leqslant p\leqslant\frac{4}{3}.\end{cases} \tag{21}$$

Proof. From the definition of the class $\mathscr{B}_p^2$, we can put

$$\frac{f(z)f'(z)}{z}=P(z), \tag{22}$$

where $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$, and P is given by (1). By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1,2,3)$ in both sides of (22), we obtain

$$a_2 = \frac{1}{3}p_1,\ a_3 = \frac{1}{4}p_2 - \frac{1}{2}a_2^2,\ a_4 = \frac{1}{5}p_3 - a_2a_3. \tag{23}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, it follows from (23) that $p_1 = 3a_2 = \frac{3}{2}p$ and $|p| \leqslant \frac{4}{3}$. In view of these facts and Lemma 1, we have

$$\begin{aligned} p_2 &= \tfrac{9}{8}(p^2 + (\tfrac{16}{9} - p^2)x), \\ p_3 &= \tfrac{9}{32}\left(3p^3 + 6(\tfrac{16}{9} - p^2)px - 3(\tfrac{16}{9} - p^2)px^2 + 4(\tfrac{16}{9} - p^2)(1 - |x|^2)y\right) \end{aligned} \tag{24}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (23) with (24), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{9}{32}(\frac{16}{9} - p^2)x + \frac{5}{32}p^2 - \frac{p}{2}\right| \\ &\leqslant \frac{9}{32}(\frac{16}{9} - p^2) + \frac{5}{32}p\left|p - \frac{16}{5}\right| \\ &= \frac{1}{16}(-7p^2 + 8p + 8), \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we have

$$\begin{aligned} |a_4 - a_3| &= \left|\frac{29}{320}p^3 - \frac{5}{32}p^2 + (\frac{16}{9} - p^2)[\frac{63}{320}px - \frac{27}{160}px^2 + \frac{9}{40}(1 - |x|^2)y - \frac{9}{32}x]\right| \\ &\leqslant \frac{16 - 9p^2}{40}\left[1 - |x|^2 + \left|\frac{p^2(50 - 29p)}{8(16 - 9p^2)} + \frac{1}{8}(10 - 7p)x + \frac{3}{4}px^2\right|\right] \\ &\leqslant \frac{16 - 9p^2}{40}Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a = \frac{p^2(50 - 29p)}{8(16 - 9p^2)},\ b = \frac{1}{8}(10 - 7p),\ c = \frac{3}{4}p.$$

(for $p = \frac{4}{3}$, we have directly that $|a_4 - a_3| = \frac{17}{270}$).
Since $p \in [0, 4/3]$, $|b| \leqslant 2(1 - c)$ is equivalent to $0 \leqslant p \leqslant \frac{6}{5}$, by Lemma 2, we have

$$Y(a,b,c) = \begin{cases} 1 + \frac{p^2(50-29p)}{8(16-9p^2)} + \frac{(10-7p)^2}{64(4-3p)},\ 0 \leqslant p \leqslant \frac{6}{5} \\ \frac{p^2(50-29p)}{8(16-9p^2)} + \frac{1}{8}(10 - 7p) + \frac{3}{4}p,\ \frac{6}{5} \leqslant p < \frac{4}{3}. \end{cases}$$

Therefore

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{2560}(-85p^3 - 400p^2 - 260p + 1424),\ 0 \leqslant p \leqslant \frac{6}{5} \\ \frac{1}{80}(-5p^3 - 10p^2 - 4p + 40),\ \frac{6}{5} \leqslant p \leqslant \frac{4}{3}. \end{cases}$$

Let

$$\mathscr{B}^{(2)+} = \bigcup_{0 \leqslant p \leqslant \frac{4}{3}} \mathscr{B}_p^{(2)} = \{f : f \in \mathscr{B}^{(2)}, f''(0) \geqslant 0\}.$$

Then by using (20) and (21) we easily get

$$sup_{f\in\mathscr{B}^{(2)+}} |a_3(f) - a_2(f)| = \frac{9}{14} = 0.64...$$

and

$$sup_{f\in\mathscr{B}^{(2)+}} |a_4(f) - a_3(f)| = \frac{1424}{2560} = 0.556...$$

□

THEOREM 5. *Let* $0 \leqslant p \leqslant 1$ *and let* $f(z) = z + a_2z^2 + a_3z^3 + ...$ *be in the class* $\mathscr{B}_p^3$. *Then we have the next sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{20}(-11p^2 + 10p + 8). \tag{25}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{600}(-53p^3 - 174p^2 - 24p + 272), & 0 \leqslant p \leqslant \frac{2}{3} \\ \frac{1}{120}(-25p^3 - 30p^2 + 8p + 48), & \frac{2}{3} \leqslant p \leqslant 1. \end{cases} \tag{26}$$

Proof. The hypothesis $f \in \mathscr{B}_p^3$ implies that there exists a function P, defined by (1) and satisfying $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$, such that

$$\left(\frac{f(z)}{z}\right)^2 f'(z) = P(z). \tag{27}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 1,2,3)$ in both sides of (27), we obtain

$$a_2 = \frac{1}{4}p_1,\ a_3 = \frac{1}{5}p_2 - a_2^2,\ a_4 = \frac{1}{6}p_3 - 2a_2a_3 - \frac{1}{3}a_2^3. \tag{28}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, by (28) we have $p_1 = 4a_2 = 2p$ and $|p| \leqslant 1$. By using these facts and Lemma 1, we get

$$\begin{aligned} p_2 &= 2[p^2 + (1-p^2)x], \\ p_3 &= 2p^3 + 4(1-p^2)px - 2(1-p^2)px^2 + 2(1-p^2)(1-|x|^2)y \end{aligned} \tag{29}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (28) with (29), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left|\frac{2}{5}(1-p^2)x - \frac{3}{20}p(10/3 - p)\right| \\ &\leqslant \frac{1}{20}(-11p^2 + 10p + 8), \end{aligned}$$

where equality occurs if $x=-1$. Similarly, we also have

$$\begin{aligned}|a_4-a_3| &= \left|\frac{1}{6}p_3-2a_2a_3-\frac{1}{3}a_2^3-a_3\right|\\ &\leqslant \frac{1}{3}(1-p^2)\left[1-|x|^2+\left|\frac{18p^2-17p^3}{40(1-p^2)}+\frac{2}{5}(3-2p)x+px^2\right|\right]\\ &\leqslant \frac{1}{3}(1-p^2)Y(a,b,c),\end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a=\frac{18p^2-17p^3}{40(1-p^2)},\, b=\frac{2}{5}(3-2p),\, c=p.$$

(for $p=1$ we have directly that $|a_4-a_3|=\frac{1}{120}$).
Since for $p\in[0,1]$, $|b|\leqslant 2(1-c)$ is equivalent to $0\leqslant p\leqslant \frac{2}{3}$, by using Lemma 2 we have

$$Y(a,b,c)=\begin{cases}1+\frac{p^2(18-17p)}{40(1-p^2)}+\frac{(3-2p)^2}{25(1-p)},\ 0\leqslant p\leqslant \frac{2}{3}\\ \frac{p^2(18-17p)}{40(1-p^2)}+\frac{1}{5}p+\frac{6}{5},\ \frac{2}{3}\leqslant p<1.\end{cases}$$

And therefore

$$|a_4-a_3|\leqslant \begin{cases}\frac{1}{600}(-53p^3-174p^2-24p+272),\ 0\leqslant p\leqslant \frac{2}{3}\\ \frac{1}{120}(-25p^3-30p^2+8p+48),\ \frac{2}{3}\leqslant p\leqslant 1.\end{cases}$$

Let

$$\mathscr{B}^{(3)+}=\bigcup_{0\leqslant p\leqslant 1}\mathscr{B}_p^{(3)}=\{f:f\in\mathscr{B}^{(3)},f''(0)\geqslant 0\}..$$

In view of (25) and (26), we easily get

$$sup_{f\in\mathscr{B}^{(3)+}}|a_3(f)-a_2(f)|=\frac{113}{220}=0.5136...$$

and

$$sup_{f\in\mathscr{B}^{(3)+}}|a_4(f)-a_3(f)|=\frac{34}{75}=0.4533.$$

□

In [24] the authors introduced the class Ω which consists of all functions $f\in\mathscr{A}$ satisfying

$$|zf'(z)-f(z)|<\frac{1}{2},(|z|<1).$$

It is proved that $\Omega\subset S^*$. Now, let

$$\Omega_p=\{f:f\in\Omega,f''(0)=p\},$$

where $|p|\leqslant 1$ (Noting that $|a_n|\leqslant\frac{1}{2(n-1)}$ when $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\Omega$[24]).

THEOREM 6. *Let $0 \leqslant p \leqslant 1$ and let $f(z) = z + a_2z^2 + a_3z^3 + ...$ be in the class Ω_p. Then we have the following sharp inequalities:*

$$|a_3 - a_2| \leqslant \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2, 0 \leqslant p \leqslant 1. \tag{30}$$

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{96}(-16p^2 + 9p + 25), \, 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{1}{12}(-2p^3 - 3p^2 + 2p + 3), \, \frac{1}{4} \leqslant p \leqslant 1. \end{cases} \tag{31}$$

Proof. By the definition of Ω, $f \in \Omega$ if and only if there exists a function $P(z)$ defined by (1) with $\mathrm{Re}P(z) > 0, z \in \mathbb{D}$, such that

$$2[P(z)+1][zf'(z) - f(z)] = z[P(z) - 1] \tag{32}$$

By using the Taylor representations for the functions f and P and comparing the coefficients of $z^n (n = 2,3,4)$ in both sides of (32), we obtain

$$a_2 = \frac{1}{4}p_1, a_3 = \frac{1}{8}p_2 - \frac{1}{4}a_2p_1, a_4 = \frac{1}{12}p_3 - \frac{1}{3}a_3p_1 - \frac{1}{6}a_2p_2. \tag{33}$$

Since $2a_2 = f''(0) = p$ and $|p_1| \leqslant 2$, by (33) we have $p_1 = 4a_2 = 2p$ and $|p| \leqslant 1$. In view of these facts and Lemma 1, we get

$$\begin{aligned} p_2 &= 2[p^2 + (1-p^2)x], \\ p_3 &= 2p^3 + 4(1-p^2)px - 2(1-p^2)px^2 + 2(1-p^2)(1-|x|^2)y \end{aligned} \tag{34}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$ and $|y| \leqslant 1$.
Combining (33) with (34), we obtain

$$\begin{aligned} |a_3 - a_2| &= \left| \frac{1}{4}(1-p^2)x - \frac{1}{2}p \right| \\ &\leqslant \frac{1}{4} + \frac{1}{2}p - \frac{1}{4}p^2, \end{aligned}$$

where equality occurs if $x = -1$. Similarly, we also have

$$\begin{aligned} |a_4 - a_3| &= \left| \frac{1}{6}(1-p^2)(1-|x|^2)y - \frac{1}{6}(1-p^2)px^2 - \frac{1}{4}(1-p^2)x \right| \\ &\leqslant \frac{1}{6}(1-p^2)\left[1 - |x|^2 + \left|\frac{3}{2}x + px^2\right|\right] \\ &\leqslant \frac{1}{6}(1-p^2)Y(a,b,c), \end{aligned}$$

where $Y(a,b,c)$ is given in (4) with

$$a = 0, b = \frac{3}{2}, c = p.$$

Since for $p \in [0,1]$, $|b| \leqslant 2(1-c)$ is equivalent to $0 \leqslant p \leqslant \frac{1}{4}$, by using Lemma 2 we have

$$Y(a,b,c) = \begin{cases} 1 + \frac{9}{16(1-p)},\ 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{3}{2} + p,\ \frac{1}{4} \leqslant p \leqslant 1. \end{cases}$$

And therefore

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{96}(-16p^2 + 9p + 25),\ 0 \leqslant p \leqslant \frac{1}{4} \\ \frac{1}{12}(-2p^3 - 3p^2 + 2p + 3),\ \frac{1}{4} \leqslant p \leqslant 1. \end{cases}$$

Let

$$\Omega^+ = \bigcup_{0 \leqslant p \leqslant 1} \Omega_p = \{f : f \in \Omega, f''(0) \geqslant 0\}.$$

In view of (30) and (31), we easily get

$$sup_{f \in \Omega^+} |a_3(f) - a_2(f)| = \frac{1}{2}$$

and

$$sup_{f \in \Omega^+} |a_4(f) - a_3(f)| = \frac{27 + 7\sqrt{21}}{216} = 0.2735...$$

□

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