One algorithm for testing annulling of mixed trigonometric polynomial functions on boundary points

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***Abstract*—This paper considers one problem of modern automated theorem provers. While there are many different numerical methods for computation of real roots of polynomials, theorem provers require higher level of assurances of result correctness. In this paper is shown one method that circumvents limitations of application of Sturm's theorem for such tasks.**

***Index Terms*—algorithm, mixed trigonometric polynomial functions, number of real roots, polynomial, proof**

I. Introduction

In this paper we consider MTP - mixed trigonometric polynomial functions

$$f(x) = \sum_{i=1}^{n} \alpha_i \, x^{p_i} \sin^{q_i} x \cos^{r_i} x, \tag{1}$$

for non zero coefficients $\alpha_i \in R\backslash\{0\}$, where $(p_i, q_i, r_i) \in N_0^3$ are variety of tuples from $N_0 = N \cup \{0\}$ while considering argument $x \in [0, \pi/2]$. MTP functions and inequalities are subject of studies in mathematics and computer science [10], [3], [4], [8], [9]. Specially, for such functions in papers [1], [5], [6], [7], as well as in dissertation [2], are considered steps of automated proving.

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Let us further assume that coefficients of MTP function are rational non zero values. For such MTP functions we consider problem of determining conditions such that

$$f(0) = 0, \quad \text{odnosno} \quad f(\pi/2) = 0.$$

is true. Stated problem can be considered numerically, but from standpoint of theory of proving it is necessary to obtain procedure that would resolve wether starting MTP function is annulling on border points. Such procedure is necessary for completely automated such MTP inequalities over segment $[0, \pi/2]$. Let us emphasize that inside of interval $(0, \pi/2)$ one such method of proving is given with paper [4] where it was reduced to level of algorithms described by pseudocode in dissertation [2].

II. Mathematical criteria

Stated considerations are related to class of MTP functions with properties that coefficients are non zero values from

$$Q(\pi) = \{P_n(\pi) : P_n \in Q[x]\},$$

where $P_n \in Q[x]$ is marking that P_n is polynomial with rational coefficients.

1. Let us for MTP function $f(x)$ divide set of indices (1) by which summation is conducted to two disjunct sets $\{j_1\}_{j_1=1}^{n_1}$ and $\{j_2\}_{j_2=1}^{n_2}$ such that

$$p_{j_1} = q_{j_1} = 0$$

and

$$p_{j_2} \neq 0 \vee q_{j_2} \neq 0$$

respectively (while still $n_1+n_2=n$). Then, based on

$$f(x) = \sum_{j_1=1}^{n_1} \alpha_{j_1} \cos^{r_{j_1}} x + \sum_{j_2=1}^{n_2} \alpha_{j_2} x^{p_{j_2}} \sin^{q_{j_2}} x \cos^{r_{j_2}} x,$$

following statement holds.

Statement 1. *It holds that*:

$$f(0)=0 \iff \sum_{j_1=1}^{n_1} \alpha_{j_1} = 0\,.$$

2. For MTP function $f(x)$ let the set of indices (1) by which summation is conducted be divided to two disjunct sets $\{k_1\}_{k_1=1}^{m_1}$ and $\{k_2\}_{k_2=1}^{m_2}$ such that

$$r_{k_1} \neq 0$$

and

$$r_{k_2} = 0$$

hold true respectively (while $m_1+m_2=n$). Then, based on

$$f(x) = \sum_{k_1=1}^{m_1} \alpha_{k_1} x^{p_{k_1}} \sin^{q_{k_1}} x \cos^{r_{k_1}} x + \sum_{j_2=1}^{m_2} \alpha_{k_2} x^{p_{k_2}} \sin^{q_{k_2}} x,$$

following holds.

Statement 2. *It holds that*:

$$f(\pi/2)=0 \iff \sum_{j_2=1}^{m_2} \alpha_{k_2} \left(\frac{\pi}{2}\right)^{p_{k_2}} = 0.$$

As we consider Statement 2, we relay on fact that π is transcendental real value, and as such is not solution to any polynomial equation with rational coefficients.

III. Algorithm

Previous mathematical conclusions can be translated to following algorithm for testing if MTP function is annuled for $x=0$

```
procedure mtp_annuled_in_0
input:
    poly:array of terms
         of MTP function
output:
    out:boolean

begin
    sum:=0
    for t in poly do
        cond_p:= t.p==0
        cond_q:= t.q==0

        if (cond_p and cond_q) do
            sum:=sum+t.alpha
    end for

    return sum==0
end
```

Similarly can be done for algorithm for testing if MTP function is annuled for $x=\frac{\pi}{2}$

```
procedure mtp_annuled_in_pi_half
input:
    poly:array of terms
         of MTP function
output:
    out:boolean

begin
    sum:=0
    for t in poly do
        cond_r:= t.r==0

        if (cond_r) do
            sum:=sum+t.alpha*pow(pi/2,t.p)
    end for

    return sum==0
end
```

IV. Examples

Let us take a look at few examples from thesis ... In all of these examples determining if values of MTP function in points 0 and $\frac{\pi}{2}$ are equal to 0 is important because it leads to different paths of proving inequalities related to these functions.

A. $f(x) = \sin^2 x \cos x + x \sin x - 2x^2 \cos x$

First example is from proving Wilkers inequality which can be transformed to proving that $f(x) = \sin^2 x \cos x + x \sin x - 2x^2 \cos x$ is greater than 0 for $x \in \left(0, \frac{\pi}{2}\right)$. Let us first look at values of values α,p,q and r for terms of function $f(x)$:

1) $term = \sin^2 x \cos x$
   $\alpha = 1$
   $p = 0$
   $q = 2$
   $r = 1$

2) $term = x \sin x$
   $\alpha = 1$
   $p = 1$
   $q = 1$
   $r = 0$

3) $term = -2x^2 \cos x$
   $\alpha = -2$
   $p = 2$
   $q = 0$
   $r = 1$

No term had values p and q both equal to 0, so sum of such terms can is automatically equal to 0. This leads to conclusion that $f(0)=0$. We can also determine that $f\left(\frac{\pi}{2}\right) \neq 0$. There is only one term with value r equal to 0 and for that term $\alpha = 1$ and $p = 1$, so sum from formula is equal to $1 \cdot \frac{\pi}{2}$. This leads to conclusion that $f\left(\frac{\pi}{2}\right)$ is not equal to 0.

B. $f(x) = \pi \sin x - x \cos x - 2x$

Second example is from proving Shafer-Fink inequality which can be transformed to proving that $f(x) = \pi \sin x - x \cos x - 2x$ is greater than 0 for $x \in \left(0, \frac{\pi}{2}\right)$. Let us first look at values of values α,p,q and r for terms of function $f(x)$:

1) $term = \pi \sin x$
   $\alpha = \pi$
   $p = 0$
   $q = 1$
   $r = 0$

2) $term = -x \cos x$
   $\alpha = -1$
   $p = 1$
   $q = 0$
   $r = 1$

3) $term = -2x$
   $\alpha = -2$
   $p = 1$
   $q = 0$
   $r = 0$

There are no terms where $p = q = 0$, which means that sum is equal to 0. This leads to conclusion that $f(0) = 0$. We can also determine that $f\left(\frac{\pi}{2}\right) = 0$. There are two terms where $r = 0$, and sum of products $\pi \cdot 1$ and $-2 \cdot \frac{\pi}{2}$ is 0. This means that $f\left(\frac{\pi}{2}\right) = 0$.

is two terms with value r equal to 0 and for those terms value of α is equal to 2 and -3 respectively. This leads to conclusion that $f\left(\frac{\pi}{2}\right)$ is not equal to 0.

V. Conclusion

Presented algorithm for deciding if MTP function is annulling at border points relates to MTP functions with rational coefficients. Those algorithms enable symbolic implementations, which would eliminate problem of numerical evaluation of values with MTP functions in border points which is first step of methodology for proving MTP inequalities, as shown in [1], [2]. With this can be achieved complete symbolic proving of such inequalities.

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