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HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. Let f be analutic in the unit disk $\mathbb{D}$ and normalized so that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. In this paper we give sharp bound of Hankel determinant of the second order for the class of analytic unctions satisfying

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma\frac{\pi}{2} \qquad (z \in \mathbb{D}),$$

for $0 < \alpha < 1$ and $0 < \gamma \le 1$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = 0 = f'(0) - 1$.

A function $f \in \mathcal{A}$ is said to be *strongly starlike of order* $\beta, 0 < \beta \le 1$ if, and only if,

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \beta\frac{\pi}{2} \qquad (z \in \mathbb{D}).$$

We denote this class by $\mathcal{S}^\star_\beta$. If $\beta = 1$, then $\mathcal{S}^\star_1 \equiv \mathcal{S}^\star$ is the well-known class of *starlike functions*.

In [1] the author introduced the class $\mathcal{U}(\alpha,\lambda)$ ($0 < \alpha$ and $\lambda < 1$) consisting of functions $f \in \mathcal{A}$ for which we have

$$\left|\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z) - 1\right| < \lambda \qquad (z \in \mathbb{D}).$$

In the same paper it is shown that $\mathcal{U}(\alpha,\lambda) \subset \mathcal{S}^\star$ if

$$0 < \lambda \le \frac{1-\alpha}{\sqrt{(1-\alpha)^2+\alpha^2}}.$$

The most valuable up to date results about this class can be found in Chapter 12 from [4].

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In the paper [2] the author considered univalence of the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma\frac{\pi}{2} \qquad (z\in\mathbb{D}) \tag{1.1}$$

for $0<\alpha<1$ and $0<\gamma\le 1$, and proved the following theorem.

Theorem A. *Let $f\in\mathcal{A}$, $0<\alpha<\frac{2}{\pi}$ and let*

$$\left|\arg\left[\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z)\right]\right| < \gamma_\star(\alpha)\frac{\pi}{2} \qquad (z\in\mathbb{D}),$$

where

$$\gamma_\star(\alpha) = \frac{2}{\pi}\arctan\left(\sqrt{\frac{2}{\pi\alpha}-1}\right) - \alpha\sqrt{\frac{2}{\pi\alpha}-1}.$$

Then $f\in\mathcal{S}^\star_\beta$, where

$$\beta = \frac{2}{\pi}\arctan\sqrt{\frac{2}{\pi\alpha}-1}.$$

2. Main result

In this paper we will give the sharp estimate for Hankel determinant of the second order for the class of analytic unctions $f\in\mathcal{A}$ which satisfied the condition (1.1).

Definition 1. Let $f\in\mathcal{A}$. Then the qth Hankel determinant of f is defined for $q\ge 1$, and $n\ge 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

Thus, the second Hankel determinant is $H_2(2) = a_2a_4 - a_3^2$.

Theorem 1. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{A}$ and satisfy the condition* (1.1). *Then we have the next sharp estimation:*

$$|H_2(2)| = |a_2a_4 - a_3^2| \le \left(\frac{2\gamma}{2-\alpha}\right)^2,$$

where $0<\alpha<2-\sqrt{2}$ and $0<\gamma\le\frac{1}{2}(\alpha^2-4\alpha+2)$.

Proof. We can write the condition (1.1) in the form

$$\left(\frac{f(z)}{z}\right)^{-(1+\alpha)} f'(z) = \left(\frac{1+\omega(z)}{1-\omega(z)}\right)^\gamma \left(= (1+2\omega(z)+2\omega^2(z)+\cdots)^\gamma\right), \tag{2.1}$$

where ω is analytic in $\mathbb{D}$ with $\omega(0)=0$ and $|\omega(z)|<1$, $z\in\mathbb{D}$. If we denote by L and R left and right hand side of equality (2.1), then we have

$$L = \left[1-(1+\alpha)(a_2z+\cdots) + \binom{-(1+\alpha)}{2}(a_2z+\cdots)^2 \right.$$
$$\left. + \binom{-(1+\alpha)}{3}(a_2z+\cdots)^3 + \cdots\right]\cdot(1+2a_2z+3a_3z^2+4a_4z^3+\cdots)$$

and if we put $\omega(z) = c_1 z + c_2 z^2 + \cdots$:

$$\begin{aligned} R = 1 &+ \gamma\left[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots\right] \\ &+ \binom{\gamma}{2}\left[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots\right]^2 \\ &+ \binom{\gamma}{3}\left[2(c_1 z + c_2 z^2 + \cdots) + 2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots\right]^3 + \cdots . \end{aligned}$$

If we compare the coefficients on z, z^2, z^3 in L and R, then, after some calculations, we obtain

$$\begin{aligned} a_2 &= \frac{2\gamma}{1-\alpha} c_1, \\ a_3 &= \frac{2\gamma}{2-\alpha} c_2 + \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)} c_1^2, \\ a_4 &= \frac{2\gamma}{3-\alpha}\left(c_3 + \mu c_1 c_2 + \nu c_1^3\right), \end{aligned} \tag{2.2}$$

where

$$\mu = \mu(\alpha,\gamma) = \frac{2(5-\alpha)\gamma}{(1-\alpha)(2-\alpha)} \quad \text{and} \quad \nu = \nu(\alpha,\gamma) = \frac{1}{3} + \frac{2}{3}\frac{(\alpha^2 - 6\alpha + 17)\gamma^2}{(1-\alpha)^3(2-\alpha)}. \tag{2.3}$$

By using the relations (2.2) and (2.3), after some simple computations, we obtain

$$H_2(2) = \frac{4\gamma^2}{(1-\alpha)(3-\alpha)}\left(c_1 c_3 + \mu_1 c_1^2 c_2 + (\frac{1}{3} - \nu_1) c_1^4 - \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} c_2^2\right),$$

where

$$\mu_1 = \frac{2\gamma}{(2-\alpha)^2}, \quad \nu_1 = \frac{(\alpha^2 - 10\alpha + 13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2},$$

and from here

$$\begin{aligned} |H_2(2)| \le \frac{4\gamma^2}{(1-\alpha)(3-\alpha)} &\Bigg(|c_1||c_3| + \mu_1 |c_1|^2 |c_2| \\ &+ \left|\frac{1}{3} - \nu_1\right| |c_1|^4 + \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} |c_2|^2 \Bigg). \end{aligned} \tag{2.4}$$

For the function $\omega(z) = c_1 z + c_2 z^2 + \ldots$ (with $|\omega(z)| < 1,\ z \in \mathbb{D}$) the next relations is valid (see, for example [3, p.128, expression (13)]):

$$|c_1| \le 1,\ |c_2| \le 1 - |c_1|^2,\ |c_3(1 - |c_1|^2) + \overline{c_1} c_2^2| \le (1 - |c_1|^2)^2 - |c_2|^2. \tag{2.5}$$

We may suppose that $a_2 \ge 0$, which implies that $c_1 \ge 0$ and instead of relations (2.5) we have the next relations

$$0 \le c_1 \le 1,\ |c_2| \le 1 - c_1^2,\ |c_3| \le 1 - c_1^2 - \frac{|c_2|^2}{1 + c_1}. \tag{2.6}$$

By using (2.6) for c_1 and c_3, from (2.4) we have

$$\begin{aligned} |H_2(2)| \le \frac{4\gamma^2}{(1-\alpha)(3-\alpha)} &\left[c_1(1 - c_1^2) + \left(\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - \frac{c_1}{1 + c_1} \right) |c_2|^2 \right. \\ &\left. + \mu_1 c_1^2 |c_2| + \left|\frac{1}{3} - \nu_1\right| c_1^4 \right]. \end{aligned} \tag{2.7}$$

Since for $0 < \alpha < 2 - \sqrt{2}$ we have $\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} \ge \frac{1}{2} \ge \frac{c_1}{1+c_1}$, then by using $|c_2| \le 1 - c_1^2$, from (2.7) after some calculations we obtain

(2.8) $$|H_2(2)| \le \frac{4\gamma^2}{(1-\alpha)(3-\alpha)}F(c_1),$$

where

(2.9) $$F(c_1) = \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} + Ac_1^2 + Bc_1^4,$$

where

$$A = \frac{2\gamma - (\alpha^2-4\alpha+2)}{(2-\alpha)^2}, B = \left|\frac{1}{3} - \nu_1\right| - \frac{2\gamma+1}{(2-\alpha)^2}.$$

Further, by using the assumptions of the theorem that $0<\alpha<2-\sqrt{2}$ and $0<\gamma\le \frac{1}{2}(\alpha^2-4\alpha+2)$, we easily conclude that $A\le 0$, while

$$0<\nu_1 = \frac{(\alpha^2-10\alpha+13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2} \le \frac{(\alpha^2-10\alpha+13)(\alpha^2-4\alpha+2)^2}{12(1-\alpha)^2(2-\alpha)^2} < \frac{13}{12}.$$

If we have that $B\le 0$, then from (2.9) we obtain that

$$F(c_1) \le \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},$$

and if $B>0$, then

$$F(c_1) \le \max\{F(0),F(1)\} = \max\left\{\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2}, \left|\frac{1}{3}-\nu_1\right|\right\} = \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},$$

since

(2.10) $$\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \left|\frac{1}{3}-\nu_1\right|$$

when $0<\alpha<2-\sqrt{2}$ and $0<\gamma\le\frac{1}{2}(\alpha^2-4\alpha+2)$ (proven later). It means that in both cases we have that

$$F(c_1) \le \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},$$

which by (2.8) implies

$$|H_2(2)| \le \left(\frac{2\gamma}{2-\alpha}\right)^2.$$

We need to prove the inequality (2.10) for appropriate α and γ. First, if $\frac{1}{3}-\nu\le 0$, i.e. if $0<\nu_1\le\frac{1}{3}$, then , since $0<\alpha<2-\sqrt{2}$, we have

$$\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \frac{1}{2} > \frac{1}{3} - \nu_1,$$

which implies that (2.10) is true. In case $\nu_1>\frac{1}{3}$, we have that inequality (2.10) is equivalent to

$$\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \frac{(\alpha^2-10\alpha+13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2} - \frac{1}{3}.$$

The last inequality is equivalent with

$$\gamma^2 < \frac{(1-\alpha)^2(4\alpha^2-16\alpha+13)}{\alpha^2-10\alpha+13}.$$

Since for $0<\alpha<2-\sqrt{2}$ we have $\gamma \le \frac{1}{2}(\alpha^2-4\alpha+2)$, then for such α we have

$$\gamma^2 \le \frac{1}{4}(\alpha^2-4\alpha+2)^2$$

and from (2.10) it is sufficient to prove that

$$\frac{1}{4}(\alpha^2-4\alpha+2)^2 \le \frac{(1-\alpha)^2(4\alpha^2-16\alpha+13)}{\alpha^2-10\alpha+13} \tag{2.11}$$

for $0<\alpha<2-\sqrt{2}$. The inequality (2.11) is equivalent to

$$(\phi(\alpha) :=)\, 4(1-\alpha)^2(4\alpha^2-16\alpha+13)-(\alpha^2-4\alpha+2)^2(\alpha^2-10\alpha+13) \ge 0, \tag{2.12}$$

where $0<\alpha<2-\sqrt{2}$. Let's put $\alpha^2-4\alpha+2=t$. Then $0<t<2$ and $\alpha=2-\sqrt{2+t}$ and from (2.11) we have

$$\phi_1(t) := \phi(2-\sqrt{2+t}) = \frac{1}{4}(2+t)\left[30+19t-t^2-(20+6t)\sqrt{2+t}\right].$$

The function ϕ_1 is continuous function in the interval $[0,2]$. It is easily to check that

$$\phi_1'(t) = \frac{1}{4}\left[68+34t-3t^2-(42+15t)\sqrt{2+t}\right]$$

and

$$\phi_1''(t) = \frac{1}{8}\left[68-12t-45\sqrt{2+t}-\frac{12}{\sqrt{2+t}}\right].$$

iN ϕ_1'', the second and the third expression reach their minimum on the segment $[0,2]$ for $t=0$, while the last expression for $t=2$. Thus

$$\phi_1''(t) < \frac{1}{8}\left(68-12\cdot 0-45\sqrt{2+0}-\frac{12}{\sqrt{2+2}}\right) = \frac{1}{8}(62-45\sqrt{2}) = -0.20\ldots < 0,$$

i.e, ϕ_1' is an decreasing function from $\phi_1'(0)=17-10.5\sqrt{2}=2.15\ldots>0$ to $\phi_1'(2)=-5<0$, which implies that the function ϕ attains its maximum in the interval $(0,2)$, so that

$$\phi_1(t) \ge \min\{\phi_1(0),\phi_1(2)\} = \min\{15-10\sqrt{2},0\} = 0.$$

This means that the inequality given by (2.12) is true.

The result of Theorem 1 is the best possible as the functions f_2, defined with

$$\left(\frac{z}{f_2(z)}\right)^{1+\alpha} f_2'(z) = \left(\frac{1+z^2}{1-z^2}\right)^{\gamma}$$

shows. In this case we have that $c_2=1$, $c_j=0$ when $j\ne 2$, and consequently, $a_2=a_4=0$, $a_3=\frac{2\gamma}{2-\alpha}$ and $H_2(2)=a_2a_4-a_3^2=-\frac{4\gamma^2}{(2-\alpha)^2}$. □

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