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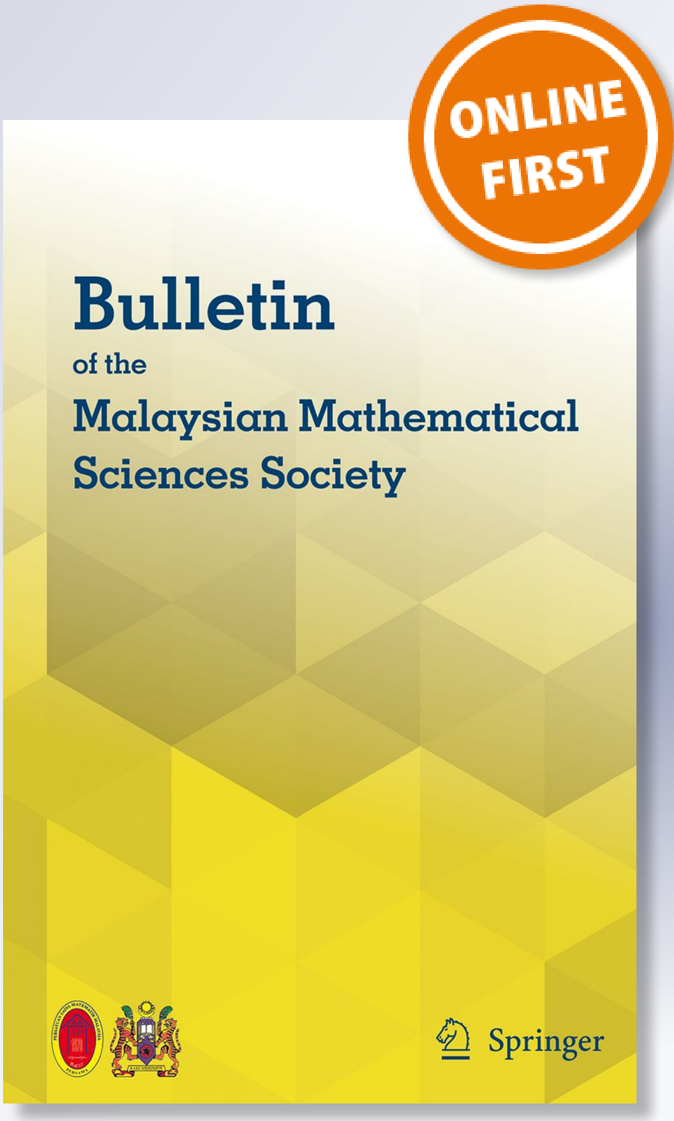

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A Class of Univalent Functions with Real Coefficients

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Abstract
In this paper, we study class $\mathcal{S}^+$ of univalent functions f such that $\frac{z}{f(z)}$ has real and positive coefficients. For such functions, we give estimates of the Fekete–Szegő functional and sharp estimates of their initial coefficients and logarithmic coefficients. Also, we present necessary and sufficient conditions for $f \in \mathcal{S}^+$ to be starlike of order $1/2$.

Keywords Univalent · Real coefficients · Fekete–Szegő · Logarithmic coefficients · Coefficient estimates

Mathematics Subject Classification 30C45 · 30C50 · 30C55

1 Introduction

Let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then, the class of starlike functions of order α, $0 \le \alpha < 1$, is defined by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D} \right\},$$

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while $\mathcal{S}^* \equiv \mathcal{S}^*(0)$ is the well-known class of starlike functions mapping the unit disk onto a starlike region D, i.e.,

$$w \in f(D) \quad \Leftrightarrow \quad tw \in f(\mathbb{D}) \text{ for all } t \in [0,1].$$

More on this classes can be found in [7] and [1].

Further, let $\mathcal{S}^+$ denote the class of univalent functions in the unit disk with the next representation

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots, \qquad b_n \geq 0, \quad n = 1, 2, 3, \ldots \tag{1}$$

For example, the Silverman class (the class with negative coefficients) is included in the class $\mathcal{S}^+$. Namely, that class consists of univalent functions of the form

$$f(z) = z - a_2 z^2 - a_3 z^3 - \cdots, \qquad a_n \geq 0, \quad n = 2, 3, \ldots$$

which implies that

$$\frac{z}{f(z)} = \frac{1}{1 - a_2 z - a_3 z^2 - \cdots},$$

i.e., $\frac{z}{f(z)}$ has the form (1). Also, the Koebe function $k(z) = \frac{z}{(1+z)^2} \in \mathcal{S}^+$. The next characterization is valid for the class $\mathcal{S}^+$ (see [2]):

$$f \in \mathcal{S}^+ \qquad \Leftrightarrow \qquad \sum_{n=2}^{\infty} (n-1) b_n \leq 1. \tag{2}$$

From the relations (1) and (2), we have that

$$b_2 + 2b_3 \leq 1 \qquad (\Rightarrow \quad 0 \leq b_2 \leq 1, \quad 0 \leq b_3 \leq 1/2). \tag{3}$$

If we put $f(z) = z + a_2 z^2 + \ldots$, then by using (1) we easily obtain that

$$b_1 = -a_2, \quad b_2 = a_2^2 - a_3. \tag{4}$$

This implies that $0 \leq b_1 \leq 2$. From (1), we obtain

$$\log \frac{f(z)}{z} = -\log(1 + b_1 z + b_2 z^2 + \cdots),$$

or

$$\sum_{n=1}^{\infty} 2\gamma_n z^n = -b_1 z + \left(\frac{1}{2} b_1^2 - b_2\right) z^2 + \left(-\frac{1}{3} b_1^3 + b_1 b_2 - b_3\right) z^3 + \cdots.$$

(We call γ_n, $n = 1, 2, \ldots$ the logarithmic coefficients of the function f.) From the last relation, we have

$$\begin{cases} 2\gamma_1 = -b_1, \\ 2\gamma_2 = \frac{1}{2}b_1^2 - b_2, \\ 2\gamma_3 = -\frac{1}{3}b_1^3 + b_1 b_2 - b_3. \end{cases} \tag{5}$$

For functions f in $\mathcal{S}^+$, we give, in most of the cases, sharp estimates of their logarithmic coefficients γ_1, γ_2 and γ_3 of f and lower and upper bound of the Fekete–Szegő functional $(a_3 - \gamma a_2^2)$. Additionally, sharp estimates of coefficients a_2, a_3, a_4 and a_5 for functions in a class containing $\mathcal{S}^+$ are given. At the end, the relation between the class $\mathcal{S}^+$ and the class of starlike functions is studied.

2 Results over the Coefficients

We start the section with a study of the Fekete–Szegő functional for the functions in the class $\mathcal{S}^+$.

Theorem 1 *For each $f \in \mathcal{S}^+$, we have*

$$-1 \le a_3 - \gamma a_2^2 \le \begin{cases} 1 + 2e^{-2\gamma/(1-\gamma)}, & 0 \le \gamma \le \frac{\nu_0}{1+\nu_0} = 0.456278\ldots \\ 2(1-\gamma)\frac{(\nu_0+1)^2}{2\nu_0+1}, & \frac{\nu_0}{1+\nu_0} \le \gamma < 1, \end{cases}$$

where $\nu_0 = 0.83927\ldots$ is the positive real root of the equation

$$2(2\nu + 1)e^{-2\nu} = 1. \tag{6}$$

The lower bound is sharp due to the function $f_1(z) = \frac{z}{1+z^2}$.

Proof We will use the same method as in the proof of Fekete–Szego theorem for the class $\mathcal{S}$ (see [1, Theorem 3.8, p. 104]). First, from the relation (4) we have that

$$-1 \le a_3 - a_2^2 = -b_2 \le 0.$$

Since a_2 and a_3 are real, we can put (as in that proof) $a_2 = \operatorname{Re} a_2 = -2\int_0^\infty \varphi(t)\mathrm{d}t$, where φ is real function and $|\varphi(t)| \le e^{-t}$. If we put

$$\int_0^\infty [\varphi(t)]^2 \mathrm{d}t = \left(\nu + \frac{1}{2}\right) e^{-2\nu}, \quad 0 \le \nu < +\infty,$$

then by Valiron–Landau lemma we have that $|a_2| \le 2(\nu + 1)e^{-\nu}$. By using the same method as in [1, p.106], we have

$$a_3 - a_2^2 = \operatorname{Re}\{a_3 - a_2^2\} = \operatorname{Re}\left\{-2\int_0^\infty e^{-2t}[k(t)]^2 \mathrm{d}t\right\},$$

where $k(t)$ is a piecewise continuous complex-valued function with $|k(t)| = 1$ for all t. If we put $k(t) = e^{i\theta(t)}$, then we obtain

$$\begin{aligned} a_3 - a_2^2 &= \mathrm{Re}\,\{a_3 - a_2^2\} = \mathrm{Re}\left\{-2\int_0^\infty e^{-2t} e^{i2\theta(t)}\mathrm{d}t\right\} \\ &= -2\int_0^\infty e^{-2t}\cos(2\theta(t))\mathrm{d}t = -2\int_0^\infty e^{-2t}\left[2\cos^2(\theta(t)) - 1\right]\mathrm{d}t \\ &= 1 - 4\int_0^\infty \left[e^{-t}\cos(\theta(t))\right]^2\mathrm{d}t = 1 - 4\int_0^\infty [\varphi(t)]^2\mathrm{d}t, \end{aligned}$$

where $\varphi(t) = e^{-t}\cos(\theta(t))$. So,

$$a_3 - a_2^2 = 1 - 4\left(\nu + \frac{1}{2}\right)e^{-2\nu} \le 0,$$

if, and only if, $0 \le \nu \le \nu_0$, where $\nu_0 = 0.83927\ldots$ is the root of Eq. (6).

Now, for $0 \le \gamma < 1$ and for $0 \le \nu \le \nu_0$ we have that

$$\begin{aligned} a_3 - \gamma a_2^2 &\le 4(1-\gamma)\left(\int_0^\infty \varphi(t)\mathrm{d}t\right)^2 - 4\int_0^\infty [\varphi(t)]^2\mathrm{d}t + 1 \\ &= 4e^{-2\nu}\left[(1-\gamma)(\nu+1)^2 - \left(\nu + \frac{1}{2}\right)\right] + 1 \\ &=: \psi(\nu). \end{aligned}$$

By using the first derivative of the function, it is an elementary fact that the function ψ has its maximum $\psi(\gamma/(1-\gamma))$ if $\frac{\gamma}{1-\gamma} \in [0, \nu_0]$ and $\psi(\nu_0)$ if $\frac{\gamma}{1-\gamma} \notin [0, \nu_0]$, which gives the right estimation in the theorem. [In the second case, we used that ν_0 satisfies Eq. (6).]

On the other hand, $a_3 - \gamma a_2^2 \ge a_3 - a_2^2 \ge -1$. □

Next, we give estimates of the first three logarithmic coefficients for functions in $\mathcal{S}^+$.

Theorem 2 *Let $f \in \mathcal{S}^+$ and let $\gamma_1, \gamma_2, \gamma_3$ be its logarithmic coefficients given by (5). Then,*

(a) $-1 \le \gamma_1 \le 0$;
(b) $-\frac{1}{2} \le \gamma_2 \le \frac{(\nu_0+1)^2}{2(2\nu_0+1)} = 0.631464\ldots$,
where $\nu_0 = 0.83927\ldots$ is the solution of Eq. (6);
(c) $-\frac{1}{4} \le \gamma_3 \le \frac{1}{3}$.

All these results, except (maybe) the upper bound of γ_2, are the best possible.

Proof (a) It is evident since $\gamma_1 = -\frac{1}{2}b_1$ [from (5)] and $0 \le b_1 \le 2$. The functions $f_1(z) = \frac{z}{1+z^2}$ and $f_2(z) = \frac{z}{(1+z)^2}$ show that the result is the best possible.

(b) From (4) and (5), we have that

$$\gamma_2 = \frac{1}{2}\left(\frac{1}{2}b_1^2 - b_2\right) = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right)$$

and the result directly follows from Theorem 1 for $\gamma = \frac{1}{2}$. For the function $f_1(z) = \frac{z}{1+z^2}$, we have that $\log \frac{f_1(z)}{z} = -\log(1+z^2) = -z^2 + \cdots$, which means that left-hand side estimate is the best possible.
We were not able to prove sharpness of the right-hand side of the inequality (the upper bound of γ_2), but it is worth pointing that the estimate goes in a line with the sharp estimate corresponding to the univalent functions, known to be (see [1, Theorem 3.8] or [7, p.136])

$$|\gamma_2| \le \frac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

(c) From (5), we have

$$2\gamma_3 = -\frac{1}{3}b_1^3 + b_1 b_2 - b_3 =: u(b_1),$$

where

$$u(t) = -\frac{1}{3}t^3 + b_2 t - b_3, \quad 0 \le t \le 2.$$

Since $u'(t) = -t^2 + b_2$ and $u'(t) = 0$ for $t_0 = \sqrt{b_2}$, then the function u attains its maximum

$$u(t_0) = u\left(\sqrt{b_2}\right) = \frac{2}{3}b_2^{3/2} - b_3 \le \frac{2}{3}(1 - 2b_3)^{3/2} - b_3 \le \frac{2}{3},$$

because $b_2 \le 1 - 2b_3$ [see (3)] and the last function is a decreasing function of b_3, $0 \le b_3 \le \frac{1}{2}$. This provides that $\gamma_3 \le \frac{1}{3}$. For the function $f_3(z) = \frac{z}{1+z+z^2}$, we have

$$\log \frac{f_3(z)}{z} = -\log(1 + z + z^2) = -z - \frac{1}{2}z^2 + \frac{2}{3}z^3 + \cdots,$$

i.e., $\gamma_3 = \frac{1}{3}$.
As for lower bound for γ_3, by using (5) and (4), we have

$$\begin{aligned} -2\gamma_3 &= \frac{1}{3}b_1^3 - b_1 b_2 + b_3 \\ &= \frac{1}{3}b_1^3 - b_1(b_1^2 - a_3) + b_3 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{3}b_1^3 + a_3 b_1 + b_3 \\
&= v(b_1),
\end{aligned}
$$

where

$$v(t) = -\frac{2}{3}t^3 + a_3 t + b_3, \quad (0 \le t \le 2).$$

From here, we have

$$v'(t) = -2t^2 + a_3.$$

If $a_3 \le 0$, then $v'(t) \le 0$, and if $a_3 > 0$ then we can write

$$v'(t) = -2(b_1^2 - a_3) - a_3 = -2b_2 - a_3$$

and also we have $v'(t) < 0$, since $0 \le b_2 \le 1$. It means that the function v is a decreasing function, which gives that

$$-2\gamma_3 \le v(0) = b_3 \le \frac{1}{2},$$

i.e., $\gamma_3 \ge -\frac{1}{4}$. The function $f_4(z) = \frac{z}{1+z^3/2}$ shows that the result is the best possible.

□

Let $\mathcal{U}(\lambda)$, $0 < \lambda \le 1$, denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < \lambda \quad (z \in \mathbb{D}). \tag{7}$$

We put $\mathcal{U}(1) \equiv \mathcal{U}$. More about classes $\mathcal{U}$ and $\mathcal{U}(\lambda)$ we can find in [3,5,7] and [4].

Let $\mathcal{U}^+(\lambda)$, $0 < \lambda \le 1$, denote the class of functions f satisfy the conditions (1) and (7). By using (2), we can conclude that $\mathcal{U}^+(\lambda) \subseteq \mathcal{U}^+(1) \equiv \mathcal{S}^+$ (see [2]). For example, the function

$$
\begin{aligned}
f_\lambda(z) &= \frac{z}{1 + (1+\lambda)z + \lambda z^2} \\
&= z - (1+\lambda)z^2 + (1+\lambda+\lambda^2)z^3 - (1+\lambda+\lambda^2+\lambda^3)z^4 + \cdots
\end{aligned} \tag{8}
$$

belongs to the class $\mathcal{U}^+(\lambda)$ and it is extremal in many cases.

Also, if $f \in \mathcal{U}^+(\lambda)$ and has the form (1), then by definition (7) we have that

$$\left| \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' - 1 \right| = \left| -\sum_{n=2}^{\infty}(n-1)b_n z^n \right| < \lambda \quad (z \in \mathbb{D}).$$

For $z = r$ $(0 < r < 1)$ and $r \to 1^-$, from the last equation we obtain

$$\left| -\sum_{n=2}^{\infty}(n-1)b_n \right| \leq \lambda,$$

or, since $b_n \geq 0$,

$$\sum_{n=2}^{\infty}(n-1)b_n \leq \lambda.$$

The last inequality implies

$$0 \leq b_2 \leq \lambda, \quad b_2 + 2b_3 \leq \lambda, \quad b_2 + 2b_3 + 3b_4 \leq \lambda, \ldots. \tag{9}$$

For the coefficients of functions from the class $\mathcal{U}^+(\lambda)$, the next theorem is valid.

Theorem 3 *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $\mathcal{U}^+(\lambda)$, $0 < \lambda \leq 1$, then we have*

$$\begin{aligned} &-(1+\lambda) \leq a_2 \leq 0, \\ &-\lambda \leq a_3 \leq 1 + \lambda + \lambda^2, \\ &-(1+\lambda+\lambda^2+\lambda^3) \leq a_4 \leq \frac{4\lambda}{3}\sqrt{\frac{2\lambda}{3}}, \\ &a_5 \geq \begin{cases} -\lambda/3, & 0 < \lambda \leq 2/27 \\ -9\lambda^2/4, & 2/27 \leq \lambda \leq 1 \end{cases}. \end{aligned}$$

All these inequalities are sharp.

Proof For $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $f \in \mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, it is shown in [4] the next sharp inequalities:

$$|a_2| \leq 1 + \lambda, \quad |a_3| \leq 1 + \lambda + \lambda^2, \quad |a_4| \leq 1 + \lambda + \lambda^2 + \lambda^3.$$

In the same paper, the authors conjectured that $|a_n| \leq \sum_{k=0}^{n-1}\lambda^k$. Since the function f_λ defined by (8) belongs to the class $\mathcal{U}^+(\lambda)$, then the lower bounds for a_2 and a_4 and the upper bounds for a_3 are valid and sharp. We only need to prove the lower bounds for a_3 and a_5 and the upper bounds for a_2 and a_4.

If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and f has the form (1), then by comparing the coefficients we easily conclude that

$$\begin{cases} a_2 = -b_1, \\ a_3 = -b_2 + b_1^2, \\ a_4 = -b_3 + 2b_1b_2 - b_1^3, \\ a_5 = -b_4 + b_2^2 + 2b_1b_3 - 3b_1^2b_2 + b_1^4. \end{cases} \tag{10}$$

From $a_2 = -b_1$ and $b_1 \geq 0$, we have $a_2 \leq 0$. Also, by using (9) and (10), we obtain

$$-a_3 = b_2 - b_1^2 \leq b_2 \leq \lambda,$$

which implies $a_3 \geq -\lambda$. The function $f_6(z) = \frac{z}{1+\lambda z^2}(= z - \lambda z^3 + \cdots)$ shows that two previous results are the best possible.

Further, from (10) we have

$$a_4 = -b_3 + 2b_1b_2 - b_1^3 \leq 2b_2b_1 - b_1^3 =: w(b_1),$$

where $0 \leq b_1 \leq 1 + \lambda$ (since $b_1 = -a_2 \leq 1 + \lambda$). It is an elementary fact to get that the function w has its maximum $\frac{4b_2}{3}\sqrt{\frac{2b_2}{3}}$ for $b_1 = \sqrt{\frac{2b_2}{3}}$. It means that

$$a_4 \leq \frac{4b_2}{3}\sqrt{\frac{2b_2}{3}} \leq \frac{4\lambda}{3}\sqrt{\frac{2\lambda}{3}},$$

since $0 \leq b_2 \leq \lambda$. The function

$$f_7(z) = \frac{z}{1 + \sqrt{\frac{2\lambda}{3}}z + \lambda z^2}$$

shows that the result is the best possible.

Finally, from (10) we also have

$$\begin{aligned} -a_5 &= b_4 - b_2^2 - 2b_1b_3 + 3b_1^2b_2 - b_1^4 \\ &\leq b_4 + 3b_1^2b_2 - b_1^4 \\ &\leq \frac{1}{3}(\lambda - b_2) + 3b_2b_1^2 - b_1^4 \\ &\leq \frac{9}{4}b_2^2 + \frac{1}{3}(\lambda - b_2) \\ &\leq \begin{cases} \lambda/3, & 0 < \lambda \leq 2/27 \\ 9\lambda^2/4 & 2/27 \leq \lambda \leq 1 \end{cases}, \end{aligned}$$

where we used the relation (9) and the same method as in the previous case. The functions

$$f_2(z) = \frac{z}{1 + \sqrt{\frac{3\lambda}{2}}z + \lambda z^2} \quad \text{and} \quad f_8(z) = \frac{z}{1 + \frac{\lambda}{3}z^4}$$

show that the result is the best possible. □

For $\lambda = 1$ from the previous theorem, we have

Corollary 1 *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belong to the class $\mathcal{S}^+$. Then, we have the next sharp inequalities*

$$-2 \le a_2 \le 0, \quad -1 \le a_3 \le 3, \quad -4 \le a_4 \le \frac{4}{3}\sqrt{\frac{2}{3}}, \quad -\frac{9}{4} \le a_5 \le 5.$$

We note that upper bound for a_5 follows from de Brange's theorem.

3 Relation with Starlike Functions

In this section, we study the relation between the class $\mathcal{S}^+$ and the class of starlike functions.

Theorem 4 *Let $f \in \mathcal{S}^+$ and let $b_1 = 0$, then $f \in \mathcal{S}^\star$.*

Proof Since $f \in \mathcal{S}^+$, then $\sum_{n=2}^{\infty}(n-1)b_n \le 1$, and since $b_1 = 0$, then also $\sum_{n=2}^{\infty}(n-1)b_n \le 1 = 1 - b_1$, which implies, by result of Reade et al. ([6]) (see the previous sited result in Theorem 6), that $f \in \mathcal{S}^\star$.

We note that if $b_1 = 0$, then $\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}$ $(z \in \mathbb{D})$ since

$$|z|^2 \cdot |z/f(z) - 1| \le |z|^2 \sum_{n=2}^{\infty} b_n \le |z|^2 \sum_{n=2}^{\infty}(n-1)b_n \le |z|^2 < 1 \quad (z \in \mathbb{D}).$$

But under the condition of this theorem we do not have that $f \in \mathcal{S}^\star(1/2)$. For example, for the function $f_1(z) = \frac{z}{1+z^2}$ we have $b_1 = 0$, but $\sum_{n=1}^{\infty}(2n-1)b_n = 3$, which means that $f_1 \notin \mathcal{S}^\star(1/2)$ (by the previous theorem). □

Theorem 5 *Let $f \in \mathcal{S}^+$. Then, the function*

$$g(z) = z + \frac{1}{2}\left(\frac{z}{f(z)} - 1 - b_1 z\right) \tag{11}$$

is univalent in $\mathbb{D}$. More precisely, $\operatorname{Re} g'(z) > 0$ $(z \in \mathbb{D})$, *$g \in \mathcal{S}^\star$ and $g \in \mathcal{U}$.*

Proof It is well known that if $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $\sum_{n=2}^{\infty} n|a_n| \le 1$, then $\operatorname{Re} f'(z) > 0$ $(z \in \mathbb{D})$ and $f \in \mathcal{S}^\star$ with $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1$ $(z \in \mathbb{D})$. It is easily to prove those statement (in the second case better to consider the form $|zf'(z) - f(z)| < |f(z)|$).

By (11), we have

$$g(z) = z + \sum_{n=2}^{\infty} \frac{1}{2} b_n z^n.$$

Since $f \in \mathcal{S}^+$ implies $\sum_{n=2}^{\infty}(n-1)b_n \le 1$ and since $\frac{n}{2(n-1)} \le 1$ for $n \ge 2$, then

$$\sum_{n=2}^{\infty} n\left(\frac{1}{2}b_n\right) = \sum_{n=2}^{\infty}(n-1)b_n \frac{n}{2(n-1)} \le \sum_{n=2}^{\infty}(n-1)b_n \le 1.$$

By previous remarks, we have $\operatorname{Re} g'(z) > 0$ $(z \in \mathbb{D})$ and $g \in \mathcal{S}^\star$. Also, $g \in \mathcal{U}$ by the result given in [5]. □

Theorem 6 *Let $f \in \mathcal{A}$ and satisfy the condition (1). Then, the condition*

$$\sum_{n=1}^{\infty}(2n-1)b_n \le 1 \tag{12}$$

is necessary and sufficient for f to be in the class $\mathcal{S}^\star(1/2)$.

Proof The sufficient condition follows from the result given in the paper of Reade, Silverman and Todorov [6].

Let us prove the necessary case. If $f \in \mathcal{S}^\star(\frac{1}{2})$, then

$$\left|\frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)}}\right| < 1 \quad (z \in \mathbb{D})$$

or equivalently

$$\frac{\left|z\left(\frac{z}{f(z)}\right)'\right|}{\left|\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)'\right|} < 1 \quad (z \in \mathbb{D})$$

and from here

$$\frac{\left|\sum_{n=1}^{\infty} nb_n z^n\right|}{\left|1 - \sum_{n=2}^{\infty}(n-1)b_n z^n\right|} < 1 \quad (z \in \mathbb{D}).$$

Since the previous relation is valid for every $z \in \mathbb{D}$, then for $z = r$ $(0 < r < 1)$ we have from the last inequality that

$$\frac{\sum_{n=1}^{\infty} nb_n r^n}{1 - \sum_{n=2}^{\infty}(n-1)b_n r^n} < 1,$$

which implies the condition

$$\sum_{n=1}^{\infty}(2n-1)b_n r^n < 1.$$

Finally, when $r \to 1$ we have

$$\sum_{n=1}^{\infty}(2n-1)b_n \le 1,$$

i.e., the relation (12). □

Remark 1 Since the class of convex functions is the subset of the class $S^{\star}(1/2)$, it follows that if a function f is convex and

$$\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$$

with $b_n \ge 0$ for $n = 1, 2, \ldots$, we have

$$\sum_{n=1}^{\infty}(2n-1)b_n \le 1.$$

The converse is not true. Namely, for the function

$$f(z) = \frac{z}{1 + \frac{1}{3}z^2},$$

we have that

$$\frac{z}{f(z)} = 1 + \frac{1}{3}z^2$$

and

$$\sum_{n=1}^{\infty}(2n-1)b_n = 1,$$

but

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 - 2z^2 + \frac{1}{9}z^4}{1 - \frac{1}{9}z^4} < 0$$

for $z = r(0 < r < 1)$ and r close to 1.

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