RACSAM (2021) 115:49
https://doi.org/10.1007/s13398-020-00977-2

ORIGINAL PAPER

Third Hankel determinant for univalent starlike functions

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Received: 11 May 2020 / Accepted: 22 November 2020

Abstract

In this paper we obtain the bound of the third Hankel determinant

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

for the class $\mathcal{S}^*$ of univalent starlike functions, i.e. the functions which satisfy in the unit disk the condition $\operatorname{Re} \frac{zf'(z)}{f(z)} > 0$. In our research we apply the correspondence between starlike functions and Schwarz functions and the results obtained by Prokhorov and Szynal and by Carlson.

Keywords Univalent functions · Starlike functions · Hankel determinant

Mathematics Subject Classification 30C45 · 30C50

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the family of all functions f analytic in $\mathbb{D}$, normalized by the condition $f(0) = f'(0) - 1 = 0$. It means that f has the expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ . \tag{1}$$

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Published online: 08 January 2021

Let $\mathcal{S}$ be the class of all univalent functions in $\mathbb{D}$ with the Taylor series expansion (1) and let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ consisting of starlike functions. It is well known that

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in \mathbb{D} \right\}. \tag{2}$$

For $q \geq 1$ and $n \geq 1$, the Hankel determinant $H_q(n)$ of a given function f of the form $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

In particular,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

and

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

are called, for brevity, the second and the third Hankel determinant, respectively.

The problem of computing the bounds, possibly sharp, of Hankel determinants in a given family of functions attracted the attention of many mathematicians. In general, for the class $\mathcal{S}$ of univalent functions, little is known. The most important result is the one of Hayman ([8]) who showed that $|H_2(n)| \leq M n^{1/2}$, where M is an absolute constant, and that this rate of growth is the best possible. However much more have been done for subclasses of $\mathcal{S}$.

For the second Hankel determinant it is worth citing the results of Janteng et al. For the following three classes: S^* of starlike functions, $\mathcal{K}$ of convex functions and $\mathcal{R}$ of functions with bounded turning they proved in [9,10] that $|H_2(2)|$ is bounded by 1, 1/8 and 4/9, respectively. The interesting results for starlike functions of order α and for strongly starlike functions of order α were obtained by Cho et al. [4,5]. They got $|H_2(2)| \leq (1-\alpha)^2$, $\alpha \in [0, 1)$ and $|H_2(2)| \leq \alpha^2$, $\alpha \in (0, 1]$, respectively. For the class $S^*(\varphi)$ of Ma-Minda starlike functions the exact bound of $|H_2(2)|$ was found by Lee et al. [13] (see also [6]).

Although the problem of finding the bound of $|H_3(1)|$ is much more complicated, a few important results were obtained in this direction. For the class $\mathcal{K}$ it was shown by Kowalczyk et al. in [12] Kowalczyk et al. that $|H_3(1)| \leq 4/135$. Lecko et al. found in [16] that 1/9 is the sharp bound of the third Hankel determinant for starlike functions of order 1/2. In [17] it was proved that $|H_3(1)| \leq 1/4$ for the class $\mathcal{U}$, which consists of univalent functions satisfying $\left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1$. What is interesting, this class does not entirely lie inside $\mathcal{S}^*$. It is worth noting that all known sharp estimates of $|H_3(1)|$ for subclasses of $\mathcal{S}$ are less than 1. On the other hand, the class $\mathcal{T}(\alpha)$ of functions which satisfy the condition $\operatorname{Re} \frac{f(z)}{z} > \alpha$, $\alpha \in [0, 1)$ is not contained in $\mathcal{S}$. For the class of this type, but for particular cases $\alpha = 0$ and $\alpha = 1/2$, Kowalczyk et al. [12] obtained the sharp bounds of $|H_3(1)|$; these values are equal to 4 and 1.

In this paper we want to improve the results concerning the third Hankel determinants for the class $\mathcal{S}^*$. The first result of this type for starlike functions was very far from accuracy.

In [1] it was proved that $|H_3(1)| \leq 16$. The result $|H_3(1)| \leq 1$ was found in [19] and it was improved by Kwon et al. [14] who obtained a better bound 8/9. In [19] it was also shown that for starlike functions with 3-fold symmetry the sharp estimate $|H_3(1)| \leq 4/9$ holds. Under the additional assumption that all coefficients of $f \in \mathcal{S}^*$ are real, Kwon and Sim derived that $-4/9 \leq H_3(1) \leq \sqrt{3}/9$ (see, [15]).

In our research we use a different approach than the usual one in which the coefficients of $f \in \mathcal{S}^*$ are expressed by the corresponding coefficients of functions with positive real part. In what follows we express the coefficients of $f \in \mathcal{S}^*$ by the corresponding coefficients of Schwarz functions, i.e. functions ω that are analytic in the unit disk, $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. Moreover, we use two lemmas.

The first one was proved by Prokhorov and Szynal [18].

Lemma 1 *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then, for any real numbers μ and ν such that*

$$(\mu, \nu) \in \left\{(\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2\,,\ \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\right\}$$

the following sharp estimate holds

$$\left|c_3 + \mu c_1 c_2 + \nu c_1^3\right| \leq 1.$$

The second lemma was obtained by Carlson [3]. The Carlson's result was published in 1940 in Arkiv för matematik, astronomi och fysik edited by the Royal Swedish Academy of Sciences. The last volume of this series appeared in 1946. Since those years this result has been cited only as a problem for a reader in the book by Goodman (Problem 16, p.78 in [7]). Unfortunately, in this book the typewritting mistake appeared. Moreover, also in the original paper of Carlson there is a small mistake in the form of one of the extremal functions. For these reasons, it is worth citing the correct version of the lemma of Carlson. Firstly, let us denote by $\mathcal{B}$ the class of analytic functions in the unit disk with the Taylor series expansion $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ and such that $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 2 *Let $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ be in $\mathcal{B}$. Then*

$$|a_{2n+1}| \leq 1 - |a_0|^2 - |a_1|^2 - \ldots - |a_n|^2, \quad n = 0, 1, \ldots \tag{3}$$

and

$$|a_{2n}| \leq 1 - |a_0|^2 - |a_1|^2 - \ldots - |a_{n-1}|^2 - \frac{|a_n|^2}{1 + |a_0|}, \quad n = 1, 2, \ldots\,. \tag{4}$$

Equality in (3) holds for

$$f(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n + \varepsilon z^{2n+1}}{1 + \left(\overline{a_n} z^{n+1} + \overline{a_{n-1}} z^{n+2} + \ldots + \overline{a_0} z^{2n+1}\right)\varepsilon}, \ |\varepsilon| = 1$$

and in (4) for

$$f(z) = \frac{a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + \frac{a_n}{1+|a_0|} z^n + \varepsilon z^{2n}}{1 + \left(\frac{\overline{a_n}}{1+|a_0|} z^n + \overline{a_{n-1}} z^{n+1} + \ldots + \overline{a_0} z^{2n}\right)\varepsilon}, \ |\varepsilon| = 1\,,$$

where $a_0 \overline{a_n}^2 \varepsilon$ is non-positive real.

We can see that the inequalities from Lemma 2 improve the classical result obtained by Wiener, i.e. $|a_n| \leq 1 - |a_0|^2$ (see, [2]).

If $f \in \mathcal{B}$, then $\omega(z) = zf(z)$ is a Schwarz function. Hence, writing $\omega(z) = c_1 z + c_2 z^2 + \cdots$, we conclude from Lemma 2 that, in particular,

$$|c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|} \quad \text{and} \quad |c_4| \leq 1 - |c_1|^2 - |c_2|^2 . \tag{5}$$

Now, we are ready to formulate our result.

Theorem 1 *Let $f \in \mathcal{S}^*$ be of the form $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. Then*

$$|H_3(1)| < \frac{5}{9} .$$

Proof Let $f \in \mathcal{S}^*$. From (2) it follows that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} ,$$

where the symbol $\prec$ denotes the subordination. It means that there exists a Schwarz function $\omega(z)$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)}, \quad z \in \mathbb{D} . \tag{6}$$

We can rewrite it equivalently as

$$zf'(z)\,[1-\omega(z)] = f(z)\,[1+\omega(z)] ,$$

and compare the coefficients at powers of z. Hence, we obtain

$$\begin{aligned} a_2 &= 2c_1 , \\ a_3 &= c_2 + 3c_1^2 , \\ a_4 &= \tfrac{2}{3}(c_3 + 5c_1c_2 + 6c_1^3) , \\ a_5 &= \tfrac{1}{6}(3c_4 + 14c_1c_3 + 43c_1^2c_2 + 30c_1^4 + 6c_2^2). \end{aligned} \tag{7}$$

Using (7) we have

$$\begin{aligned} H_3(1) &= \frac{1}{18}\Big[3c_1^4c_2 + 6c_1^3c_3 + 10c_1c_2c_3 - 8c_3^2 - 11c_1^2c_2^2 + 9(c_2 - c_1^2)c_4\Big] \\ &= \frac{1}{18}\Big[-7c_3\left(c_3 - \tfrac{10}{7}c_1c_2\right) - c_3^2 + 3c_1^2c_2\left(c_1^2 - c_2\right) + 6c_1^3c_3 - 8c_1^2c_2^2 + 9(c_2 - c_1^2)c_4\Big]. \end{aligned}$$

If we apply the triangle inequality and $\left|c_3 - \frac{10}{7}c_1c_2\right| \leq 1$ (the case $\mu = -\frac{10}{7}$, $\nu = 0$ in Lemma 1) and Formulae (5) we get

$$\begin{aligned} |H_3(1)| &\leq \frac{1}{18}\Big[7|c_3| + |c_3|^2 + 3|c_1|^2|c_2| + 6|c_1|^3|c_3| + 8|c_1|^2|c_2|^2 + 9\left(|c_2| + |c_1|^2\right)|c_4|\Big] \\ &\leq \frac{1}{18}\Big[(7 + 6|c_1|^3)\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + \left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right)^2 \\ &\quad + 3|c_1|^2|c_2| + 8|c_1|^2|c_2|^2 + 9\left(|c_2| + |c_1|^2\right)(1 - |c_1|^2 - |c_2|^2)\Big] =: \frac{1}{18}h(|c_1|, |c_2|). \end{aligned}$$

The function $h(x, y)$ can be written as

$$h(x, y) = h_1(x, y) + h_2(x, y) + h_3(x, y) ,$$

where

$$h_1(x,y) = \left[\tfrac{y}{(1+x)^2} - 1\right] y^3 \tag{8}$$

$$h_2(x,y) = 6(1-x^2)y - \tfrac{9-x^2+7x^3}{1+x} y^2 \tag{9}$$

$$h_3(x,y) = 3y - 8y^3 + 2(1-x^2)(4+4x^2+3x^3)\ . \tag{10}$$

Now, we shall maximize $h(x, y)$ in the set

$$\Omega = \{(x,y) : x \geq 0,\, y \geq 0,\, y \leq 1-x^2\}\ .$$

The inequality $y \leq 1 - x^2$ follows immediately from (5).

Clearly,

$$h_1(x,y) \leq 0\ . \tag{11}$$

Since h_2 is a quadratic function of a variable y, it takes its greatest value at $y_* = \frac{3(1-x)(1+x)^2}{9-x^2+7x^3}$. It is easy to check that $y_* < 1 - x^2$ for all $x \in [0, 1]$ and

$$h_2(x,y) \leq h_2(x,y_*) = \frac{9(1-x)^2(1+x)^3}{9-x^2+7x^3} =: g_2(x)\ .$$

We find the critical points of g_2. We have

$$g_2'(x) = \frac{9(1-x)(1+x)^2(9-43x-22x^2+10x^3-14x^4)}{(9-x^2+7x^3)^2}$$

and $(9 - 43x - 22x^2 + 10x^3 - 14x^4)' = -43 - 44x + 30x^2 - 56x^3 < 0$ for all $x \in [0, 1]$. For this reason, the only solution of $9 - 43x - 22x^2 + 10x^3 - 14x^4 = 0$ in $(0, 1)$, i.e. $x_2 = 0.191\ldots$, is a critical point of g_2 in this interval. Hence,

$$h_2(x,y) \leq g_2(x_2) = 1.104\ldots\ . \tag{12}$$

For h_3 we immediately obtain

$$h_3(x,y) \leq h_3(x, \tfrac{1}{2\sqrt{2}}) = \tfrac{1}{\sqrt{2}} + 2(1-x^2)(4+4x^2+3x^3) =: g_3(x)\ .$$

But $g_3'(x) = 2x^2(9 - 16x - 15x^2)$, so $g_3(x) \leq g_3(x_3)$, where

$$x_3 = \frac{\sqrt{199}-8}{15} = 0.407\ldots\ .$$

Consequently,

$$h_3(x,y) \leq g_3(x_3) = \frac{1}{\sqrt{2}} + \frac{112256 + 137708\sqrt{199}}{253125} = 8.825\ldots\ . \tag{13}$$

Combining (11)–(13), we obtain

$$h(x,y) \leq 9.929\ldots < 10\ ,$$

so

$$|H_3(1)| < \frac{5}{9}\ .$$

□

Remark 1 The method applied in the proof of Theorem 1 seems to be good for many similar problems involving coefficients of Schwarz functions. In any case, it is necessary not only to apply Formulae (5), or in general, Lemma 2, but also the optimal reduction of a maximized expression with help of Lemma 1 in its general form.

Funding The project/research was financed in the framework of the project Lublin University of Technology-Regional Excellence Initiative, funded by the Polish Ministry of Science and Higher Education (contract no. 030/RID/2018/19).

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