Bull. Korean Math. Soc. **57** (2020), No. 4, pp. 839–850
https://doi.org/10.4134/BKMS.b190520
pISSN: 1015-8634 / eISSN: 2234-3016

SHARP BOUNDS FOR INITIAL COEFFICIENTS AND THE SECOND HANKEL DETERMINANT

Rosihan M. Ali, See Keong Lee, and Milutin Obradović

Reprinted from the
Bulletin of the Korean Mathematical Society
Vol. 57, No. 4, July 2020

Bull. Korean Math. Soc. **57** (2020), No. 4, pp. 839–850
https://doi.org/10.4134/BKMS.b190520
pISSN: 1015-8634 / eISSN: 2234-3016

SHARP BOUNDS FOR INITIAL COEFFICIENTS AND THE SECOND HANKEL DETERMINANT

Rosihan M. Ali, See Keong Lee, and Milutin Obradović

Abstract. For functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belonging to particular classes, this paper finds sharp bounds for the initial coefficients a_2, a_3, a_4, as well as the sharp estimate for the second order Hankel determinant $H_2(2) = a_2a_4 - a_3^2$. Two classes are treated: first is the class consisting of $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in the unit disk $\mathbb{D}$ satisfying

$$\left|\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z) - 1\right| < \lambda, \quad 0 < \alpha < 1,\ 0 < \lambda \leq 1.$$

The second class consists of Bazilevič functions $f(z) = z+a_2z^2+a_3z^3+\cdots$ in $\mathbb{D}$ satisfying

$$\operatorname{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right\} > 0, \quad \alpha > 0.$$

1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. Further, let $\mathcal{S}^\star$ denote its familiar subset consisting of univalent starlike functions. The Bazilevič class $\mathcal{B}_1(\alpha)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right\} > 0,\ z \in \mathbb{D}, \alpha \geq 0.$$

R. Singh investigated this class in his paper [13]. It is well-known that $\mathcal{B}_1(\alpha)$, $\alpha \geq 0$, consists of normalized univalent functions in $\mathbb{D}$. Indeed, for $\alpha = 0$,

Received May 23, 2019; Accepted June 4, 2020.
2010 *Mathematics Subject Classification.* Primary 30C45, 30C50.
Key words and phrases. Coefficient estimates, Hankel determinants, univalent functions, Bazilevič functions.
The first author gratefully acknowledged support from a USM research university grant 1001.PMATHS.8011101. The second author acknowledged support from a USM research university grant 1001.PMATHS.8011038. The work of the third author was supported by MNZZS Grant, No. ON174017, Serbia.

they are starlike functions, that is, $\mathcal{B}_1(\alpha) \subset \mathcal{S}^\star$. For $\alpha = 1$, these functions are close-to-convex, which analytically satisfy the condition

$$\operatorname{Re}\{f'(z)\} > 0,\ z \in \mathbb{D}.$$

Closely related is the class $\mathcal{U}(\alpha, \lambda)$ introduced by Obradović in [6]. This class consists of functions $f \in \mathcal{A}$ for which

$$(1) \qquad \left| \left(\frac{z}{f(z)} \right)^{1+\alpha} f'(z) - 1 \right| < \lambda,\ z \in \mathbb{D},\ 0 < \alpha < 1,\ 0 < \lambda \leq 1.$$

Thus $\mathcal{U}(\alpha, \lambda)$ is linked to $\mathcal{B}_1(\alpha)$ for $\alpha < 0$. It is shown in [6] that $\mathcal{U}(\alpha, \lambda) \subset \mathcal{S}^\star$ if

$$0 < \lambda \leq \frac{1-\alpha}{\sqrt{(1-\alpha)^2 + \alpha^2}} := \lambda_\star.$$

In the limiting cases when $\lambda = 1$, and either $\alpha = 0$ or $\alpha = 1$, functions in the class (1) satisfy respectively

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad \text{or} \quad \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1.$$

The former is a subclass of $\mathcal{S}^\star$, while functions in the latter class are univalent (see [1, 9]). Considerable interest has gone into the studies of this latter class and its generalization, see for example, the works in [8, 14]. The univalency problem for the class $\mathcal{U}(\alpha, \lambda)$ when α is a complex number was studied by Fournier and Ponnusamy in [2].

For functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{A}$, $n = 1, 2, \ldots$ and $q = 1, 2, \ldots,$ the Hankel determinants $H_q(n)$ are defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

It is evident that the second order Hankel determinant is $H_2(2) = a_2 a_4 - a_3^2$. The Hankel determinants are important in the study of singularities and power series with integral coefficients [10, 11]. Lee *et al.* in [4] gave a survey on Hankel determinants and obtained bounds for $H_2(2)$ for several classes defined by subordination.

In this paper, for $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belonging to the class $\mathcal{U}(\alpha, \lambda)$, sharp bounds for the initial coefficients a_2, a_3, a_4, as well as the sharp estimate for the second order Hankel determinant $H_2(2) = a_2 a_4 - a_3^2$ are obtained. Different extremal function occurs for each coefficient bound. Similar problems are studied for Bazilevič functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{B}_1(\alpha)$.

The following representation result is needed in the sequel.

Lemma 1.1 ([7]). *Let $f \in \mathcal{U}(\alpha, \lambda)$, $0 < \alpha < 1$, $0 < \lambda \leq 1$. Then*

$$\left(\frac{z}{f(z)}\right)^{\alpha} = 1 - \alpha\lambda z^{\alpha} \int_0^z \frac{\omega(t)}{t^{\alpha+1}} dt, \tag{2}$$

where ω is analytic in $\mathbb{D}$, $\omega(0) = 0$, and $|\omega(z)| < 1$.

A main tool used in this paper is a result of Prokhorov and Szynal [12, Lemma 2], which centers on the class Ω consisting of analytic self-maps $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ in $\mathbb{D}$. The authors obtained sharp estimates on the functional

$$\Psi(\omega) = |c_3 + \mu c_1 c_2 + \nu c_1^3|$$

within the class of all $\omega \in \Omega$. The actual result is very elaborate. However, we do not require the full extent of the result, and the ensuing lemma captures the portion needed in our sequel.

We shall adopt the following notations used in [12]: For μ and ν real numbers, let

$$\begin{aligned}
D_1 &= \left\{(\mu, \nu) : |\mu| \leq \frac{1}{2},\ |\nu| \leq 1\right\}, \\
D_2 &= \left\{(\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2,\ \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\right\}, \\
D_5 &= \{(\mu, \nu) : |\mu| \leq 2,\ |\nu| \geq 1\}, \\
D_6 &= \left\{(\mu, \nu) : 2 \leq |\mu| \leq 4,\ \nu \geq \frac{1}{12}(\mu^2 + 8)\right\}, \\
D_7 &= \left\{(\mu, \nu) : |\mu| \geq 4,\ \nu \geq \frac{2}{3}(|\mu| - 1)\right\}.
\end{aligned}$$

Lemma 1.2 ([12]). *Let $\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in \Omega$. For real μ and ν, let*

$$\Psi(\omega) = \left|c_3 + \mu c_1 c_2 + \nu c_1^3\right|.$$

Then the sharp estimate $\Psi(\omega) \leq \Phi(\mu, \nu)$ holds, where

$$\Phi(\mu, \nu) = \begin{cases} 1, & (\mu, \nu) \in D_1 \cup D_2 \cup \{(2, 1)\}; \\ |\nu|, & (\mu, \nu) \in D_5 \cup D_6 \cup D_7. \end{cases}$$

2. Coefficient bounds and the second Hankel determinant

Theorem 2.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belong to the class $\mathcal{U}(\alpha, \lambda)$, $0 < \alpha < 1$, $0 < \lambda \leq 1$. Then the following sharp estimates hold*:

(a) $|a_2| \leq \frac{\lambda}{1-\alpha}$.

(b) $|a_3| \leq \begin{cases} \frac{\lambda}{2-\alpha}, & 0 < \lambda \leq \frac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)}, \\ \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}, & \frac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)} \leq \lambda \leq 1. \end{cases}$

(c) $|a_4| \leq \begin{cases} \frac{\lambda}{3-\alpha}, & 0 < \lambda \leq \min\{\lambda_\nu, 1\}, \\ \frac{(1+\alpha)(1+2\alpha)\lambda^3}{6(1-\alpha)^3}, & \lambda_\nu \leq \lambda \leq 1,\ \alpha_\nu \leq \alpha < 1, \end{cases}$

where

$$\lambda_\nu = \sqrt{\frac{6(1-\alpha)^3}{(1+\alpha)(1+2\alpha)(3-\alpha)}},$$

and $\alpha_\nu = 0.123956\ldots$ *is the root of the equation*

$$4\alpha^3 - 15\alpha^2 + 26\alpha - 3 = 0.$$

Proof. Let $\omega(z) = c_1 z + c_2 z^2 + \cdots \in \Omega$ in (2). Upon integration,

$$\left(\frac{z}{f(z)}\right)^\alpha = 1 - \alpha\lambda \sum_{n=1}^{\infty} \frac{c_n}{n-\alpha} z^n,$$

that is,

$$\frac{f(z)}{z} = \left(1 - \alpha\lambda \sum_{n=1}^{\infty} \frac{c_n}{n-\alpha} z^n\right)^{-\frac{1}{\alpha}} \tag{3}$$

(the principal value is used here).

From the relation (3), some calculations show that

$$\sum_{n=1}^{\infty} a_{n+1} z^n = \sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n + \frac{1+\alpha}{2}\left(\sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n\right)^2 + \frac{(1+\alpha)(1+2\alpha)}{6}\left(\sum_{n=1}^{\infty} \frac{\lambda c_n}{n-\alpha} z^n\right)^3 + \cdots.$$

By comparing coefficients,

$$\begin{aligned} a_2 &= \frac{\lambda}{1-\alpha} c_1; \\ a_3 &= \frac{\lambda}{2-\alpha} c_2 + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} c_1^2; \\ a_4 &= \frac{\lambda}{3-\alpha} c_3 + \frac{(1+\alpha)\lambda^2}{(1-\alpha)(2-\alpha)} c_1 c_2 + \frac{(1+\alpha)(1+2\alpha)\lambda^3}{6(1-\alpha)^3} c_1^3. \end{aligned} \tag{4}$$

Since $|c_1| \leq 1$, (4) easily leads to $|a_2| \leq \frac{\lambda}{1-\alpha}$. Also by using $|c_2| \leq 1 - |c_1|^2$, (4) shows that

$$\begin{aligned} |a_3| &\leq \frac{\lambda}{2-\alpha}|c_2| + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}|c_1|^2 \\ &\leq \frac{\lambda}{2-\alpha}(1 - |c_1|^2) + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}|c_1|^2 \\ &= \frac{\lambda}{2-\alpha} + \left(\frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} - \frac{\lambda}{2-\alpha}\right)|c_1|^2. \end{aligned}$$

The assertion 2.1(b) now follows from the cases when the factor $\frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} - \frac{\lambda}{2-\alpha}$ is either positive or negative, and $|c_1| \le 1$.

From (4), the coefficient a_4 can be expressed in the form

$$|a_4| = \frac{\lambda}{3-\alpha}|c_3 + \mu c_1 c_2 + \nu c_1^3|, \tag{5}$$

where

$$\mu = \mu(\alpha,\lambda) = \frac{(1+\alpha)(3-\alpha)\lambda}{(1-\alpha)(2-\alpha)},\ \nu = \nu(\alpha,\lambda) = \frac{(1+\alpha)(1+2\alpha)(3-\alpha)\lambda^2}{6(1-\alpha)^3}. \tag{6}$$

It is evident that μ and ν are positive for $0 < \alpha < 1$. Lemma 1.2 will be used in this case. First, the values α and λ are constrained so that $0 < \mu \le \frac{1}{2}$, $\mu \le 2$, $\mu \le 4$, and $\nu \le 1$. From (6), the following equivalences hold:

$$0 < \mu \le \frac{1}{2} \Leftrightarrow \lambda \le \frac{(1-\alpha)(2-\alpha)}{2(1+\alpha)(3-\alpha)} := \lambda_{1/2};$$

$$\mu \le 2 \Leftrightarrow \lambda \le \frac{2(1-\alpha)(2-\alpha)}{(1+\alpha)(3-\alpha)} := \lambda_2;$$

$$\mu \le 4 \Leftrightarrow \lambda \le \frac{4(1-\alpha)(2-\alpha)}{(1+\alpha)(3-\alpha)} := \lambda_4;$$

$$\nu \le 1 \Leftrightarrow \lambda \le \sqrt{\frac{6(1-\alpha)^3}{(1+\alpha)(1+2\alpha)(3-\alpha)}} := \lambda_\nu.$$

It can also be shown that the functions $\lambda_{1/2}, \lambda_2, \lambda_4, \lambda_\nu$ are decreasing for $\alpha \in (0,1)$, and that

$$0 < \lambda_{1/2} < \frac{1}{3},\ 0 < \lambda_2 < \frac{4}{3},\ 0 < \lambda_4 < \frac{8}{3},\ 0 < \lambda_\nu < \sqrt{2}.$$

We also require these previous values to lie in the interval $(0,1]$. Note that

$$0 < \lambda_2 \le 1, \qquad \alpha \in [\alpha_2, 1),$$

where $\alpha_2 = \frac{4-\sqrt{13}}{3} = 0.13148\ldots$,

$$0 < \lambda_4 \le 1, \qquad \alpha \in [\alpha_4, 1),$$

where $\alpha_4 = \frac{7-\sqrt{24}}{5} = 0.42020\ldots$, and

$$0 < \lambda_\nu \le 1, \qquad \alpha \in [\alpha_\nu, 1),$$

where $\alpha_\nu = 0.123956\ldots$ is the unique real root of the equation $4\alpha^3 - 15\alpha^2 + 26\alpha - 3 = 0$ in the interval $(0,1)$. Also, it is clear that

$$0 < \lambda_{1/2} < \lambda_2 < \lambda_4,$$

and that

$$\lambda_{1/2} \le \lambda_\nu, \qquad \alpha \in (0, \alpha'_\nu],$$

where $\alpha'_\nu = 0.96739\ldots$ is the root of the equation $22\alpha^3 - 65\alpha^2 - 28\alpha + 68 = 0$ (of course $\lambda_\nu \le \lambda_{1/2}$ for $\alpha \in [\alpha'_\nu, 1)$). Further,

$$\lambda_\nu \le \lambda_2 \le 1,\ \alpha \in [\alpha_2, 1), \quad \text{and} \quad \lambda_\nu \le \lambda_4 \le 1,\ \alpha \in [\alpha_4, 1),$$

where α_2 and α_4 are defined earlier.

For the proof of Statement 2.1(c), two cases are considered: $\lambda \le \lambda_\nu$ and $\lambda \ge \lambda_\nu$.

Case 1 ($\lambda \le \lambda_\nu$). The condition $\lambda \le \lambda_\nu$ implies $\nu \le 1$. If $\alpha \in (0, \alpha_\nu]$ and $0 < \lambda \le \lambda_{1/2}$, the inequality $\lambda_{1/2} \le \lambda_\nu$ implies $0 < \mu \le \frac{1}{2}$ and $0 < \nu \le 1$, which by Lemma 1.2 gives $\Phi(\mu, \nu) = 1$. If $\alpha \in (0, \alpha_\nu]$ and $\lambda_{1/2} \le \lambda \le 1$, then $\frac{1}{2} \le \mu < 2$, $0 < \nu \le 1$ and $\frac{4}{27}(\mu+1)^3 - (\mu+1) \le \nu$, and so Lemma 1.2 implies $\Phi(\mu, \nu) = 1$.

To verify that the latter inequality

$$\frac{4}{27}(\mu+1)^3 - (\mu+1) \le \nu \tag{7}$$

holds for $\alpha \in (0, \alpha_\nu]$ and $\lambda_{1/2} \le \lambda \le 1$, write

$$\mu = \mu(\alpha, \lambda) = \frac{(1+\alpha)(3-\alpha)\lambda}{(1-\alpha)(2-\alpha)} = A\lambda,$$

$$\nu = \nu(\alpha, \lambda) = \frac{(1+\alpha)(1+2\alpha)(3-\alpha)\lambda^2}{6(1-\alpha)^3} = B\lambda^2,$$

where

$$A = A(\alpha) = \frac{(1+\alpha)(3-\alpha)}{(1-\alpha)(2-\alpha)}, \quad B = B(\alpha) = \frac{(1+\alpha)(1+2\alpha)(3-\alpha)}{6(1-\alpha)^3}.$$

Here A and B are increasing functions of α. From the value $\lambda_{1/2} = \frac{(1-\alpha)(2-\alpha)}{2(1+\alpha)(3-\alpha)} = \frac{1}{2A}$, it is seen that for $\alpha \in (0, \alpha_\nu]$,

$$\frac{3}{2} < A \le 1.967\ldots < 2,\ \frac{1}{2} < B \le 1,\ 0.254\ldots \le \lambda_{1/2} < \frac{1}{3},\ \lambda_{1/2} = \frac{1}{2A}.$$

Evidently the inequality (7) is equivalent to

$$\psi(\lambda, \alpha) =: 4(A\lambda+1)^3 - 27(A\lambda+1) - 27B\lambda^2 \le 0.$$

Now

$$\psi'_\lambda(\lambda, \alpha) = 12A^3\lambda^2 + 6(4A^2 - 9B)\lambda - 15A,$$

and since $4A^2 - 9B > 0$ for $\alpha \in (0, \alpha_\nu]$ (because $4A^2 > 9$, $9B \le 9$ for $\alpha \in (0, \alpha_\nu]$), this implies ψ'_λ is an increasing function of λ. The endpoint values are

$$\psi'_\lambda(\lambda_{1/2}, \alpha) = -\frac{27B}{A} < 0,$$

while

$$\psi'_\lambda(1, \alpha) = 12A(A+1)^2 - 27A - 54B > 0.$$

Thus the function ψ has its minimum in the interval $(\lambda_{1/2}, 1]$. Therefore,

$$\psi(\lambda, \alpha) \le \max\{\psi(\lambda_{1/2}, \alpha), \psi(1, \alpha)\}. \tag{8}$$

Since

$$\psi(\lambda_{1/2}, \alpha) = -54 - 27B\lambda_{1/2}^2 < 0,$$

and

$$\psi(1, \alpha) = -\frac{1}{6}(888 + 3636\alpha + 212\alpha^2 - 11937\alpha^3 + 5400\alpha^4 - 1080\alpha^5 + 216\alpha^6) < 0$$

for $\alpha \in (0, \alpha_\nu]$, (8) shows that $\psi(\lambda, \alpha) < 0,\ \alpha \in (0, \alpha_\nu]$.

Following similar arguments above, it is seen that the same conclusion holds in the case when $\alpha \in [\alpha_\nu, 1)$.

Case 2 ($\lambda \geq \lambda_\nu$, $\alpha \in [\alpha_\nu, 1)$). In this case, readily $\nu \geq 1$. If $\lambda_\nu \leq \lambda \leq \lambda_2$, $\alpha \in [\alpha_\nu, 1)$, then $0 < \mu \leq 2$, $\nu \geq 1$, which by Lemma 1.2 implies $\Phi(\mu, \nu) = \nu$. If $\lambda_2 \leq \lambda \leq \lambda_4$, $\alpha \in [\alpha_2, 1)$, then $2 \leq \mu \leq 4$, $\nu \geq \frac{1}{12}(\mu^2 + 8)$, and Lemma 1.2 again gives $\Phi(\mu, \nu) = \nu$. Finally, if $\lambda \geq \lambda_4$, $\alpha \in [\alpha_4, 1)$, then $\mu \geq 4$, $\nu \geq \frac{2}{3}(\mu - 1)$ and Lemma 1.2 yields $\Phi(\mu, \nu) = \nu$.

From (5), we deduce that $|a_4| \leq \frac{\lambda}{3-\alpha}$ in Case 1, or $|a_4| \leq \frac{\lambda}{3-\alpha}\nu$ in Case 2, and from there the assertion (c) follows. All these results are best possible as shown by taking $\omega(z) = z$, $\omega(z) = z^2$, or $\omega(z) = z^3$ in (3) for the extremal functions f. □

In the next theorem, a sharp bound on the Hankel determinant of the second order for the class $\mathcal{U}(\alpha, \lambda)$ is obtained for certain α and λ.

Theorem 2.2. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belong to the class $\mathcal{U}(\alpha, \lambda)$, where $0 < \alpha \leq 2 - \sqrt{2}$ and $\lambda^2 \leq \frac{12(1-\alpha)^2}{(1+\alpha)(3-\alpha)(2-\alpha)^2}$. Then the sharp estimate*

$$|H_2(2)| = |a_2 a_4 - a_3^2| \leq \left(\frac{\lambda}{2-\alpha}\right)^2$$

holds.

Proof. By using the relation (4), after some simple computations, we obtain

$$\begin{aligned} H_2(2) &= a_2 a_4 - a_3^2 \\ &= \frac{\lambda^2}{(1-\alpha)(3-\alpha)} \left(c_1 c_3 - \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} c_2^2 - \frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2} c_1^4 \right), \end{aligned}$$

and so

$$\begin{aligned} |H_2(2)| \leq{}& \frac{\lambda^2}{(1-\alpha)(3-\alpha)} \\ & \left(|c_1||c_3| + \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2}|c_2|^2 + \frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2}|c_1|^4 \right). \end{aligned}$$

For the function $\omega(z) = c_1 z + c_2 z^2 + \cdots$ with $|\omega(z)| < 1$, $z \in \mathbb{D}$, the Carathéodory-Toeplitz inequalities holds:

$$|c_1| \leq 1,\ |c_2| \leq 1 - |c_1|^2,\ \left|c_3(1 - |c_1|^2) + \bar{c}_1 c_2^2\right| \leq (1 - |c_1|^2)^2 - |c_2|^2. \tag{9}$$

We may suppose that $a_2 \geq 0$, which implies that $c_1 \geq 0$. Thus the relations (9) take the form

$$0 \leq c_1 \leq 1,\ |c_2| \leq 1 - c_1^2,\ |c_3| \leq 1 - c_1^2 - \frac{|c_2|^2}{1+c_1}. \tag{10}$$

It follows from (6) and (10) that

$$\begin{aligned} |H_2(2)| \leq \frac{\lambda^2}{(1-\alpha)(3-\alpha)} & \left(c_1(1-c_1^2) + \left(\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - \frac{c_1}{1+c_1} \right) |c_2|^2 \right. \\ & \left. + \frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2} c_1^4 \right). \end{aligned} \tag{11}$$

Since $0 < \alpha \leq 2 - \sqrt{2}$, then $\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} \geq \frac{1}{2} \geq \frac{c_1}{1+c_1}$ for every $0 \leq c_1 \leq 1$, and from (11),

$$\begin{aligned} |H_2(2)| \leq \frac{\lambda^2}{(1-\alpha)(3-\alpha)} & \left(c_1(1-c_1^2) + \left(\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - \frac{c_1}{1+c_1} \right) (1-c_1^2)^2 \right. \\ & \left. + \frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2} c_1^4 \right) := \frac{\lambda^2}{(1-\alpha)(3-\alpha)} F(c_1), \end{aligned}$$

where

$$\begin{aligned} F(c_1) = \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} & - \left(2\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - 1) \right) c_1^2 \\ & + \left(\frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2} - \frac{1}{(2-\alpha)^2} \right) c_1^4. \end{aligned}$$

Since

$$2\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - 1 \geq 0,$$

and the assumption of the theorem yields

$$\frac{(1+\alpha)(3-\alpha)\lambda^2}{12(1-\alpha)^2} - \frac{1}{(2-\alpha)^2} \leq 0,$$

we deduce that

$$F(c_1) \leq \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2}.$$

Hence

$$|H_2(2)| \leq \frac{\lambda^2}{(1-\alpha)(3-\alpha)} \cdot \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} = \left(\frac{\lambda}{2-\alpha} \right)^2.$$

The function f obtained from $\omega(z) = z^2$ in (3) shows the result is sharp. □

3. The second Hankel determinant for Bazilevič functions

Singh in his paper [13] obtained sharp bounds for the first four coefficients for the class $\mathcal{B}_1(\alpha)$. Recent advances on coefficient estimates for $\mathcal{B}_1(\alpha)$ are the partial estimates for the fifth and sixth coefficients obtained by Marjono *et al.* in [5].

The same approach used in Theorem 2.1 provides an alternate proof to the result of Singh [13]. We provide an outline of the proof.

Theorem 3.1 ([13]). *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belong to the class $\mathcal{B}_1(\alpha)$, $\alpha > 0$. Then the following sharp estimates hold*:

(a) $|a_2| \leq \frac{2}{1+\alpha}$, $\alpha > 0$;

(b) $|a_3| \leq \begin{cases} \frac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2}, & 0 < \alpha \leq 1, \\ \frac{2}{2+\alpha}, & \alpha \geq 1; \end{cases}$

(c) $|a_4| \leq \begin{cases} \frac{2}{3+\alpha} + 4\frac{(1-\alpha)(5+3\alpha+\alpha^2)}{3(2+\alpha)(1+\alpha)^3}, & 0 < \alpha \leq 1, \\ \frac{2}{3+\alpha}, & \alpha \geq 1. \end{cases}$

Proof. We give a sketch of the proof. If $f \in \mathcal{B}_1(\alpha)$, $\alpha > 0$, then

$$\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

where $\omega \in \Omega$. Thus

$$\left(\frac{f(z)}{z}\right)^{\alpha} = 1 + \frac{2\alpha}{z^\alpha}\int_0^z t^{\alpha-1}\frac{\omega(t)}{1-\omega(t)}dt,$$

and

$$\frac{f(z)}{z} = \left(1 + \frac{2\alpha}{z^\alpha}\int_0^z t^{\alpha-1}(\omega(t)+\omega^2(t)+\cdots)dt\right)^{\frac{1}{\alpha}}. \tag{12}$$

If $\omega(z) = c_1z + c_2z^2 + \cdots \in \Omega$ in (12), comparing coefficients yield

$$\begin{aligned} a_2 &= \frac{2}{1+\alpha}c_1, \\ a_3 &= \frac{2}{2+\alpha}c_2 + \left(\frac{2}{2+\alpha} + \frac{2(1-\alpha)}{(1+\alpha)^2}\right)c_1^2, \\ a_4 &= \frac{2}{3+\alpha}\left(c_3 + \mu c_1c_2 + \nu c_1^3\right), \end{aligned} \tag{13}$$

where

$$\mu = 2 + 2\frac{(1-\alpha)(3+\alpha)}{(1+\alpha)(2+\alpha)} = \frac{2(\alpha+5)}{(\alpha+1)(\alpha+2)},$$

$$\nu = 1 + 2\frac{(1-\alpha)(3+\alpha)(5+3\alpha+\alpha^2)}{3(2+\alpha)(1+\alpha)^3} = \frac{\alpha^4+5\alpha^3+11\alpha^2+19\alpha+36}{3(2+\alpha)(1+\alpha)^3}.$$

Using the inequalities $|c_1| \leq 1$ and $|c_2| \leq 1 - |c_1|^2$ in (13), we easily obtain the first two results in Theorem 3.1.

For the estimate of $|a_4|$, it follows from (13) that

$$|a_4| = \frac{2}{3+\alpha}\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| \le \frac{2}{3+\alpha}\Phi(\mu,\nu), \tag{14}$$

where $\Phi(\mu,\nu)$ is given in Lemma 1.2. It can be verified that μ and ν are decreasing and positive functions of α. Further,

$$2 \le \mu < 5,\ 1 \le \nu < 6 \quad \text{for } \alpha \in (0,1],$$

and

$$0 < \mu \le 2,\ \frac{1}{3} < \nu \le 1 \quad \text{for } \alpha \in [1,+\infty).$$

Using the same method as in the proof of Theorem 2.1 leads to the deductions that $\Phi(\mu,\nu) = \nu$ for $\alpha \in (0,1]$, and $\Phi(\mu,\nu) = 1$ for $\alpha \in [1,+\infty)$. The desired result now follows from (14).

We note that in establishing certain inequalities, it is sometimes more expedient to put $\mu = 2 + 2\mu_1$ and $\nu = 1 + 2\nu_1$, where

$$\mu_1 = \frac{(1-\alpha)(3+\alpha)}{(1+\alpha)(2+\alpha)}, \quad \nu_1 = \frac{(1-\alpha)(3+\alpha)(5+3\alpha+\alpha^2)}{3(2+\alpha)(1+\alpha)^3}.$$

□

The final result presented here is the sharp bound for the second order Hankel determinant for the class $\mathcal{B}_1(\alpha)$. Krishna and Ramreddy in [3] earlier obtained this sharp bound for $\alpha \in [0,1]$. The method of proof presented here is different, and establishes the result for all $\alpha > 0$.

Theorem 3.2. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ belong to the class $\mathcal{B}_1(\alpha)$, $\alpha > 0$. Then the sharp bound for the second order Hankel determinant is*

$$|H_2(2)| = |a_2 a_4 - a_3^2| \le \left(\frac{2}{2+\alpha}\right)^2.$$

Proof. By using the relations (13) and (14), $H_2(2)$ is

$$H_2(2) = \frac{4c_1}{(1+\alpha)(3+\alpha)}(c_3 + \mu_2 c_1 c_2 + \nu_2 c_1^3) - \left(\frac{2}{2+\alpha}\right)^2 c_2^2, \tag{15}$$

where

$$\mu_2 = \frac{2}{(2+\alpha)^2}, \qquad \nu_2 = \frac{\alpha^4 + 6\alpha^3 + 12\alpha^2 + 2\alpha - 9}{3(1+\alpha)^2(2+\alpha)^2}. \tag{16}$$

Thus

$$|H_2(2)| \le \frac{4}{(1+\alpha)(3+\alpha)}\left(|c_1||c_3| + \mu_2|c_1|^2|c_2| + \nu_2|c_1|^4 + \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2}|c_2|^2\right).$$

If we apply (10) for c_2 and c_3, then the previous inequality yields

$$|H_2(2)| \le \frac{4}{(1+\alpha)(3+\alpha)}\left(c_1(1-c_1^2) + \left(\frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} - \frac{c_1}{1+c_1}\right)|c_2|^2 + \mu_2 c_1^2|c_2| + |\nu_2|c_1^4\right)$$

$$\leq \frac{4}{(1+\alpha)(3+\alpha)}\left(\frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} - Ac_1^2 + Bc_1^4\right)$$
$$:= \frac{4}{(1+\alpha)(3+\alpha)}\Phi(c_1),$$

where

$$\Phi(c_1) = \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} - Ac_1^2 + Bc_1^4,\ 0 \leq c_1 \leq 1,$$

and

$$A = 2\frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} - 1 - \mu_2, \tag{17}$$
$$B = \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} + |\nu_2| - 1 - \mu_2 = |\nu_2| - \frac{3}{(2+\alpha)^2}.$$

We need to maximize $\Phi(c_1)$ over $0 \leq c_1 \leq 1$. First the function $\varphi(\alpha) = \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2}$ is an increasing function of $\alpha > 0$ and $\frac{3}{4} < \varphi(\alpha) < 1$ for $\alpha > 0$. Also, the function ν_2 defined by (16) is an increasing function of $\alpha > 0$, and $-\frac{3}{4} < \nu_2 < \frac{1}{3}$, which implies that $|\nu_2| < \frac{3}{4}$ for every $\alpha > 0$. By using these facts, we deduce that A defined by (17) is positive, and also that

$$\frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} > |\nu_2|, \quad \alpha > 0. \tag{18}$$

If $|\nu_2| \leq \frac{3}{(2+\alpha)^2}$, then (17) shows that $B \leq 0$, and it is evident that $\Phi(c_1) \leq \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2}$. Hence

$$|H_2(2)| \leq \frac{4}{(1+\alpha)(3+\alpha)} \cdot \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2} = \left(\frac{2}{2+\alpha}\right)^2.$$

If $|\nu_2| \geq \frac{3}{(2+\alpha)^2}$, then $B \geq 0$ and

$$\max \Phi(c_1) = \max\{\Phi(0), \Phi(1)\} = \max\left\{\frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2}, |\nu_2|\right\} = \frac{(1+\alpha)(3+\alpha)}{(2+\alpha)^2}$$

because of the relation (18). In this case also,

$$|H_2(2)| \leq \left(\frac{2}{2+\alpha}\right)^2.$$

The result is best possible as shown by the function f given in (12) with $\omega(z) = z^2$. □

References

[1] L. A. Aksent'ev, *Sufficient conditions for univalence of regular functions*, Izv. Vyssh. Uchebn. Zaved. Mat. **1958** (1958), no. 3 (4), 3–7.

[2] R. Fournier and S. Ponnusamy, *A class of locally univalent functions defined by a differential inequality*, Complex Var. Elliptic Equ. **52** (2007), no. 1, 1–8. `https://doi.org/10.1080/17476930600780149`

[3] D. V. Krishna and T. RamReddy, *Second Hankel determinant for the class of Bazilevic functions*, Stud. Univ. Babeş-Bolyai Math. **60** (2015), no. 3, 413–420. https://doi.org/10.1080/17476933.2015.1012162

[4] S. K. Lee, V. Ravichandran, and S. Supramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl. **2013** (2013), 281, 17 pp. https://doi.org/10.1186/1029-242X-2013-281

[5] Marjono, J. Sokół, and D. K. Thomas, *The fifth and sixth coefficients for Bazilevič functions* $\mathcal{B}_1(\alpha)$, Mediterr. J. Math. **14** (2017), no. 4, Paper No. 158, 11 pp. https://doi.org/10.1007/s00009-017-0958-y

[6] M. Obradović, *A class of univalent functions*, Hokkaido Math. J. **27** (1998), no. 2, 329–335. https://doi.org/10.14492/hokmj/1351001289

[7] ———, *A class of univalent functions. II*, Hokkaido Math. J. **28** (1999), no. 3, 557–562. https://doi.org/10.14492/hokmj/1351001237

[8] M. Obradović, S. Ponnusamy, and K.-J. Wirths, *Geometric studies on the class* $\mathcal{U}(\lambda)$, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 3, 1259–1284. https://doi.org/10.1007/s40840-015-0263-5

[9] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc. **33** (1972), 392–394. https://doi.org/10.2307/2038067

[10] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. (2) **41** (1966), 111–122. https://doi.org/10.1112/jlms/s1-41.1.111

[11] ———, *On the Hankel determinants of univalent functions*, Mathematika **14** (1967), 108–112. https://doi.org/10.1112/S002557930000807X

[12] D. V. Prokhorov and J. Szynal, *Inverse coefficients for* (α, β)*-convex functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **35** (1981), 125–143 (1984).

[13] R. Singh, *On Bazilevič functions*, Proc. Amer. Math. Soc. **38** (1973), 261–271. https://doi.org/10.2307/2039275

[14] A. Vasudevarao and H. Yanagihara, *On the growth of analytic functions in the class* $U(\lambda)$, Comput. Methods Funct. Theory **13** (2013), no. 4, 613–634. https://doi.org/10.1007/s40315-013-0045-8

Rosihan M. Ali
School of Mathematical Sciences
Universiti Sains Malaysia
11800 Penang, Malaysia
Email address: rosihan@usm.my

See Keong Lee
School of Mathematical Sciences
Universiti Sains Malaysia
11800 Penang, Malaysia
Email address: sklee@usm.my

Milutin Obradović
Department of Mathematics
Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73
Belgrade 11000, Serbia
Email address: obrad@grf.bg.ac.rs